The Geometry Behind Paradoxes of Voting Power

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Abstract

Despite the many useful applications of power indices, the literature on power indices is raft with counterintuitive results or paradoxes, as well as real-life institutions that exhibit these behaviors. This has led to a cataloging of sorts where new and different paradoxes are calculated and then shown to exist in nature. Even though the paradoxes sound different from one another with names like the paradox of redistribution, the donor and transfer paradoxes, the paradox of quarreling members, the paradox of a new member, and the paradox of large size, they can be classified by the underlying geometric properties that induce the counterintuitive results. Perhaps surprisingly, analyzing the geometry behind the paradoxes for three voters is sufficient to understand the geometry behind the paradoxes. Voting power induces a partition on games where two games are in the same part if each player i has the same power in each game. The paradoxes are a result of three geometric ideas and how they interact with the partition: a point passing a hyperplane thereby changing parts, moving hyperplanes that change the size or number of parts in a partition, and changing the dimension of the space by adding or subtracting a voter.

Key words : Voting Power, Paradoxes, Geometry

Power indices are used to measure the a priori distribution of power among voters under a given voting rule. Many of these power indices uniquely satisfy different sets of axioms, including the most commonly used indices by Penrose (1946), Shapley and Shubik (1954) and Banzhaf (1965), as well as others. Such axiomatic approaches have been used to create new indices, as well as to champion one power index over others. Generalized power indices, such as semivalues (Carreras, Freixas, and

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Puente, 2003, Laruelle and Valenciano, 2003a, and Saari and Sieberg, 2001), measure power in broader classes of cooperative games, often following the same axiomatic development.

As a tool, power indices have been used to examine weighted voting in institutions including the International Monetary Fund (Dreyer and Schotter, 1980 and Leech, 2002c), the Electoral College (Mann and Shapley, 1964), the European Union Council of Ministers (Johnston, 1995 and Leech, 2002b), and the Israeli Knesset (Laruelle, 2001). Not only have power indices been used to analyze existing institutions, but they have been part of the debate about the design of new institutions. For example, Turnovec (1996) and Widgren (1994) use power indices to model the effects of institutional reforms on, and the introduction of new members into, the European Union. Because power indices rarely agree on the measure of power for a voter, let alone on the ranking of the power of voters (cf. Saari and Sieberg, 2001), the selection of a power index is paramount. Although a productive way to generate power indices, the axiomatic approach has not been successful in comparing how the power indices differ and when one power index is more applicable than another.

Despite the many useful applications of power indices, the literature on power indices is raft with counterintuitive results or paradoxes, as well as real-life institutions that exhibit these behaviors. This has led to a cataloging of sorts where new and different paradoxes are calculated and then shown to exist in nature. Felsenthal and Machover (1995, 1998) divide power indices according to their ability to measure 'P-power' (the power to share a purse) and 'I-power' (the power to influence) and use the paradoxes (often described as postulates, when an index is not susceptible to the paradox) as a way to compare power indices. They cast doubt on the importance of some paradoxes, offer new perspectives on other paradoxes, and generate new paradoxes. Laruelle and Valenciano (2003b) also distinguish between power indices by introducing two measures (factual success and decisiveness) that utilize the voting rule, as well as voters' behavior.

Even though the paradoxes sound different from one another with names like the paradox of redistribution (Dreyer and Schotter, 1980 and Schotter, 1981), the donor and transfer paradoxes (Felsenthal and Machover, 1998), the paradox of quarreling members (Kilgour, 1974), the paradox of a new member (Brams, 1975 and Brams and Affuso, 1976), the paradox of large size (Brams, 1975 and Shapley, 1973), the fattening paradox (Felsenthal and Machover, 1998), etc., they can be classified by the underlying geometric properties that induce the counterintuitive result.

To provide a geometric setting, the discrete space of simple weighted-voting games are viewed as points on a simplex. The voting rule partitions the simplex into different regions or parts where the power of all games in a part yield the same power index. The counterintuitive results described as paradoxes can be classified according to three geometric properties: a change in the simple weighted-voting game causes the game to switch to another part of the partition (geometrically, a point passes a hyperplane that partitions the space), the voting rule changes or restrictions are placed on what coalitions can form (geometrically, the size and/or shape of the parts of the partition change), and voters are introduced, consolidated, or deleted from the game (geometrically, the dimension changes by adding or subtracting a voter). Combining these geometric ideas in succession results in other paradoxes, e.g., fattening paradox (Felsenthal and Machover, 1998).

Perhaps surprisingly, analyzing the geometry behind the paradoxes for three voters is sufficient to understand the geometry behind the paradoxes for any number of voters. We review simple weighted-voting games, view power as a discrete map, and introduce the geometry for three voters in Section 2. In Section 3, we explain the relationship between geometry and classes of paradoxes. Because of the low dimension and the inherent symmetry, the examples often are proof that all power indices suffer from a particular paradox (e.g., the paradox of redistribution).

1 Simple Weighted-Voting Games, Power, and Geometry

Cooperative game theory models how groups or coalitions form to achieve a particular goal (e.g., passing legislation) and the value received if their objective is met. Notationally, a coalition S is a subset of a finite set of voters $N = \{1, 2, \ldots n\}$ and the utility derived by S is denoted as $v(S)$ where the real-valued function v has as its domain the power set of N and satisfies $v(\emptyset) = 0$ and super-additivity $[v(S \cup T) \geq v(S) + v(T)]$. Intuitively, these two conditions are that a coalition of no size has no value and that the sum of the whole is at least as great as the sum of its parts or that two coalitions can get at least as much done together as they could apart.

A cooperative game is simple if, for each $S \subseteq N$, either $v(S) = 0$ or $v(S) = 1$, where a coalition S is viewed either as a losing coalition, i.e., $v(S) = 0$, or a winning coalition, i.e., $v(S) = 1$. For a simple voting game, winning coalitions can pass measures and enact legislation. These games offer a minimal number of restrictions of what subsets can be winning coalitions. Let the collection of all winning subsets of a finite set N of voters be denoted by $\mathcal{W}$ where

1. $N \in \mathcal{W}$
2. $\emptyset \notin \mathcal{W}$
3. Monotonicity: If $X \in \mathcal{W}$ and $X \subseteq Y \subseteq N$, then $Y \in \mathcal{W}$.

A priori power is determined by the structure of the institution and which subsets of voters can coalesce to form winning or losing coalitions. This is markedly different than looking at the voting behavior for a particular issue.

Although power can be defined for any simple voting game where the outcome only depends on which subsets of N are winning coalitions, we will assume that each voters' vote has a weight associated to it. Many of the paradoxes relate changes in weights to the corresponding change in power. A simple weighted-voting game is a set of n voters, where voter i's vote carries the weight w_i, and a quota, a value that if the sum of the voters' weights in a coalition is greater than or equal to the quota, q, then the coalition is a winning coalition. Denote a simple weighted-voting game by $[q; w_1, w_2, \ldots, w_n]$. Hence,

$$v(S) = \begin{cases} 0 & \text{if} \quad \sum_{i \in S} w_i < q, \\ 1 & \text{if} \quad \sum_{i \in S} w_i \geq q. \end{cases} \tag{1}$$

The w_i's are restricted usually to be nonnegative integers and the sum of the weights $w = w_1 + w_2 + \cdots + w_n$ is fixed, e.g., representing a fixed number of shares of stock or a fixed number of seats in a senate. The weight of a voter is a crude form of measuring how important, or how much power, an individual brings to a coalition, whereas power indices calculate a voter's contribution to a political process.

For simple weighted-voting games to be well-defined, q must satisfy $\frac{w}{2} < q \leq w$. In words, for a coalition to pass a measure, the weights of the voters in the coalition must be more than a majority of the total weight of all voters. Otherwise, two coalitions with less than a majority of the total weight of voters could pass conflicting legislation. The weights in the simple weighted-voting game $[q; w_1, w_2, \ldots, w_n]$ can be normalized and viewed as a point on the $(n-1)$-dimensional simplex

$$S_{n-1} = \left\{ (x_1, x_2, \ldots, x_n) \mid w_1 + w_2 + \cdots w_n = w \text{ and } x_i = \frac{w_i}{w} \geq 0 \text{ for all } i \right\}.$$

The quota induces the hyperplane $\sum_{i \in S} x_i = \dfrac{q}{w}$ to divide the simplex of all simple weighted-voting games into those that have S as a winning and losing coalition (Eq. 1). The collection of all hyperplanes forms a partition of the simplex, where the number and size of the parts of the partition depend on the quota.

Because most of the paradoxes can be understood by an analysis of simple weighted-voting games with only three voters, we consider these games in detail. For a game with 3 voters, the normalized weights of the three voters can be viewed as a point on the 2-simplex. The 2-simplex is the intersection of the plane $x_1 + x_2 + x_3 = 1$ and the positive octant where $x_i \geq 0$ for all i; this can be viewed as an equilateral triangle in the plane as shown in Figure 1.

For 3-voter simple weighted-voting games, the hyperplanes associated with a fixed normalized quota q where $\frac{1}{2} < q < 1$ partition the simplex $S_2 = \{(x_1, x_2, x_3) :$

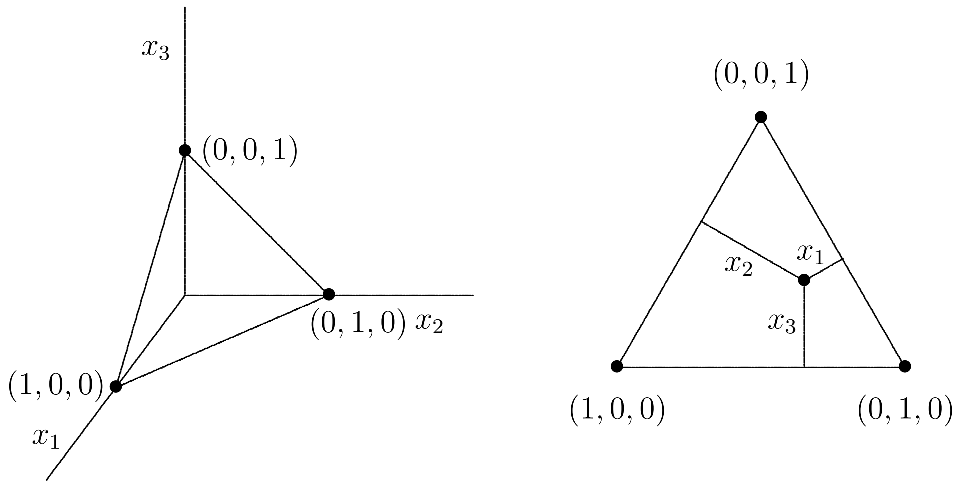


Figure 1: The plane $x_1 + x_2 + x_3 = 1$ and the 2-simplex $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0$, and $x_3 \geq 0.\}$

Region	R_i; $i = 1 - 3$	R_{i+3}; $i = 1 - 3$	R_{i+6}; $i = 1 - 3$	R_{10}
MWCs	$\{i\}$	$N/\{i\}$	$\{i, j\}, \{i, k\}$ where $i \neq j \neq k$	$\{1, 2\}, \{1, 3\}, \{2, 3\}^*$ or $\{1, 2, 3\}^{**}$

Table 1: Regions and their corresponding minimal winning coalitions (MWCs). The coalition structure for R_{10} depends on whether $q \leq \frac{2}{3}^*$ or $q > \frac{2}{3}^{**}$.

$x_1 + x_2 + x_3 = 1$ and $x_i \geq 0\}$ into ten regions $R_1 - R_{10}$ (Table 1 and Figure 2). The games in each region form an equivalence class because each game has the same sets of winning and losing coalitions. For example, in Figure 2, the only winning coalitions in games in region R_7 are $\{1, 2, 3\}$, $\{1, 2\}$, and $\{1, 3\}$; this follows because

$$\begin{array}{lll} x_1 + x_2 + x_3 \geq q & x_1 + x_2 \geq q & x_1 + x_3 \geq q \\ & x_2 + x_3 < q & x_i < q \text{ for all } i. \end{array}$$

Notice that the inequalities $x_1 + x_2 \geq q$, $x_2 + x_3 < q$ and $x_1 + x_3 \geq q$ can be rewritten as $x_3 \leq 1 - q$, $x_1 > 1 - q$, and $x_2 \leq 1 - q$. Hence, the lines $x_1 = 1 - q$, $x_2 = 1 - q$, and $x_3 = 1 - q$ are parallel to the sides of the equilateral triangle (where $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$). This holds in general: for 3-voter simple weighted-voting games with $q < 1$, the hyperplanes that partition the simplex are lines parallel to the sides of the equilateral triangle. When $q = 1$, there are four regions: R_i where player i is the dictator (for $i = 1$ to 3) and R_{10} where all voters must be part of a coalition for it to be winning.

A power index is a discrete map from the space of normalized n-voter, simple weighted-voting games to vectors in $\mathbb{R}^n$ where the i^{th} entry of the vector represents

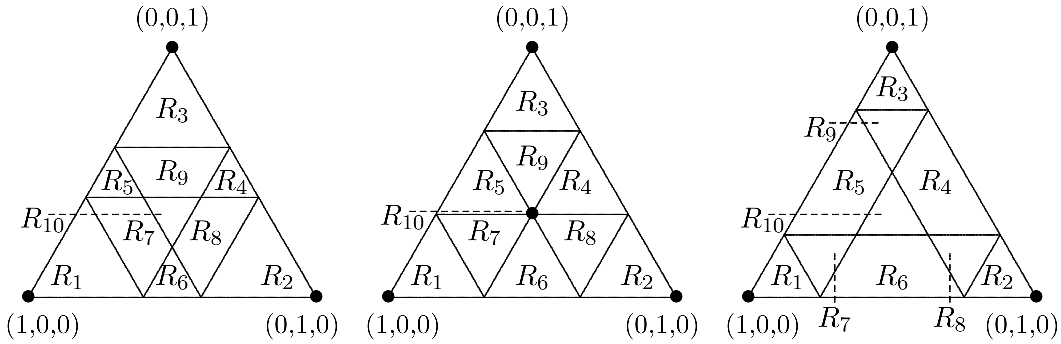


Figure 2: Shape of regions for $\frac{1}{2} < \frac{q}{w} < \frac{2}{3}$ (left), $\frac{q}{w} = \frac{2}{3}$ (middle), and $\frac{2}{3} < \frac{q}{w} < 1$ (right).

the power of the i^{th} voter. For a fixed quota, let

$$P_q : S_{n-1} \to \mathbb{R}^n \tag{2}$$

represent a power index. Because there are many specialized power indices (e.g., Banzhaf, 1965, Coleman, 1971, Deegan and Packel, 1982, Penrose, 1946, and Shapley and Shubik, 1954) that measure different aspects of power, I refrain from giving too many details for specific power indices or measures. Regardless of the method of measuring power, the geometry of the domain and the partition that slices the simplex into parts (that indicate the winning and losing coalitions) are the same. Also, note that we are measuring a priori power that is independent of the position of the voters on a particular issue. It considers all possible coalitions that can form and may weigh the outcome according to size (as semi-values do) or other characteristics. The partitioned regions of the simplex are equivalence classes where the voters' powers are preserved for games in the region.

To be well defined, power indices must also satisfy certain regularity conditions. For example, a power index should not be biased toward a voter: a permutation of the weights of the voters should result in the same permutation of the resulting powers. P_q must satisfy the following conditions:

1. (Invariance) If σ is a permutation of the set of voters N, then voter i's power in $[q; x_1, \ldots, x_n]$ should be the same as voter $\sigma(i) = j$ in the permuted game $[q; y_1, \ldots, y_n]$ where $y_j = x_i$. Equivalently,

   $P_q(x_1, x_2, \ldots, x_n)_i = P_q(y_1, y_2, \ldots, y_n)_j$ where $\sigma(i) = j$ and $y_j = x_{\sigma(i)}$ for all i.

2. (Symmetry) If two voters are members of the identical winning coalitions, then they have the same power.

3. (Dummy voter) If a voter a is never part of a minimal winning coalition, then voter a's power is 0.

Felsenthal and Machover (1998) distinguish between measures that satisfy the above conditions and power indices that are normalized so that the elements of the resulting power vector sum to 1.

To demonstrate possible paradoxes, it is helpful to have specific examples. I review two of the most commonly used power indices: the Banzhaf and Shapley-Shubik power indices. In 1965, Banzhaf introduced his power index in a lawsuit while examining the fairness of voting involving the Nassau County (NY) Board of Supervisors (Banzhaf, 1965). The Banzhaf index counts the number of times that a voter is necessary to be part of a coalition for a measure to pass. This is referred to as a critical voter. The i^{th} component of the Banzhaf power index is given by

$$B_q(\mathbf{x})_i = \sum_{S \subseteq N} \left[v(S) - v(S/\{i\}) \right]. \tag{3}$$

The Shapley-Shubik power index (Shapley and Shubik, 1954) extends the Shapley value (Shapley, 1953) to simple weighted-voting games. The i^{th} component of the Shapley-Shubik power index is given by

$$S_q(\mathbf{x})_i = \sum_{S \subseteq N} \left[v(S) - v(S/\{i\}) \right] (|S| - 1)! \tag{4}$$

Intuitively, Shapley-Shubik power index measures the power of a voter given every sequence of 'yes' votes. The notion is that the voters could join the coalition in any order and in $(|S| - 1)!$ of these orders, voter i joined last and made the coalition a winning coalition. Voter i is often referred to as the pivotal voter. Under the Shapley-Shubik power index, a voter i has to be critical in the Banzhaf sense above, but the value of being critical depends on the number of elements in S. For both indices, the power of voter i depends on the number and/or size of the winning coalitions for which i is critical or pivotal (when $v(S) - v(S/\{i\}) = 1$). For three voters with fixed weights, the Shapley-Shubik and Banzhaf power indices agree on the relative ranking of the voters' power (Saari and Sieberg, 2001). To get a sense of the calculation for a 3-voter game, consider the following example.

Example 1. The simple weighted-voting game $[3; 2, 1, 1]$ normalizes to $[\frac{3}{4}; \frac{1}{2}, \frac{1}{4}, \frac{1}{4}]$ and has winning coalitions $\{1, 2, 3\}$, $\{1, 2\}$, and $\{1, 3\}$. Hence, voter 1's power is determined by potentially nonzero terms $v(S) - v(S/\{i\})$ in Eq. 3 (so, $v(S)$ must be 1 for the difference to be nonzero) and

$$\begin{aligned} B_{\frac{3}{4}}\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)_1 &= [v(\{1,2,3\}) - v(\{2,3\})] + [v(\{1,2\}) - v(\{2\})] \\ &\quad + [v(\{1,3\}) - v(\{3\})] = 1 + 1 + 1 = 3. \end{aligned}$$

Similarly, voter 2 has Banzhaf power

$$B_{\frac{3}{4}}\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)_2 = [v(\{1,2,3\}) - v(\{1,3\})] + [v(\{1,2\}) - v(\{1\})] = 0 + 1 = 1.$$

And, by symmetry, voter 3's power is also 1. And, $B_{\frac{3}{4}}(\frac{1}{2},\frac{1}{4},\frac{1}{4}) = (3,1,1)$. The normalized Banzhaf power index is $\frac{3}{5}$:$\frac{1}{5}$:$\frac{1}{5}$. It follows from Eq. 4 that $S_{\frac{3}{4}}(\frac{1}{2},\frac{1}{4},\frac{1}{4}) = (4,1,1)$. The normalized Shapley-Shubik power index is then $\frac{4}{6}$:$\frac{1}{6}$:$\frac{1}{6}$.

Due to the superadditivity assumption, only the minimal winning coalitions are necessary to generate all winning coalitions. From Example 1, the coalition of voters 1 and 2 is a minimal winning coalition because both are necessary (to vote "yes") to pass a measure. However, the grand coalition of all voters is not minimal because either voter 2 or voter 3 could exit the coalition (thereby voting "no") and the remaining voters could still pass the measure.

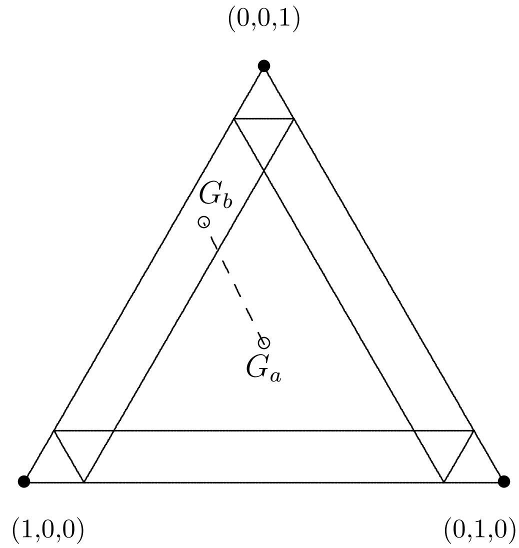
2 Geometry of Paradoxes of Voting Power

For games with more voters, higher dimensional simplices represent the domain of power indices. Similarly, the quota partitions the simplex into regions where the voters' powers are constant. When a simple weighted-voting game is in the interior of the partition, then a small perturbation may not cause the power to change. Only by changing parts in the partitions can the power change. Changing parts requires passing a hyperplane, the boundary of the part.

2.1 Domain Effects

The paradox of redistribution compares the change in a voter's weight to the corresponding change in the voter's power. The counterintuitive outcome is that a voter's weight may increase, yet its power decreases, or a voter's weight may decrease, yet its power increases, or both of these situations may occur. For three voters, only the one-sided paradox can occur, not both. The geometry of the simplex readily explains why even the more general paradox is true. Because simple weighted-voting games are domain points on the simplex, a change in one voter's weight (or coordinate) must be met with changes in at least another voter's weight, too. As described, there is a lot of freedom in how the other voters' weights can be adjusted. So, the paradox may not seem too remarkable.

The paradox of redistribution was first noted by Fischer and Schotter (1978). Schotter (1981) uses simplices to determine the likelihood of the paradox for the



$G_a = [\frac{7}{8}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ with power index $\frac{1}{3}$:$\frac{1}{3}$:$\frac{1}{3}$
$G_b = [\frac{14}{16}; \frac{5}{16}, \frac{1}{16}, \frac{10}{16}]$ with power index $\frac{1}{2}$:0:$\frac{1}{2}$

Figure 3: The paradox of redistribution as an effect of passing a hyperplane. When $G_a \to G_b$, voter 1's weight decreases but its power increases. When $G_b \to G_a$, voter 1's weight increases but its power decreases.

Banzhaf and Shapley-Shubik power indices. However, the paradox is not an artifact of the particular power index used, as described in the following example.

Example 2. (The ubiquity of the paradox of redistribution) Consider the effect of voter 1's weight increasing from $\frac{1}{3}$ to $\frac{5}{16}$ in the games $G_a = [\frac{7}{8}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ and $G_b = [\frac{14}{16}; \frac{5}{16}, \frac{1}{16}, \frac{10}{16}]$. Figure 3 shows how with the quota fixed at $\frac{7}{8}$, changing the weights of the voters results in passing a hyperplane into another part of the partition. Specifically, the game moves from region R_{10} (using the notation from Figure 2) into R_5. Notice that because all three voters are necessary to form a winning coalition in G_a, the normalized power index for G_a is $\frac{1}{3}$:$\frac{1}{3}$:$\frac{1}{3}$, regardless of the specific power index. This is due to the invariance under of permutations of the voters. Similarly, because voters 1 and 3 are part of all the same winning coalitions in G_b and voter 2 is a dummy voter (her vote never changes a losing coalition to a winning coalition), the power index is $\frac{1}{2}$:0:$\frac{1}{2}$. Due to symmetry, every power index will exhibit this paradox under these changes (cf. Felsenthal and Machover, 1998).

Felsenthal and Machover (1998) consider a more surprising version of the paradox of redistribution called the donation paradox. They show that if the power index doesn't satisfy a monotonicity condition, then it is possible for a voter to donate some of its weight to another voter and the donor's power increases while the recipient's

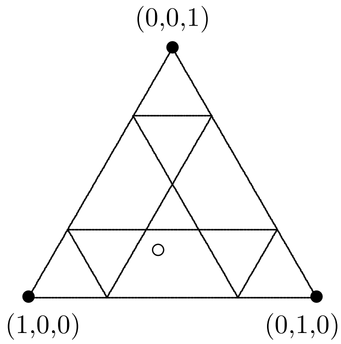


Figure 4: As the quota decreases from $q_1 = \frac{8}{11}$ (left), $q_2 = \frac{7}{11}$ (middle), and $q_3 = \frac{6}{11}$ (right) point on the simplex representing the weights $\left(\frac{5}{11}, \frac{4}{11}, \frac{2}{11}\right)$ is in region R_6, R_7, and R_{10}, respectively.

power decreases. Although this requires a nonmonotone power index, the geometry behind the paradox remains the same: a perturbation in the weights of the voters causes the game to pass a hyperplane.

2.2 Partition Effects

So far we have considered the effect of changing the voters' weights in the simple weighted-voting game. However, it is possible to achieve paradoxical outcomes by leaving the weights fixed and changing the shape and number of partitions. Figure 2 and Table 1 indicate a geometric consequence to changing the value of the quota: the size, shape, and characteristics of partitions of the simplex may change. Institutions that have changed or considered changing their requirements for a measure to pass (by changing q) have been analyzed. For example, Dreyer and Schotter (1980) consider quota effects on the distribution of power in the International Monetary Fund.

In general, the quota affects the size and number of parts in the partition of the simplex. It seems as if lowering the quota benefits the voter whose vote has the largest weight. Winning coalitions from before the changed quota will be retained. However, the critical voters may change. And, new winning coalitions may form. We see from Figure 2 that a point may fall into different regions as the quota changes. The voter with the largest weight may benefit from such a change or not. The following example demonstrates two scenarios where the same weights are used to show how the quota affects the voter with the largest weight.

Example 3. (The Shapley-Shubik power index and the quota paradox) Consider the effect of the quota decreasing from $\frac{8}{11}$ to $\frac{7}{11}$ to $\frac{6}{11}$ for the game with voters 1-3 with weights $\frac{5}{11}, \frac{4}{11}, \frac{2}{11}$, respectively. Under the Shapley-Shubik power index, these three games (with quota decreasing) have resulting power indices of $\frac{1}{2}$:$\frac{1}{2}$:0, $\frac{4}{6}$:$\frac{1}{6}$:$\frac{1}{6}$ and $\frac{1}{3}$:$\frac{1}{3}$:$\frac{1}{3}$. The voter with the largest weight initially benefits from a decrease in the quota, but a further decrease in the quota lowers voter 1's power. The quota effect appears in Figure 4. The fixed game is in regions R_6, R_7, and R_{10} as the quota decreases.

Realize that the Shapley-Shubik power index is not the only paradox susceptible to this quota effect. Due to symmetry, the normalized power index for the games $\left[\frac{8}{11}; \frac{5}{11}, \frac{4}{11}, \frac{2}{11}\right]$ and $\left[\frac{6}{11}; \frac{5}{11}, \frac{4}{11}, \frac{2}{11}\right]$ always is $\frac{1}{2}$:$\frac{1}{2}$:0 and $\frac{1}{3}$:$\frac{1}{3}$:$\frac{1}{3}$. A decrease in the quota has adversely affected the voter with the largest weight. However, a decrease in the quota can also have a positive effect on the voter with the largest weight. For a general power index, this occurs when the quota decreases from $\frac{8}{11}$ to $\frac{7}{11}$ resulting in the game $\left[\frac{7}{11}; \frac{5}{11}, \frac{4}{11}, \frac{2}{11}\right]$, as long as the index gives more power to voter 1 which is reasonable as voter 1 is in two minimal winning coalitions while voters 2 and 3 are each in one minimal winning coalition.

There are other ways to adjust the size and number of parts of the partition of the simplex. Kilgour introduced the paradox of quarreling members where restricting which coalitions can form increases the power of the quarreling members. Specifically, if two voters quarrel, they will never both vote "yes" on a measure. Even though they cannot be part of the same winning coalition, it is possible that one of these voters power increases. Quarreling restricts the freedom of the quarreling members, thereby decreasing their options. It seems paradoxical that the additional restriction can help the quarreling members. But, quarreling also restricts the non-quarreling voters' options, too.

Measuring the power of the voters when certain coalitions cannot form requires modifications of the power indices. Modifying power indices can be viewed as restricting the power index to the coalitions that can form. For quarreling members, realize that we do not assume that the two quarreling members are always on opposite sides of a vote, but that they would not be both part of a winning coalition. The following example demonstrates how quarreling members reduce the number of regions in the partition.

Example 4. (The paradox of quarreling members) The simple weighted-voting game $[\frac{3}{4}; \frac{2}{3}, \frac{1}{6}, \frac{1}{6}]$ is in region R_7 in Figure 2 and has minimal winning coalitions are $\{1, 2, 3\}$, $\{1, 2\}$, and $\{1, 3\}$. Consequently, as in Example 1, the normalized Banzhaf power index for this game is $\frac{3}{5}$:$\frac{1}{5}$:$\frac{1}{5}$. If voters 2 and 3 quarrel, then the winning coalition $\{1, 2, 3\}$ is restricted from forming. Figure 5 indicates the coarser partition that results from quarreling. Voter 1 is critical twice while voters 2 and 3 are each critical once. Modifying the normalized Banzhaf power index for the restricted set of coalitions, the game with quarreling has a power index of $\frac{1}{2}$:$\frac{1}{4}$:$\frac{1}{4}$. In this case,

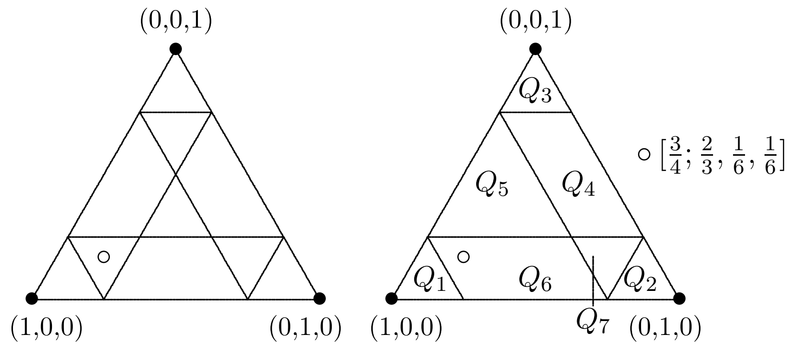


Figure 5: The paradox of quarreling members: Voters 2 and 3 quarrel resulting in the removal of a hyperplane that decreases the number of parts in the partition.

the counterintuitive result is that voter 3's power increased because of its quarreling with voter 2.

2.3 Dimensional Effects

So far, we have only considered the geometry for simple weighted-voting games with 3 voters. To consider the effect of introducing or removing a voter from a game, we also consider the simplex generated by 2 voters. When normalized, these games are on the unit interval $[0, 1]$ where x_1 is represented by the distance from 0 and x_2 is represented by the distance from 1. Naturally, $x_1 + x_2 = 1$ as required. Adding a voter to a simple weighted-voting game increases the dimension of the space. Brams (1975) and Brams and Affuso (1976, 1985a, 1985b) consider the paradox of a new member where a new voter is introduced into the game while the relative weights of the other voters is constant (that is, the weights of the "old" voters are proportional), yet an old voter's weight increases. Felsenthal and Machover (1998) mathematically represent this paradox in the following way: when the game $[q; u_1, u_2, \ldots, u_n]$ changes to $[q; v_1, v_2, \ldots, v_n, v_{n+1}]$ where $v_{n+1} \in [0, 1]$ and $v_i = (1 - v_{n+1})u_i$ for $i = 1$ to n and one of voters 1 through n has its power increase. This seems paradoxical because the introduction of the new voter would seem to take power away from the other voters. Introducing a new member to an organization can have unanticipated consequences. Researchers have applied power indices to see the effect of proposed expansion of the European Union (e.g., Turnover, 1996 and Widgrén, 1994). The following example demonstrates the paradox.

Example 5. (Paradox of a new member) Consider the 2-voter game $[0.75; 0.7, 0.3]$. Clearly, the power under any index is $\frac{1}{2}$:$\frac{1}{2}$ as both voters are necessary for a coalition to be winning. The line in Figure 6 shows the possible games for which a third voter can be added and the ratio of voter 1's and voter 2's weights held constant. Notice

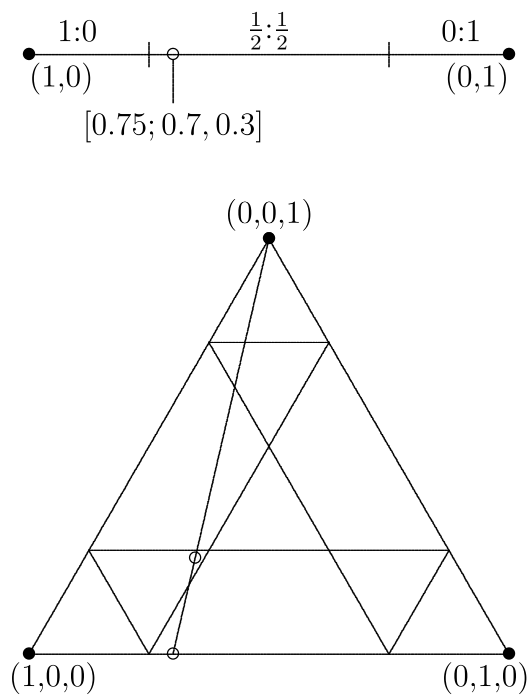


Figure 6: The paradox of a new member: Voter 1's power increases despite the introduction of a new voter.

that this line intersects region R_7, in which case voter 1's power increases. As a representative game in this region, the game $[0.75; \frac{7}{13}, \frac{3}{13}, \frac{3}{13}]$ satisfies the conditions. Under the Shapley-Shubik power index, voter 1's power is $\frac{4}{6}$ in the new 3-voter game while under the Banzhaf power index voter 1's power is $\frac{3}{5}$; both are greater than $\frac{1}{2}$.

Brams (1975) coined the term paradox of large size: if voters decide to form a bloc, then the power of the bloc cannot be smaller than the sum of the power of its members. We consider this paradox when one voter annexes another voter (absorbsing its weight). Yet, by increasing its weight, the voter's power decreases. Again, 3 voters is sufficient to demonstrate that this paradox is independent of the measurement of power. The following example considers this paradox.

Example 6. (Paradox of large size) Consider the simple weighted-voting game $G_a = [\frac{3}{4}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$. By symmetry, each voter's power is $\frac{1}{3}$. If voter 1 receives the entirety of voter 3's weight, the resulting game is $G_b = [\frac{3}{4}; \frac{2}{3}, \frac{1}{3}]$; as both voters are necessary to form a winning coalition, the resulting power is $\frac{1}{2}$ for each voter. The paradox is that the aggregate power of voter 1 and voter 2 before becoming a single player was $\frac{2}{3}$ while the power of the combined voter decreases to $\frac{1}{2}$. Figure 7 shows that combining voters 1 and 3 can be viewed as projecting the game in the interior to one on the boundary (representing games where voter 2 has weight 0). Notice that

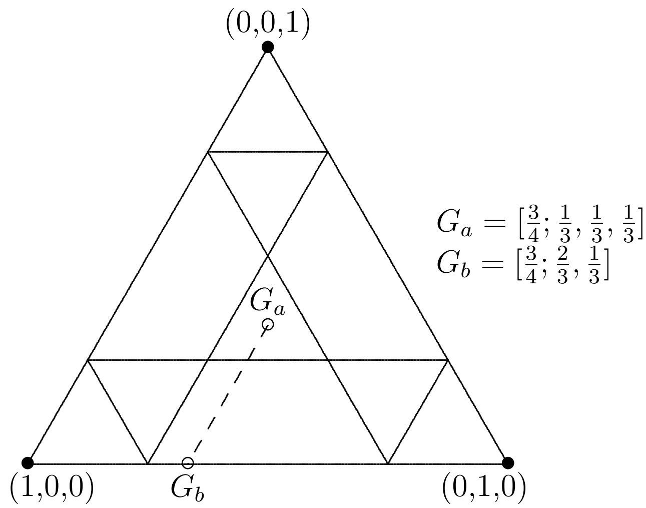


Figure 7: The Paradox of large size: Voter 3 coalesces with voter 1 and their cumulative power decreases.

every projection that re-allocates voter 3's weight to voters 1 and 2 results in the same power index $\frac{1}{2}$:$\frac{1}{2}$, as both voters are necessary in the only winning coalition.

Saari and Sieberg (2001) show that complete reversals of rankings of voters under power indices can occur when adding or subtracting a voter. When viewed as a projection, there are many seemingly natural ways to project from the $(n+1)$-voter simplex to the n-voter simplex. These different methods result in different powers in the projected games.

2.4 Combining Geometric Elements

Felsenthal and Machover (1998) introduce the fattening paradox where increasing a voter's weight while keeping the other voters' weights fixed results in a decrease in power for the (un)lucky recipient of the extra weight. This can be viewed as changing the position in the simplex of the weights. Consider the example where voter 1's weight increases from 4 in the game $G_a = [8; 4, 4, 1, 1, 1]$ to 5, resulting in $[8; 5, 4, 1, 1, 1]$ (Felsenthal and Machover, 1998). Under the normalized Banzhaf power index, voter 1's power is $\frac{1}{2}$ in G_a (due to symmetry, as voters 1 and 2 are the only two critical voters). In G_b, voter 1's power decreases to approximately 0.474 under the normalized Banzhaf power index. (Leech (2002a) provides algorithms for computing various power indices.)

This paradox combines two geometric properties. Not only have the weights been changed, but the normalized quota has changed too, from $\frac{8}{11}$ to $\frac{8}{12}$. Decreasing the normalized quota results in a change in the number and/or size of parts in the partition. This can be viewed as moving the hyperplane at the same time as the increase in one voter's weight redistributes the normalized weights. These two actions cause the game to pass a hyperplane. For the above example,

$$\begin{array}{ccc} [8;4,4,1,1,1] & \rightarrow & [8;5,4,1,1,1] \\ \downarrow & & \downarrow \\ \left[\frac{8}{11};\frac{4}{11},\frac{4}{11},\frac{1}{11},\frac{1}{11},\frac{1}{11}\right] & \rightarrow & \left[\frac{8}{12};\frac{5}{12},\frac{4}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12}\right] \end{array}$$

In general, if voter 1's weight increases by k from x_1 to x_1+k, the above diagram becomes

$$\begin{array}{ccc} [q;x_1,x_2,\ldots,x_n] & \rightarrow & [q;x_1+k,x_2,\ldots,x_n] \\ \downarrow & & \downarrow \\ \left[\frac{q}{X};\frac{x_1}{X},\frac{x_2}{X},\ldots,\frac{x_n}{X}\right] & \rightarrow & \left[\frac{q}{X+k};\frac{x_1+k}{X+k},\frac{x_2}{X+k},\ldots,\frac{x_n}{X+k}\right] \end{array}$$

where $X = \sum_{i=1}^{n} x_i$. The normalized quota has decreased from $\frac{q}{X}$ to $\frac{q}{X+k}$ at the same time as the game moves proportionally in the direction of the $(1,0,\ldots,0)$-vertex of the simplex. This is comparable to adding or subtracting a player. Hence, the fattening paradox has elements of each of the geometric properties.

3 Conclusion

The geometry that arises from the partition on the simplex of simple weighted-voting games is a natural way to classify paradoxical outcomes in voting power. Although the paradoxes' names do not indicate the geometry behind the paradox, three geometric properties: changing regions in a partition by passing a hyperplane, altering the number and/or size of the parts of a partition, and adding or subtracting a voter leads to the paradoxes are the basis for the paradoxes. Not only does the geometry provide a tool to analyze paradoxes, but also to construct new ones.

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