Removal Independence for β-closed Systems of Sets

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We begin with some needed terminology and notation. Let $\beta > 1$ be an integer and E a finite set with $5 \leq \beta + 3 \leq |E|$. A β-closed system of sets on E is a collection H of subsets of E (commonly called *clusters*) having the property that

$A \in H$ implies $|A| \geq \beta$;

$A, B \in H$ with $|A \cap B| \geq \beta \Longrightarrow A \cap B \in H$;

$E \in H$.

We let E_β denote the subsets of E with cardinality β. For $A \subset E$ with $|A| \geq \beta$, we define $H_A = \{A, E\}$, with $H_0 = \{E\}$. We define the removal restriction of H with respect to A by $H|_A = \{X \cap A : X \in H, \beta \leq |X \cap A| < |A|\}$. We shall be working in a collection $\mathcal{W}$ of β-closed systems of E having the property that $H \in \mathcal{W}$ implies that $H_A \in \mathcal{W}$ and $H|_A \in \mathcal{W}$ whenever $|A| \geq \beta$.

There are many examples of β-closed systems of sets on E. The non-singleton clusters of any hierarchical tree form a 2-system, as do the non-singleton clusters of any closed weak hierarchy. Somewhat trivial examples are constructed by looking at $E_\beta \cup \{E\}$ or the collection of all subsets of E having cardinality at least β. The notion of a closed weak hierarchy may be generalized to values of $\beta \neq 2$, by looking at a β-closed system H having the property that the intersection of any $(\beta + 1)$-clusters of H is also the intersection of a β-member subfamily. The closed weak hierarchies are then the special case where $\beta = 2$. Weak orders arise when $\beta = 1$. Details of these and other examples may be found

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in a paper by Crown and Janowitz (An Injective Set Representation of Closed Systems of Sets, DIMACS Series in Discrete Mathematics and Theoretical Computer Science **63**, 2003, pp. 67-79).

Let $n > 1$ be an integer, and $N = \{1, 2, \ldots, n\}$. A *consensus function* is then a mapping $C : \mathcal{W}^n \to \mathcal{W}$. A *profile* is an n-tuple $P = (H_1, H_2, \ldots, H_n)$ of members of $\mathcal{W}$. It will be convenient to introduce some special profiles. We let P_0 denote the profile having H_0 for each of its components, and for $|A| \geq \beta$ and $J \subseteq N$, let $P_{A;J}$ be defined by $P(j) = H_A$ for $j \in J$, and H_0 otherwise. We agree to call C nontrivial if there is a profile P such that $C(P) \neq H_0$. We say that it is (removal)-independent if

$$P|_A = P'|_A \Longrightarrow C(P)|_A = C(P')|_A. \tag{1}$$

Our discussion will make strong use of a commonly used set representation for a β-closed system of sets H. We define

$$\gamma_\beta(H) = \{(B, c) \colon B \in E_\beta, \text{ and for some } X \in H,\ B \subseteq X \text{ and } c \notin X\}$$

It is routine to observe that for $(B, c) \in E_\beta \times E$,

$$(B, c) \notin \gamma_\beta(H) \Longleftrightarrow B \subseteq X \text{ with } X \in H \Longrightarrow c \in X\}.$$

This representation is extremely useful for hierarchical trees and closed weak hierarchies because in these important instances, H is completely determined by $\gamma_2(H)$.

Our goal is to establish the next Theorem.

Theorem Let C be nontrivial and independent with $C(P_0) = H_0$. There then exists a nonempty subset M of N such that

$$\gamma_\beta(C(P)) = \bigcup_{i \in M} \gamma_\beta(P(i)),$$

for all profiles P.

This generalizes a Theorem of Powers (Consensus n-trees and Removal Independence, Journal of the Korean Mathematical Society, 37, No. 3, 473-490, 2000). where it was shown that for hierarchical trees, nontrivial removal independent cluster methods are strong dictators in that there is an index i having the property that $C(P) = P(i)$ for every profile P. We also obtain this result for closed weak hierarchies.

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