New results on interval comparison

Meltem Öztürk, Alexis Tsoukiàs

Abstract

In this paper we propose a general framework for the interval comparison which finds different classes of preference structures having an interval representation. First of all we propose an axiomatisation for an interval representation and then we present some results which generalise the characterisation of some classical properties such as transitivity, Ferrers relation, etc. We make use of these results to find all the preference structures which satisfy our axioms and which may be represented with classical intervals (interval defined with two extreme points) and with 3-points intervals (intervals with an intermediate point).

Key words : preference modelling, intransitivity, interval representation

1 Introduction

The comparison of objects is an important issue in the modelling of real life situations. In decision aiding, such comparisons are done through the use of binary relations defined on the set of alternatives (A) and the set of these relations is called *preference structure*. A common way is to consider two relations, P and I, called respectively strict preference and indifference. There exist in the literature other types of preference structures having more than two relations. An interested reader can find more details about such structures in [DDF84, RV84, Roy85, Vin88, TV95, TV98].

In this paper we are interested in complete preference structures having two binary relations (P is an asymmetric relation, I is the symmetric complement of P, the relation $P \cup I$ is complete and the relation $P \cap I$ is empty). Using complete structures we suppose that the decision maker is able to compare the alternatives (we can have aPb, bPa or aIb).

*LAMSADE, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris cedex 16, France. {ozturk,tsoukias}@lamsade.dauphine.fr

Let us remember that there exist also a number of studies done on partial orders where a third symmetric and irreflexive relation J, ($aJb \Longleftrightarrow not(aPb), not(bPa), not(aIb)$) called incomparability, is used ([DM41, Fis85, TV97]).

Linear orders and weak orders are the most elementary complete structures. A linear order consists of a ranking of alternatives from the best one to the worst one without any ex aequo while a weak order defines indifference is an equivalence relation. A weak order is indeed a total order of the equivalence (indifference) classes of A. Following the well-known works of Luce ([Luc56]), a new line of research appeared in preference modelling with the introduction of an intransitive indifference relation. A number of studies have shown that the transitivity of the indifference can be empirically falsifiable in some context. Undoubtedly the most famous example on this subject is that of [Luc56], with a cup of sweetened coffee. Before him, some authors have already suggested this phenomenon (see [Arm39, GR36, Fec60, Hal55, Poi05]) and Fishburn and Monjardet ([FM92]) have presented some historical comments on the subject. Relaxing the property of transitivity of indifference results in different structures. It results that the semiorders are the simplest ones and there exist other ordered sets generalizing semiorders. Fishburn, in [Fis97], distinguishes ten nonequivalent ordered sets having an intransitive indifference relation. These are semiorders, interval orders, split semiorders, split interval orders, tolerance orders, bitolerance orders, unit tolerance orders, bisemiorders, semitransitive orders and subsemitransitive orders.

In real life situation intransitivity of indifference is generally related to the presence of threshold as it can be seen in the following example:

Example 1 *Suppose that we have to define preference relations between three alternatives a, b and c which are evaluated with respect to their performance such that:*

- *the performance of a: 1000*

- *the performance of b: 1020*

- *the performance of c: 1040.*

Let us imagine that the decision maker gives some additional information such that: he prefers one alternative to another one if the first one is greater than the second one and if the difference between their performance is greater than thirty (threshold); otherwise he is indifferent between them. It is easy to see that he is indifferent between a and b and between b and c while he prefers c to a for a maximization problem.

The presence of a threshold can be represented by an interval. It is sufficient to note that associating a value $g(x)$ (evaluation of the alternative) and a strictly positive value q (threshold) to each element x of A is equivalent to associating two values: $f_1(x) = g(x)$ (representing the left extreme of an interval) and $f_2(x) = g(x) + q$ (representing the right extreme of the interval; obviously: $f_2(x) > f_1(x)$ always holds). In the example 1, the alternative x will be preferred to the alternative y if and only if the interval associated to x is completely to the right (no intersection) of the interval associated to y.

As this small example shows, intervals appear as appropriate tools for the representation of intransitivity, thus Fishburn showed that the majority of the ten classes presented in his paper [Fis97] has an interval representation.

Although there exists a large literature on the preference modelling and intransitivity, a unification of representation for such structures is missing. For that reason in this paper we propose a general framework where a special interval representation is proposed. The aim of our framework is to propose a common language for the study of preference structures having an interval representation (defining a mapping to a class of preference structures through a function from the set of objects to the set of intervals) and to find some new results about these structures.

2 Basic Notions

Let us consider a countable finite set A of objects. Each object x of A is represented by an interval and we associate to each interval a finite number n of ordered points. We call such an object a *n-points interval*. If not otherwise mentioned, we will use indifferently the notation x for the object x and its associated interval. We denote a n-points interval x by a vector of n elements: $\langle f_1(x), \cdots, f_n(x)\rangle$, with for all x in A and i in $\{1, \ldots, n-1\}$, $f_i(x) < f_{i+1}(x)$. Let us remark that elements f_i are not necessarily equi-distanced. Figure 1 illustrates the graphical representation of a n-points interval.

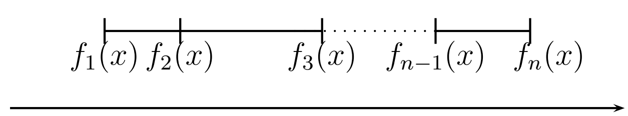


Figure 1: n-points interval representation

We call a "*relative position*" and denote by $\varphi(x, y)$ the position of the interval x with respect to the interval y ($\varphi(x, y) \neq \varphi(y, x)$). We define this notion by the help of a n-tuple as in the following:

Definition 1 (Relative position) *The relative position $\varphi(x, y)$ is a n-tuple ($\varphi_1(x, y), \ldots,$ $\varphi_n(x, y)$) where $\varphi_i(x, y)$ represents the number of j such that $f_i(x) \leq f_j(y)$.*

Intuitively, φ represents to what extend the position of two intervals is close to the case of two disjoint intervals, case which guaranties a strict preference.

Example 2 *Let x and y be two 3-points intervals represented in figure 2, then $\varphi(x, y) = (1, 0, 0)$ since there is only $f_3(y)$ being greater than $f_1(x)$, and $f_2(x)$ and $f_3(x)$ are greater than all the points of y.*

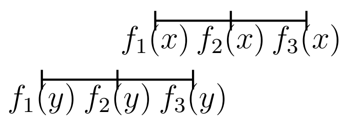


Figure 2: Relative position $\varphi(x, y) = (1, 0, 0)$

It is easy to see that the relative position shows the closest points of one interval with respect to the points of another one: $\forall i$, $f_{n-\varphi_i(x,y)}(y) < f_i(x) < f_{n+1-\varphi_i(x,y)}(y)$.

The number of all possible relative positions depends on n: the biggest n, the biggest the number of relative positions.

Proposition 1 *[TV04] Let x and y be two n-points intervals then the number of possible relative positions $\varphi(x, y)$ is $m = \frac{(2n)!}{(n!)^2}$.*

For example in the case of 2-points intervals, we get: $m = (2 * 2)!/(2!)^2 = 6$; which means that two 2-points intervals can have 6 different situations: the interval x completely to the right of the interval y, the intervals x and y have a non empty intersection (x being to the right of y), the interval x included in the interval y and the symmetric cases of these three situations.

The table 1 shows the number of possible relative positions depending on the number n. The relative position of x with respect to y can be calculated by using the relative

	$n = 2$	$n = 3$	$n = 4$	n
Relative positions	6	20	70	$\frac{(2n)!}{(n!)^2}$

Table 1: Number of relative positions depending on n

position of y with respect to x (we use the notation $\varphi^T(x, y) = \varphi(y, x)$).

Proposition 2 *Let $\varphi(x, y)$ be the relative position of n-points interval x with respect to the n-points interval y then*

$$\forall i, \ \varphi_i^T(x, y) = n - |j, \varphi_j \geq (n + 1 - i)|$$

Example 3 *Let us retake example 2. We already showed that the relative position of x with respect to y is $(1,0,0)$. We calculate now the n-tuple associated to its converse:*

$$\varphi_1^T(x,y) = 3 - |\{j, \varphi_j \geq (4-1)\}| = 3 - 0 = 3,$$
$$\varphi_2^T(x,y) = 3 - |\{j, \varphi_j \geq (4-2)\}| = 3 - 0 = 3,$$
$$\varphi_3^T(x,y) = 3 - |\{j, \varphi_j \geq (4-3)\}| = 3 - 1 = 2.$$

Hence we get $\varphi^T(x,y) = (3,3,2)$, which means that the relative position of y with respect to x is $\varphi(y,x) = (3,3,2)$.

We will use the notion of relative position in order to characterize preference structures. We analyze first of all the relation between different relative positions since one relative position may give more argument for a preference relation than another one. For example, for the representation of a strict preference the case of two disjoint intervals may be more suitable than a case with two intervals, one included in the other. For that reason we introduce a new binary relation, called *"stronger than"*, on the set of relative positions.

Definition 2 ("Stronger than" relation) *Let φ and φ' be two relative positions, then we say that φ is "stronger than" φ' and note $\varphi \rhd \varphi'$ if $\forall i \in \{1,\ldots,n\},\ \varphi_i \leq \varphi'_i$.*

We present an example showing how we define a ‘stronger than” relation.

Example 4 *Let $\varphi(x,y)$ and $\varphi(x,t)$ be two relative positions of the figure 3. We have $\varphi(x,y) = (1,1,0)$, $\varphi(x,t) = (2,1,0)$. We get "$\varphi(x,y)$ is stronger than $\varphi(x,t)$" since $1 \leq 2,\ 1 \leq 1$ and $0 \leq 0$.*

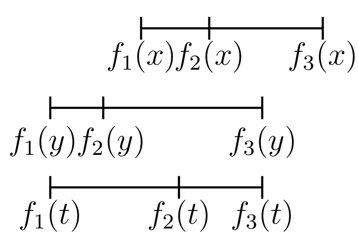


Figure 3: Example: $(1,1,0) \rhd (2,1,0)$

The "stronger than" relation characterised as in definition 2 satisfies some classical properties:

Proposition 3 *The "stronger than" relation is a partial order (reflexive, antisymmetric and transitive) defining a lattice on the set of possible relative positions.*

Proof. $\rhd$ is a partial order since it is induced from the relation "$<$" which is reflexive, antisymmetric and transitive. ■

Let us remark that the relation $\rhd$ is not complete. We present in figure 4 the graph of the relation $\rhd$ associated to 3-points intervals. It is easy to see, for example, that we have $(2,0,0) \not\rhd (1,1,0)$ and $(1,1,0) \not\rhd (2,0,0)$.

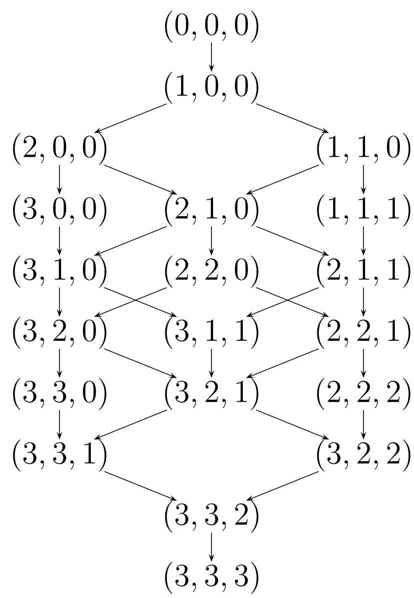


Figure 4: Graph of "stronger than" relation for $n = 3$

3 Preference structures with n-points intervals

We are ready now to define preference structure with a strict preference relation P and an indifference relation I, and having an n-points interval representation. Intuitively, we will define P as a set of relative positions satisfying some constraints, and construct I as the complement of P. In order to define the constraints of the construction, we do an axiomatization:

Axiom 1 *The relation $P \cup I$ is complete and I is the complement of P (*i.e. $I(x,y) \iff \neg P(x,y) \wedge \neg P(y,x)$ *).*

Axiom 2 *The relations $P(x, y)$ and $I(x, y)$ depends only on the relative position of x and y.*

Axiom 3 *If a relative position φ is in the set of the strict preference P then all the relative positions which are stronger than φ are also in the set of P.*

Axiom 4 *If forall i, $f_i(x) < f_i(y)$ then $P(x, y)$ is not satisfied.*

Axiom 5 *The set of relative positions forming P has one and only one weakest relative position (the relative position which is dominated in the sense of "stronger than" relation by all the relative positions of the set of P).*

Axiom 1 shows that P and I are exhaustive and exclusive, axiom 2 presents the comparison parameters and axiom 3 guaranties the monotonicity. Every relative position is not a good candidate to represent a strict preference. Axiom 4 eliminates some undesired situations for the relation P. The role of the weakest relative position of a set of P is very important since we can determine all the other elements of the set by the help of the weakest one (axiom 3). Using this reasoning, the axiom 5 guarantees a unique representation for the strict preference relations by forbidding the existence of more than one weakest relative positions in their set (each set of P can be represented by its weakest relative position).

According to this axiomatisation each relative position, excepts the one violating the axiom 4, will be once and only once the weakest relative position of the set of relative positions forming the strict preference relation. This remark helps us to calculate the number of sets satisfying our five axioms.

Proposition 4 *The number of sets satisfying axioms 1-5 for n-points intervals is $\frac{(2n)!}{(n!)^2} - \frac{1}{n+1}\binom{2n}{n}$.*

It means that there are respectively 4, 15 and 56 sets satisfying our axiomatisation when 2-points, 3-points and 4-points intervals are used for the representation.

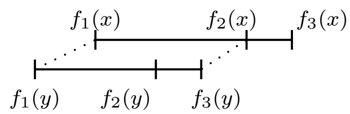
We can present now the n-points interval representation of a preference structure satisfying axioms 1-5. In this representation we denote by $P_{\leq\varphi}$ the preference relation having the relative position φ as the weakest one. In the same way we denote by $I_{\leq\varphi}$ the symmetric complement of the relation $P_{\leq\varphi}$.

Definition 3 *Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a n-tuple in $\{0, 1, \dots, n\}$, and x and y two n-points intervals. The preference relations $P_{\leq\varphi}$, $I_{\leq\varphi}$ associated to φ where $\varphi \not\trianglerighteq (n, n-1, n-2, \dots, 1)$ is defined as*

$$\begin{aligned} P_{\leq\varphi}(x, y) &\iff \varphi(x, y) \rhd \varphi \\ I_{\leq\varphi}(x, y) &\iff \neg P_{\leq\varphi}(x, y) \wedge \neg P_{\leq\varphi}(y, x) \end{aligned}$$

It is easy to verify that the preference structure associated to a n-tuple φ characterised as in definition 3 verifies the axioms 1, 2, 3, 4, 5 and satisfies the conditions that we defined in the beginning of the paper, *i.e.* P is an asymmetric relation, I is the symmetric complement of P, the relation $P \cup I$ is complete and the relation $P \cap I$ is empty.

Now, consider the strict preference relation, presented in figure 3, having the relative position (2,0,0) as the weakest relative position.



$P_{\leq(2,0,0)} : (0,0,0) \cup (1,0,0) \cup (2,0,0)$

Then $P_{\leq(2,0,0)}(x,y)$ iff $f_1(y) < f_1(x), f_2(y) < f_2(x), f_3(y) < f_2(x)$ and $f_3(y) < f_3(x)$. We can remark that the second and the last inequalities are redundant. In order to avoid such redundancies we introduce a new notion that we call *the component set* of a n-tuple φ.

Definition 4 *Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a n-tuple in $\{0, 1, \ldots, n\}$ the component set $Cp_{\leq\varphi}$ associated to φ is the set of couples ($n - \varphi_i, i$) such that there is no $i' < i$ with $\varphi_{i'} \leq \varphi_i$.*

For instance $Cp_{\leq(2,0,0)} = \{(1,1)(3,2)\}$. Hence, $Cp_{\leq\varphi}$ represents the set of couples of points that are necessary and sufficient to be compared. Conditions on the elements of $Cp_{\leq\varphi}$ guarantee the minimality of the representation. The set $Cp_{\leq\varphi}$ contains all the information concerning the preference structure. The relations between different component sets give information about relation between different preference relation.

Proposition 5 *if $Cp_{\leq\varphi} \subseteq Cp_{\leq\varphi'}$ then $P_{\leq\varphi'} \subseteq P_{\leq\varphi}$ and $I_{\leq\varphi} \subseteq I_{\leq\varphi'}$.*

Following the definition 4, one can also say that the axiom 4 is verified by $P_{\leq\varphi}$ iff $Cp_{\leq\varphi}$ contains at least one (i, j) with $i \geq j$.

4 Some results

In this section we present some results on the definition of some classical properties of binary relations in our framework. We will then show the use of these results for the characterisation of different preference structures having an n-points interval representation.

We begin by the property of transitivity. The transitivity of the strict preference relation is required for the majority of preference structures. However some preference structures having an intransitive strict preference relation are also studied by some researchers.

For instance, Abbas proposed in its PhD thesis ([Abb94]) a new structure, called *tangential circle orders*, having an intransitive preference relation. He showed that it is possible to characterise these structures with a geometrical representation in such a way that every object is presented by a circle and when two circles have an empty intersection the preference relation P is verified, otherwise the indifference relation I is verified. The way that we define the preference structure associated to a n-tuple φ (definition 3) does not necessarily provide a transitive strict preference. We present in the following the condition that a preference structure associated to a φ must satisfy in order to have a transitive $P_{\leq\varphi}$.

Proposition 6 *$P_{\leq\varphi}$ is transitive if and only if $\forall(i,j) \in Cp_{\leq\varphi}\ i \geq j$.*

Proof.

- If $\forall(i,j) \in Cp_{\leq\varphi}\ i \geq j$ then $P_{\leq\varphi}$ is transitive: obvious.

- If $P_{\leq\varphi}$ is transitive then $\forall(i,j) \in Cp_{\leq\varphi}\ i \geq j$: We prove this result by showing that if $\exists(i,j) \in Cp_{\leq\varphi}\ i < j$ then $\exists x, y, z,\ P_{\leq\varphi}(x,y) \wedge P_{\leq\varphi}(y,z)$ and$\neg P_{\leq\varphi}(x,z)$. ■

We present now an example of intransitive strict preference relation which satisfies the definition 3.

Example 5 *Let $P^3_{\leq(3,2,0)}$ be a strict preference relation ($P^3_{\leq(3,2,0)}(x,y) \iff (f_3(y) < f_3(x)) \wedge (f_1(y) < f_2(x))$ and $Cp_{\leq\varphi} = \{(2,1)(3,3)\}$). Consider three 3-points intervals, x, y and z such that:*

$$f_1(x) < f_1(y) < f_2(x) < f_1(z) < f_2(y) < f_2(z) < f_3(z) < f_3(y) < f_3(x)$$

It is easy to see that we have $P^3_{\leq(3,2,0)}(x,y)$ and $P^3_{\leq(3,2,0)}(y,z)$ but not $P^3_{\leq(3,2,0)}(x,z)$ (see figure 5).

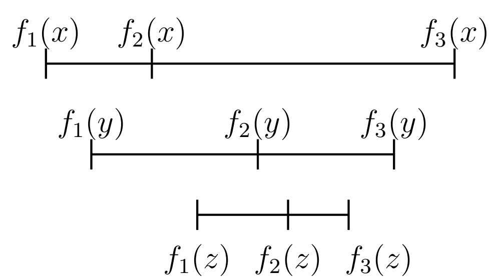


Figure 5: $P^3_{\leq(3,2,0)}(x,y)$, $P^3_{\leq(3,2,0)}(y,z)$ and not $P^3_{\leq(3,2,0)}(x,z)$

We called *intransitive set* an order having an intransitive preference relation P.

Definition 5 *The relation $P_{\leq\varphi} \cup I_{\leq\varphi}$ on a finite set A, is an intransitive set if its asymmetric part $P_{\leq\varphi}$ is intransitive.*

We showed that there is a number of preference structures which do not have a transitive indifference. We present now the conditions that a preference structure associated to a φ must satisfy in order to have a transitive indifference relation.

Proposition 7 *$I_{\leq\varphi}$ is transitive if and only if $\exists i \in \{1, \ldots, n\},\ Cp_{\leq\varphi} = \{(i, i)\}$.*

Proof.

- If $Cp_{\leq\varphi} = \{(i, i)\}$ then $I_{\leq\varphi}$ is transitive: obvious.

- If $I_{\leq\varphi}$ is transitive then $Cp_{\leq\varphi} = \{(i, i)\}$: We prove this result by contradiction. Supposing that $I_{\leq\varphi}$ is transitive we analyse two cases: $\exists(i, j) \in Cp_{\leq\varphi}, i \neq j$ and $\forall(i, j) \in Cp_{\leq\varphi}, i = j$ and $|Cp_{\leq\varphi}| > 1$. We show that these two cases are contradictory with the transitivity of $I_{\leq\varphi}$. ■

This result shows that, the only structures having transitive indifference are the ones where two points of the same level are compared (intuitively it means that we do not really need intervals for the representation, points are sufficient. Let us remark that it is the case for linear orders and weak orders). Such a result is coherent with the use of interval representation for intransitive indifference since we need more than one point to represent intransitivity ($I_{\leq\varphi}$ is intransitive if and only if $\exists i \in \{1, \ldots, n\},\ Cp_{\leq\varphi} = \{(i, j)\}$ where $i \neq j$ or $|Cp_{\leq\varphi}| > 1$).

Such a result allows us to determine the representation of weak orders with our approach.

Corollary 1 *$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a weak order if and only if $\exists i \in \{1, \ldots, n\},\ Cp_{\leq\varphi} = \{(i, i)\}$.*

Proof.

- obvious. ■

Such a result allows the existence of different representations for weak orders when n-points intervals are used. It is easy to verify the following assertion.

Proposition 8 *Let φ be a relative position of two n-points intervals. The number of $Cp_{\leq\varphi}$ representing a weak order is n.*

For example with 2-points intervals, there exist two ways to characterize weak orders: $P_{\leq(1,1)}(x, y) \iff f_1(y) < f_1(x)$ ($Cp_{\leq\varphi} = ((1, 1))$) and $P_{\leq(2,0)}(x, y) \iff f_2(y) < f_2(x)$ ($Cp_{\leq\varphi} = ((2, 2))$).

We can generalise the result of the corollary 1 on the characterisation of d-weak orders. A d-weak order is defined as the intersection of d weak orders. It is shown that these structures have an interval representation.

Proposition 9 ([Özt05]) *$P \cup I$ is a d-weak order if and only if there exists d real-valued functions g_i ($i \in \{1, \dots, d\}$) defined on A such that $\forall x, y \in A$,*

$$\begin{cases} xPy \Longleftrightarrow \forall i \in \{1, \dots, d\},\ g_i(x) > g_i(y); \\ \forall x,\ \forall i \in \{1, \dots, d-1\},\ g_{i+1}(x) \geq g_i(x). \end{cases} \tag{1}$$

Such a representation corresponds to the use of d-points intervals with our approach. We generalize this result to the case of n-points intervals where n can be greater than or equal to d.

Corollary 2 *$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a d-weak order if and only if $|Cp_{\leq\varphi}| = d$ and $\forall (i,j) \in Cp_{\leq\varphi}, i = j$.*

Let us remind that a 2-weak order corresponds to what Fishburn calls a biweak order (let us remind that a biweak order is equivalent to a bilinear order (see [Fis97]). We present in the following one of the 4-points interval representation of a 3-weak order.

Example 6 *The relation $P_{\leq(3,3,1,0)} \cup I_{\leq(3,3,1,0)}$ is a 3-weak order (we have $Cp_{\leq(3,3,1,0)} = ((1,1)(3,3)(4,4))$).*

As it can be seen in the previous example, when n-points intervals ($n > d$) are used, there exist more than one representation for d-weak orders. The number of all possible representations depends on n and is calculated easily as in the following proposition.

Proposition 10 *Let φ be a relative position of two n-points intervals. The number of $Cp_{\leq\varphi}$ representing a d-weak order is $\binom{n}{d}$.*

For example, bilinear orders have one (resp. three) representation when 2-points intervals (resp. 3-points intervals) are used.

Another class of preference structures is the one of interval orders where the indifference is not transitive. The relation $P \cup I$ is an interval order if and only if it is reflexive, complete and a Ferrers relation. Let us give first of all the definition of a Ferrers relation:

Definition 6 *Let R be a binary relation defined on the set $A \times A$. R is a Ferrers relation iff $\forall x, y, z, w \in A, \quad (xRy \wedge zRw) \Longrightarrow (xRw \vee zRy)$.*

It is shown that for a complete preference structure where P and I are exhaustive and exclusive, the relation $P \cup I$ is Ferrers if and only if $P.I.P \subseteq P$ (*i.e.* $\forall x, y, z, w \in A, \quad (xPy \wedge yIw \wedge wPz) \Longrightarrow xPw$).

Using this remark we present now the condition of a Ferrers relation in our approach.

Proposition 11 $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subseteq P_{\leq\varphi}$ *if and only if* $Cp_{\leq\varphi} = \{(i,j)\}$ *with* $i \geq j$.

Proof.

- If $Cp_{\leq\varphi} = \{(i,j)\}$ where $i \geq j$ then $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subseteq P_{\leq\varphi}$: obvious.

- If $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subseteq P_{\leq\varphi}$ then $Cp_{\leq\varphi} = \{(i,j)\}$ where $i \geq j$: We do our analysis in two parts.

- first of all if $Cp = (i,j)$ with $i < j$ then $P_{\leq\varphi}$ is not transitive. Since $I_{\leq\varphi}$ is reflexive $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subseteq P_{\leq\varphi}$ is not satisfied when $P_{\leq\varphi}$ is not transitive.
- Here we analyze two cases where $|Cp_{\leq\varphi}| > 1$: $\exists (i,j) \in Cp_{\leq\varphi}, i < j$ and $\forall (i,j) \in Cp_{\leq\varphi}, i \geq j$.
  - The first one provides an intransitive $P_{\leq\varphi}$.
  - The key point of the analysis of the second case is the definition of $I_{\leq\varphi}$ when $|Cp_{\leq\varphi}| > 1$: let $(i,j), (l,m)$ be two elements of $Cp_{\leq\varphi}$. Thus, we have $f_i(x) \geq f_j(y) \wedge f_l(y) \geq f_m(x)$ with $(i,j) \neq (l,m) \Longrightarrow I_{\leq\varphi}(x,y)$. In this case we have one point of x greater than one point of y and one point of y greater than one point of x. In such a case one can always find four elements w, x, y, z such that $P_{\leq\varphi}(w,x), I_{\leq\varphi}(x,y), P_{\leq\varphi}(y,z)$ and $\neg P_{\leq\varphi}(w,z)$. ■

It is easy to see that the proposition 11 is equivalent to the following assertion because of the axiom 4.

Proposition 12 $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subseteq P_{\leq\varphi}$ *if and only if* $|Cp_{\leq\varphi}| = 1$.

We are able now to characterize an interval order by the help of our approach:

Corollary 3 $P_{\leq\varphi} \cup I_{\leq\varphi}$ *is an interval order if and only if* $|Cp_{\leq\varphi}| = 1$.

Proof.

- If $Cp_{\leq\varphi} = \{(i,j)\}$ where $i \geq j$ then $P_{\leq\varphi} \cup I_{\leq\varphi}$ is an interval order: We prove that $P_{\leq\varphi} \cup I_{\leq\varphi}$ is reflexive and complete and $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subseteq P_{\leq\varphi}$.

- If $P_{\leq\varphi} \cup I_{\leq\varphi}$ is an interval order then $Cp_{\leq\varphi} = (i,j)$ where $i \geq j$: if $P_{\leq\varphi} \cup I_{\leq\varphi}$ is an interval order then $P_{\leq\varphi}.I_{\leq\varphi}.P_{\leq\varphi} \subseteq P_{\leq\varphi}$ which implies $|Cp_{\leq\varphi}| = 1$. ■

In the following proposition we give the number of possible representations of interval orders when n-points intervals are used.

Proposition 13 *Let φ be a relative position of two n-points intervals. The number of $Cp_{\leq\varphi}$ representing an interval order is $\frac{n(n-1)}{2}$.*

For instance there is only one representation of interval order when 2-points intervals are used while there are three representations when 3-points intervals are used.

Like for the case of weak orders we generalise the result of the corollary 3 to the case of d-interval orders. A d-interval order is defined as the intersection of d intervals orders and we showed in [Özt05] that they have an interval representation.

Proposition 14 ([Özt05]) *$P \cup I$ is d-interval order if and only if there exist $2d$ real-valued functions g_i ($i \in \{1, \ldots, 2d\}$) defined on A, such that ,*

$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \forall k \in \{1, \ldots, d\},\ g_{(2k-1)}(x) > g_{(2k)}(y), \\ \forall x,\ \forall i \in \{1, \ldots, d-1\},\ g_{i+1}(x) \geq g_i(x). \end{cases}$$

Similar to the case of d-weak order, the interval representation that we suggested in proposition 14 for d-interval orders makes use of $2d$ functions g_i defined on disjoint sets. Such a representation corresponds to the use of $2d$-points intervals with our approach. In addition with such a representation every ordered index belonging to $Cp_{\leq\varphi}$ is formed by two consecutive points g_i and g_{i+1}. We generalize this result to the case of n-points intervals where n can be greater than or equal to $2d$.

Corollary 4 *$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a d-interval order if and only if $|Cp_{\leq\varphi}| = d$ and $\forall (i, j) \in Cp_{\leq\varphi}, i \geq j$.*

Let us remind that a special class of d-interval orders is the class of 2-interval orders which is identical to the class of trapezoid orders and to the class of bitolerance orders ([Fis85]).

Corollary 5 *$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a bitolerance order if and only if it is a d-interval order and $|Cp_{\leq\varphi}| = 2$.*

The number of all possible n-points interval representation for bitolerance orders is given in the following.

Proposition 15 *Let φ be a relative position of two n-points intervals. The number of $Cp_{\leq\varphi}$ representing a bitolerance order is $2\binom{n}{4} + \binom{n}{3}$.*

When 3-points intervals are used, there is just one representation for bitolerance orders which is the one of split interval orders.

Finally we are interested in a preference structure which is defined as the intersection of a weak order and an interval order. We called such a structure a *triangle order*. Like in the case of d-weak orders and d-interval orders, these structures have an interval representation.

Proposition 16 ([Özt05]) *$P \cup I$ is a triangle order if and only if there exists 3 real-valued functions g_i ($i \in \{1, 2, 3\}$) defined on A, such that*

$$\begin{cases} \forall x, y \in A,\ xPy \Longleftrightarrow \begin{cases} g_1(x) > g_1(y), \\ g_2(x) > g_3(y), \end{cases} \\ \forall x,\ \forall i \in \{1, 2\},\ g_{i+1}(x) \geq g_i(x). \end{cases} \tag{2}$$

We generalize this result to the case of n-points intervals where $n \geq 3$.

Proposition 17 *$P_{\leq\varphi} \cup I_{\leq\varphi}$ is a triangle order if and only if $Cp_{\leq\varphi} = \{(l, l), (i, j)\}$, where $i \geq j$.*

Let us remark that if the index l of the ordered index of the weak order is smaller then the indexes of interval order ($l < j$) then the resulting triangle is oriented to the left, otherwise it is oriented to the right.

The number of all the possible n-points interval representations that a triangle order can have is given hereafter.

Proposition 18 *Let φ be a relative position of two n-points intervals. The number of $Cp_{\leq\varphi}$ representing a triangle order is $2\binom{n}{3}$.*

Triangle orders have two representations when 3-points intervals are used.

We recapitulate in table 2 the number of n-points interval representations of some preference structures for 2, 3, 4 and n-points intervals.

The results of this section provide an exhaustive view of the comparison of 2-points and 3-points intervals.

5 2-points intervals and 3-points intervals

We present in this section all the preference structures having a 2-points and 3-points interval representation.

	$n = 2$	$n = 3$	$n = 4$	$n > 4$
Weak Order	2	3	4	n
Bilinear Order	1	3	6	$\frac{n(n-1)}{2}$
Interval Order	1	3	6	$\frac{n(n-1)}{2}$
Bitolerance Order	0	1	6	$2\binom{n}{4} + \binom{n}{3}$
Triangle Order	0	2	8	$2\binom{n}{3}$

Table 2: Number of representations for different structures

5.1 2-points intervals

The 2-points intervals are the simplest intervals that our approach makes use of since only the extreme points of intervals are considered. Let us remind that four of six different relative positions of 2-points intervals satisfy our axiomatisation. We show in this subsection that three different preference structures can be represented with the help of these four relative positions: weak orders, bilinear orders and interval orders. Table 3 recapitulates all these preference structures.

Preference Structure	$\langle P_{\leq\varphi}, I_{\leq\varphi}\rangle$ *interval representation*
Weak Orders	$Cp_{\leq(1,1)} = \{(1,1)\}$ $Cp_{\leq(2,0)} = \{(2,2)\}$
Bi-weak Orders	$Cp_{\leq(1,0)} = \{(1,1),(2,2)\}$
Interval Orders	$Cp_{\leq(0,0)} = \{(2,1)\}$

Table 3: Preference structures with 2-points interval representation

It is easy to remark that 2-points intervals propose a minimal representation for a bilinear order and an interval order since for both of them we need two different points. However each representation of weak orders makes use of only one point of 2-points intervals. The other point does not have any role for the construction of the ordered set. This is the reason for which the simplest representation of weak orders with our approach is obtained with 1-point intervals which is equivalent to the classical numerical representation of weak orders where every object is evaluated by one function. The use of n-points intervals ($n > 1$) for the representation of weak orders can be correct but will be more complicated than its classical numerical representation.

5.2 3-points intervals

In this section we present the results of the analysis of the fifteen sets of P which satisfy our axiomatisation. Our study shows that from the fifteen sets of P, seven different preference structures can be defined, some of them having more than one 3-points interval representation: weak orders and bi-weak orders have three different 3-points interval representations while three-weak orders have one, interval orders have three, split interval orders have one, triangle orders have two and intransitive orders have two. Table 4 represents the general form of these representations by the help of component sets.

Preference Structure	$\langle P_{\leq\varphi}, I_{\leq\varphi}\rangle$ *interval representation*
Weak Orders	$Cp_{\leq(3,3,0)} = \{(3,3)\}$ $Cp_{\leq(3,1,1)} = \{(2,2)\}$ $Cp_{\leq(2,2,2)} = \{(1,1)\}$
Bi-weak Orders	$Cp_{\leq(3,1,0)} = \{(2,2),(3,3)\}$ $Cp_{\leq(2,1,1)} = \{(1,1),(2,2)\}$ $Cp_{\leq(2,2,0)} = \{(1,1),(3,3)\}$
Three-Weak Orders	$Cp_{\leq(2,1,0)} = \{(1,1),(2,2),(3,3)\}$
Interval Orders	$Cp_{\leq(0,0,0)} = \{(3,1)\}$ $Cp_{\leq(3,0,0)} = \{(3,2)\}$ $Cp_{\leq(1,1,1)} = \{(2,1)\}$
Split Interval Orders	$Cp_{\leq(1,0,0)} = \{(3,2),(2,1)\}$
Triangle Orders	$Cp_{\leq(1,1,0)} = \{(2,1),(3,3)\}$ $Cp_{\leq(2,0,0)} = \{(1,1),(3,2)\}$
Intransitive Orders	$Cp_{\leq(3,2,0)} = \{(3,3),(1,2)\}$ $Cp_{\leq(2,2,1)} = \{(1,1),(2,3)\}$

Table 4: Preference structures with 3-points interval representation

Let us remark that the classical representation of the majority of these structures do not make use of intervals (intervals can be seen as vectors of some ordered points). For instance, weak orders use simple numbers while bi-weak orders (resp. three-weak orders) utilize two (resp. three), not necessarily ordered numbers. Triangle orders are represented by triangles and intransitive orders by circles. Our study shows that all these seven structures have a 3-points interval representation.

It is easy to remark that 3-points intervals propose a mininal representation for three-weak orders, split interval orders, triangle orders and intransitive orders.

6 Conclusion

In this paper, we proposed a framework for the interval comparison. This framework is general in the sense that it generates all the possible comparisons respecting certain axioms that we judge natural and not very restrictive. It allowed us to generalize the definitions of certain properties such as transitivity, Ferrers relation etc. and it provided an exhaustive view of the comparison with 2-points and 3-points intervals. It results that 2-points intervals may be used in the simplest representation of bilinear orders and interval orders while 3-points intervals provide the simplest representation for three-weak orders, split interval orders, triangle orders and intransitive orders. We showed that certain structures may have more than one representation even if these ones are minimal (like in the case of triangle orders). Our framework showed also that there exist an easiest way to represent some structures. For instance, the classical representation of triangle orders and intransitive orders make use of geometric figures (dimension two). As a result they have more complicated comparison rules (for example the comparison of circles is done by a quadratic function). By proposing a 3- points interval representation we facilitate the comparison rules and the preference representation.

Some possible interesting extensions of this paper would be the exhaustive study of 4-points interval comparison (let us remark that in this case there are 56 sets of P satisfying our axiomatisation and some of them are already known thanks to the propositions that we presented in this paper) and the integration of some coherence condition.

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