On Influence and Power Indices*

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Abstract. In the paper we investigate the Hoede-Bakker index - the notion which computes the overall decisional 'power' of a player in a social network. It is assumed that each player has an inclination (original decision) to say 'yes' or 'no' which, due to influence of other players, may be different from the final decision of the player. The main drawback of the Hoede-Bakker index is that it hides the actual role of the influence function, analyzing only the final decision in terms of success and failure. In this paper, we further investigate the Hoede-Bakker index, proposing an improvement which fully takes into account the mutual influence among players. A global index which distinguishes an influence degree from a power index is analyzed. We define weighted influence indices, in particular, a possibility influence index which takes into account any possibility of influence, and a certainty influence index which expresses certainty of influence. We consider different influence functions and study their properties.

Keywords: Hoede-Bakker index, weighted influence index, possibility influence index, certainty influence index, equidistributed influence index, power index, influence function

1 Introduction

In cooperative game theory, a *decisional power* has been proposed by Hoede and Bakker [4], and later generalized and modified by Rusinowska and de Swart [8]. The Hoede-Bakker index computes the overall decisional 'power' of a player in a social network. It is assumed that a decision of a player may be influenced by decisions of other players. Specifically, it is considered that each player has an *inclination* (original decision) to say 'yes' (coded by $+1$) or 'no' (coded by -1). For each possible configuration i of individual inclinations, it is supposed that after mutual influence the actual decision Bi of all players is made. Then, a group decision $gd(Bi)$ is given.

The main drawback of the Hoede-Bakker index is that it hides the actual role of the influence function B, analyzing only the final decision in terms of successes and failures. The aim is then to provide alternative ways putting into lights the role of the influence function B.

In the paper, we propose a general form of the index which enables the analysis of influence among players. This index fills a gap between power indices which are classical in voting games (e.g. the Banzhaf index), and the Hoede-Bakker index. The general idea

* This research has been initiated during a Short Term Scientific Mission of Michel Grabisch to Tilburg University. The authors gratefully acknowledge support of this mission by COST Action 274, TARSKI.

is to compute a weighted number of times an individual j makes another individual k change his decision, and more generally, the number od times a group S makes an individual $k \notin S$ change his decision. Two particular ways of of weighting lead to a possibility influence index which takes into account any possibility of influence, and to a certainty influence index which expresses certainty of influence.

Concerning conventions of notations, cardinality of sets $S, T, \ldots$ will be denoted by the corresponding lower case $s, t, \ldots$. We omit braces for sets, e.g., $\{k, m\}$, $N \setminus \{j\}$, will be written km, $N \setminus j$, etc.

2 The Hoede-Bakker index

Let us first recapitulate the original Hoede-Bakker index as introduced in [4], and its generalization given in [8]. The general framework is the following. We consider a social network with the set of all players (agents, actors, voters) denoted by $N = \{1, ..., n\}$. The players have to make a certain acceptance-rejection decision. Each player has an inclination either to say 'yes' (denoted by $+1$) or 'no' (denoted by -1). An *inclination vector*, denoted by i, is an n-vector consisting of ones and minus ones. Let I be the set of all n-vectors. It is assumed that players may influence each others, and due to the influences in the network, the final decision of a player may be different from his original inclination. In other words, each inclination vector $i \in I$ is transformed into a *decision vector* Bi, where $B : I \to I$ is the influence function. The set of all influence functions will be denoted by $\mathcal{B}$. The decision vector Bi is an n-vector consisting of ones and minus ones and indicating the decisions made by all players. Let $B(I)$ denote the set of all decision vectors. Furthermore, the *group decision function* $gd : B(I) \to \{+1, -1\}$ is introduced, having the value $+1$ if the group decision is 'yes', and the value -1 if the group decision is 'no'. The set of all group decision functions will be denoted by $\mathcal{G}$.

The following definition has been introduced ([4]):

Definition 1 *Given $B \in \mathcal{B}$ and $gd \in \mathcal{G}$, the decisional power index (the Hoede-Bakker index) $HB : \mathcal{B} \times \mathcal{G} \to [0, 1]^n$ is given by*

$$HB_k(B, gd) := \frac{1}{2^{n-1}} \sum_{\{i|i_k=+1\}} gd(Bi) \quad for\ k \in N. \tag{1}$$

In [8], a certain generalization of the Hoede-Bakker index has been proposed:

Definition 2 *Given $B \in \mathcal{B}$ and $gd \in \mathcal{G}$, the generalized Hoede-Bakker index $GHB : \mathcal{B} \times \mathcal{G} \to [0, 1]^n$ is given by*

$$GHB_k(B, gd) := \frac{1}{2^n} \left(\sum_{\{i|i_k=+1\}} gd(Bi) - \sum_{\{i|i_k=-1\}} gd(Bi) \right) \quad for\ k \in N. \tag{2}$$

First of all, we notice that neither in the original definition of the Hoede-Bakker index nor in its generalization mentioned above, the functions B and gd are considered separately. When calculating the (original or generalized) Hoede-Bakker index, only the relation between an inclination vector i and the group decision $gd(Bi)$ is taken into account. If we do not separate the two functions B and gd, we may define *Success*, *Failure* and *Decisiveness* of a player starting not from the final decision of the player in question, but from his inclination (for an analysis of success and decisiveness of a player in voting situations, see [6]). Consequently, we may say that a player is *successful* if his inclination coincides with the group decision. Adopting such a definition of being successful, if all inclination vectors are equally probable, then the generalized Hoede-Bakker index is a kind of a 'net' Success (see [7]), i.e., it is equal to 'Success − Failure = Decisiveness', where Success, Decisiveness, and Failure of a player are defined as a probability that the player is successful, is decisive, and fails, respectively. Consequently, if a successful player is defined as a player having the inclination equal to the group decision, and if all inclination vectors are equally probable, then the generalized Hoede-Bakker index coincides with the absolute Banzhaf index; see [8].

3 The influence indices

In order to take fully into account the mutual influence among players and to separate the functions B and gd in the Hoede-Bakker setting, we introduce

$$\textit{global index} = (d, \phi) \tag{3}$$

where d determines the influence degree and ϕ is the power index. We will investigate these two components separately.

When analyzing the 'influence part', the first question may appear how to measure a degree of influence of a player (or a coalition) on the other voters. The answer is not necessarily that straightforward if we can just observe the inclinations and the final decisions of the players in a multi-player social network. Suppose the final decision of player A is different from his inclination, but this decision coincides with the inclinations of two other players in the network, say, agents B and C. Was voter A's decision different from his inclination because of the unique influence of player B, or the unique influence of player C, or maybe A voted like this only because he faced an influence of the strong two-party coalition? These are the questions that not always can be answered univocally if apart from knowing the function B, we are not able to observe a 'real act of influencing among players'. Consequently, we introduce a family of *influence indices*.

Before doing this, we introduce several notations for convenience. Let for each $S \subseteq N$ and $j \in N$

$$I_{S\to j} := \{i \in I \mid \forall k \in S\ [i_k = -i_j]\} \tag{4}$$

$$I^*_{S\to j}(B) := \{i \in I_{S\to j} \mid i_j = -(Bi)_j\} = \{i \in I \mid \forall k \in S\ [i_k = -i_j = (Bi)_j]\}. \tag{5}$$

$I_{S\to j}$ and $I^*_{S\to j}(B)$ denote the set of all inclination vectors of *potential influence* of coalition S on player j, and the set of all inclination vectors of *observed influence* of S on j under given $B \in \mathcal{B}$, respectively. Of course,

$$|I_{S\to j}| = 2^{n-s}.$$

For every $i \in I_{S\to j}$, we put $i_S := i_k$, for some $k \in S$, and we also introduce

$$n^*(i, S) := |\{m \in N \mid \forall k \in S\ [i_m = i_k]\}| \geq s. \tag{6}$$

It is the number of players with the same inclination as players of S under $i \in I_{S\to j}$ (including the players from S).

3.1 The possibility influence index

Definition 3 *Given $B \in \mathcal{B}$, for each $S \subseteq N$ and $j \in N$, the possibility influence index of coalition S on player j is defined as*

$$\overline{d}(B, S \to j) := \frac{|I^*_{S\to j}|}{|I_{S\to j}|}. \tag{7}$$

For each $S \subseteq N$, $j \in N$, $B \in \mathcal{B}$

$$\overline{d}(B, S \to j) \in [0,1], \quad and \quad \overline{d}(B, S \to j) = 0 \ for \ j \in S.$$

$\overline{d}(B, S \to j)$ measures a degree of influence player k has on player j, taking into account any possibility of influence.

3.2 The certainty influence index

Switching to another extreme way of calculating influence degree gives us the definitions of the certainty influence index.

Definition 4 *Given $B \in \mathcal{B}$, for each $S \subseteq N$ and $j \in N$, the certainty influence index of coalition S on player j is given by*

$$\underline{d}(B, S \to j) := \frac{|\{i \in I^*_{S\to j} \mid \forall p \notin S \ [i_p = i_j]\}|}{2}. \tag{8}$$

$\underline{d}(B, S \to j) \in \{0, \frac{1}{2}, 1\}$ expresses certainty of influence, i.e., it measures a degree of a certain influence player k has on player j.

3.3 The weighted influence index

Finally, we propose a more general definition of the influence index.

For each $S \subseteq N$, $j \in N \setminus S$ and $i \in I_{S\to j}$, we introduce a *weight* $\alpha_i^{S\to j} \in [0,1]$ *of influence of coalition S on $j \notin S$ under the inclination vector $i \in I_{S\to j}$*. We assume that for each $S \subseteq N$ and $j \in N \setminus S$ there exists $i \in I_{S\to j}$ such that $\alpha_i^{S\to j} > 0$. We assume that the weights $\alpha_i^{S\to j}$ of influence of coalition S on $j \notin S$ do not depend on the influence function, but they only depend on $n^*(i, S)$. In particular, for each $S \subset N$, $j \notin S$ and $i \in I_{S\to j}$,

$$\alpha_i^{S\to j} = \alpha_{-i}^{S\to j}. \tag{9}$$

Definition 5 *Given $B \in \mathcal{B}$, for each $S \subseteq N$, $j \in N \setminus S$, the weighted influence index of coalition S on player j is defined as*

$$d_\alpha(B, S \to j) := \frac{\sum_{i \in I^*_{S\to j}(B)} \alpha_i^{S\to j}}{|\{i \in I_{S\to j} \mid \alpha_i^{S\to j} > 0\}|}. \tag{10}$$

The possibility and certainty influence indices are recovered as follows. For each $S \subseteq N$, $j \in N \setminus S$ and $B \in \mathcal{B}$

$$\overline{d}(B, S \to j) = d_{\overline{\alpha}}(B, S \to j), \ where \ \overline{\alpha}_i^{S\to j} = 1 \ for \ each \ i \in I_{S\to j} \tag{11}$$

$$\underline{d}(B, S \to j) = d_{\underline{\alpha}}(B, S \to j), \; where\ for\ each\ i \in I_{S \to j}$$

$$\underline{\alpha}_i^{S \to j} = \begin{cases} 1\ if\ \forall p \notin S\ [i_p = i_j] \\ 0 \qquad otherwise \end{cases}. \tag{12}$$

Apart from the possibility and certainty influence indices, we specify the equidistributed influence index. Given $B \in \mathcal{B}$, for each $S \subseteq N$, $j \in N \setminus S$

$$d^*(B, S \to j) = d_{\alpha^*}(B, S \to j), \; where\ \alpha_i^{*S \to j} = \frac{1}{2^{n^*(i,S)} - 1}\ for\ each\ i \in I_{S \to j}. \tag{13}$$

For each $S \subseteq N$ and $j \notin S$, we introduce:

$$D_\alpha(S \to j) := \max_{B \in \mathcal{B}} d_\alpha(B, S \to j) \tag{14}$$

in particular,

$$D^*(S \to j) := \max_{B \in \mathcal{B}} d^*(B, S \to j) \tag{15}$$

3.4 Influential null player

We introduce the following definition:

Definition 6 *Player $k \in N$ is said to be an influential null player under $B \in \mathcal{B}$ if*

$$\forall j \in N \setminus k\ \forall i \in I_{k \to j}\ [(Bi)_j = (Bi^{-k})_j], \tag{16}$$

where i^{-k} is given by

$$i_m^{-k} = \begin{cases} i_m & if\ m \neq k \\ -i_m & if\ m = k \end{cases} \quad for\ each\ m \in N. \tag{17}$$

Fact 1 *If $k \in N$ is an influential null player under $B \in \mathcal{B}$, and B both satisfies $B(1, \ldots, 1) = (1, \ldots, 1)$ and $B(-1, \ldots, -1) = (-1, \ldots, -1)$, then*

$$\forall j \in N\ [\underline{d}(B, k \to j) = 0]. \tag{18}$$

Let us consider the following monotonicity condition:

$$\forall i, i' \in I\ [i \leq i' \;\Rightarrow\; Bi \leq Bi'] \tag{19}$$

where

$$Bi \leq Bi' \iff \{k \in N \mid (Bi)_k = +1\} \subseteq \{k \in N \mid (Bi')_k = +1\}. \tag{20}$$

This condition will be violated, for instance, if there is a kind of 'opposite influence' ('My vote is (always) different from your inclination'). Nevertheless, one may suppose that in many situations the condition (19) holds.

Fact 2 *If $\widetilde{k} \in N$ is an influential null player under $B \in \mathcal{B}$, and condition (19) is satisfied, then for each $j \in N$*

$$\widetilde{k} = \arg \min_{k \in N \setminus j} d_\alpha(B, k \to j) \tag{21}$$

4 The influence functions

Let us study some properties of the influence functions $B \in \mathcal{B}$. Before we focus of the influence functions, we remark properties of some related concepts. First, we introduce for any $\emptyset \neq S \subseteq N$ the set

$$I_S := \{i \in I \mid \forall k, j \in S \ [i_k = i_j]\}. \tag{22}$$

We denote by i_S the value i_k for some $k \in S$, $i \in I_S$. The following properties are immediate.

(i) $I_k = I = 2^N$ for all $k \in N$.
(ii) Letting $I_\emptyset := 2^N$, $I : 2^N \to 2^{2^N}$ is an antitone function.
(iii) For any $S, T \neq \emptyset$: $I_{S\cap T} \supseteq I_S \cup I_T \supseteq I_S \cap I_T \supseteq I_{S\cup T}$ and

$$I_{S\cap T} = I_S \cup I_T \quad iff \quad (\min_{K\in\{S,T\}} |K| = 1 \ or \ S \subseteq T \ or \ T \subseteq S) \tag{23}$$

$$I_S \cup I_T = I_S \cap I_T \quad iff \quad (\max_{K\in\{S,T\}} |K| = 1 \ or \ S = T) \tag{24}$$

$$I_S \cap I_T = I_{S\cup T} \quad iff \quad S \cap T \neq \emptyset. \tag{25}$$

Definition 7 *Let $\emptyset \neq S \subseteq N$ and $B \in \mathcal{B}$. The set of followers of S under B is defined as*

$$F_B(S) := \{j \in N \mid \forall i \in I_S \ [(Bi)_j = i_S]\}. \tag{26}$$

The set of anti-followers of S under B is defined as

$$\overline{F}_B(S) := \{j \in N \mid \forall i \in I_S, (Bi)_j = -i_S\}. \tag{27}$$

Letting $F_B(\emptyset) := \emptyset$, F_B is a mapping from 2^N to 2^N.

Proposition 1 *Let $B \in \mathcal{B}$. Then the following holds:*

(i) Whenever $S \cap T = \emptyset$, $F_B(S) \cap F_B(T) = \emptyset$.
(ii) F_B is an isotone function. Consequently, if $F_B(N) = \emptyset$, then $F_B \equiv \emptyset$.
(iii) For each $\widetilde{j} \in F_B(S) \setminus S$

$$\widetilde{j} = \arg\max_{j\in N\setminus S} d_\alpha(B, S \to j). \tag{28}$$

Moreover,

$$\overline{d}(B, S \to \widetilde{j}) = \underline{d}(B, S \to \widetilde{j}) = 1 \tag{29}$$

$$d^*(B, S \to \widetilde{j}) = \frac{1}{2^{n-s-1}} \sum_{p=0}^{n-s-s} \binom{n-s-1}{p} \frac{1}{2^{p+s}-1}. \tag{30}$$

Assume F_B is not identically the empty set. Then the *kernel* of B is the following collection of sets:

$$\mathcal{K}(B) := \{S \in 2^N \mid F_B(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow F_B(S') = \emptyset\}.$$

The kernel is well defined due to isotonicity. It it the set of “true” influential coalitions.

Definition 8 *Let S, T be two disjoint subsets of N. B is said to be a purely influential function of S upon T if it satisfies for all $i \in I_S$:*

$$(Bi)j = \begin{cases} i_S, & \text{if } j \in T \\ i_j, & \text{otherwise.} \end{cases} \tag{31}$$

The set of such functions is denoted $\mathcal{B}_{S \to T}$.

Note that these functions are arbitrary on $I \setminus I_S$.
In each $\mathcal{B}_{S \to T}$, there are 3 particular members. The minimal one is such that $Bi = -1_N$ for all $i \in I \setminus I_S$, the maximal one is such that $Bi = 1_N$ for all $i \in I \setminus I_S$. More interesting is the one which is the identity function on $I \setminus I_S$. We call it the *canonical pure influential function of S upon T*, and we denote it by $B_{S \to T}$.

Proposition 2 *Let S, T be two disjoint subsets of N. Then the following holds:*

(i) For all $B \in \mathcal{B}_{S \to T}$, $F_B(S) = S \cup T$.
(ii) For each $B \in \mathcal{B}_{S \to T}$, $j \in N \setminus S$

$$d_\alpha(B, S \to j) = \begin{cases} D_\alpha(S \to j) & \text{if } j \in T \\ 0 & \text{if } j \in N \setminus (S \cup T) \end{cases} \tag{32}$$

where

$$D_\alpha(S \to j) := \max_{B \in \mathcal{B}} d_\alpha(B, S \to j).$$

(iii) For each $B \in \mathcal{B}$ there exist two disjoint $S, T \subset N$ such that $B \notin \mathcal{B}_{S \to T}$

Lastly, we define several influence functions $B \in \mathcal{B}$ as illustration.

(i) *Majority function* - Let $n \geq t \geq \lfloor \frac{n}{2} \rfloor + 1$, and introduce

$$i^+ := |\{k \in N \mid i_k = +1\}| \tag{33}$$

We define $B \in \mathcal{B}$ such that for each $i \in I$

$$Bi := \begin{cases} 1_N & \text{if } i^+ \geq t \\ -1_N & \text{if } i^+ < t \end{cases} \tag{34}$$

(ii) *Dictatorship* - Let $\widetilde{k} \in N$. We define $B \in \mathcal{B}$ such that for each $i \in I$ and $j \in N$

$$(Bi)_j = i_{\widetilde{k}} \tag{35}$$

(iii) *The identity function (no influence)*, i.e., for each $i \in I$

$$Bi = i \tag{36}$$

(iv) Let $t \in [0, n]$. Functions satisfying for each $i \in I$

$$\text{if } i^+ \geq t, \text{ then } i \leq Bi \tag{37}$$

(effect of mass psychology when i has a sufficient number of $+1$)

(v) Let $t \in [0, n]$. Functions satisfying for each $i \in I$

$$if\ i^+ \leq t,\ then\ i \geq Bi \tag{38}$$

(effect of the empty restaurant when i has a low number of +1)

(vi) Let $t \in (0, n]$. Functions mixing the last two cases, i.e., satisfying for each $i \in I$

$$i \leq Bi\ iff\ i^+ \geq t,\ and\ i \geq Bi\ iff\ i^+ < t \tag{39}$$

We have investigated properties of these influence functions. Due to space limitations, we limit ourselves to the two first cases.

Proposition 3 *Let $n \geq t \geq \lfloor \frac{n}{2} \rfloor + 1$ and the influence function B be defined by (34). Then the following holds:*

(i) For each $\emptyset \neq S \subseteq N$ and $j \in N \setminus S$

$$d_\alpha(B, S \to j) = \begin{cases} D_\alpha(S \to j) & if\ \ s \geq t \\ \frac{D_\alpha(S \to j)}{2} & if\ \ s < t = n \\ \frac{\sum_{i \in I^+_{S \to j, \geq t}} \alpha_i^{S \to j} + \sum_{i \in I^-_{S \to j, < t}} \alpha_i^{S \to j}}{|\{i \in I_{S \to j} | \alpha_i^{S \to j} > 0\}|} & if\ \ s < t < n \end{cases} \tag{40}$$

(ii) For each $S \subseteq N$

$$F_B(S) = \begin{cases} N & if\ \ s \geq t \\ \emptyset & if\ \ s < t \end{cases} \tag{41}$$

Consequently, the kernel is $\mathcal{K}(B) = \{S \mid |S| = t\}$.

Proposition 4 *Let $\widetilde{k} \in N$ and the influence function B be defined by (35). Then the following holds:*

(i) For each $\emptyset \neq S \subseteq N$ and $j \in N \setminus (S \cup \{\widetilde{k}\})$

$$d_\alpha(B, S \to j) = \begin{cases} D_\alpha(S \to j) & if\ \ \widetilde{k} \in S \\ \frac{D_\alpha(S \to j)}{2} & if\ \ \widetilde{k} \notin S \end{cases} \tag{42}$$

(ii) For each $S \subseteq N$

$$F_B(S) = \begin{cases} N & if\ \ \widetilde{k} \in S \\ \emptyset & if\ \ \widetilde{k} \notin S \end{cases} \tag{43}$$

Consequently, the kernel is $\mathcal{K}(B) = \{\widetilde{k}\}$.

5 The power indices

Let us consider the 'power part' of the global index. We define *Success*, *Failure*, *Luck* and *Decisiveness* of a player starting from the final decision of the player, not as before from the inclination. For instance, a player is said to be *successful* if his decision coincides with the group decision.

Given a probability distribution $p : I \to [0,1]$ over all inclination vectors, and $B \in \mathcal{B}$, we define $p_B : I \to [0,1]$ such that for each $b \in I$

$$p_B(b) = \begin{cases} \sum_{\{i|Bi=b\}} p(i) & if\ b \in B(I) \\ 0 & if\ b \notin B(I) \end{cases} \tag{44}$$

where $B(I)$ is the set of all decision vectors. Of course,

$$\sum_{b \in I} p_B(b) = 1. \tag{45}$$

Since some decision vectors may never appear after applying the influence function, we define now the group decision function $gd : I \to \{+1, -1\}$ on the set of all n-vectors, assigning (as before) the value $+1$ if the group decision is 'yes', and -1 if the group decision is 'no'. Moreover, for each $b \in I$ and $k \in N$, let $b^{-k} \in I$ be given by

$$b_j^{-k} = \begin{cases} b_j & if\ j \neq k \\ -b_j & if\ j = k \end{cases}. \tag{46}$$

Definition 9 *Given $gd \in \mathcal{G}$, $p_B : I \to [0,1]$, we define for each $k \in N$*

- *Player k's Success*

$$SUC_k(gd, p_B) := Prob(k\ is\ successful) = \sum_{\{b \in I | b_k = gd(b)\}} p_B(b) \tag{47}$$

- *Player k's Failure*

$$FAIL_k(gd, p_B) := Prob(k\ fails) = \sum_{\{b \in I | b_k = -gd(b)\}} p_B(b) \tag{48}$$

- *Player k's Decisiveness*

$$DEC_k(gd, p_B) := Prob(k\ is\ decisive) = \sum_{\{b \in I | b_k = gd(b) = -gd(b^{-k})\}} p_B(b) \tag{49}$$

- *Player k's Luck*

$$LUCK_k(gd, p_B) := Prob(k\ is\ lucky) = \sum_{\{b \in I | b_k = gd(b) = gd(b^{-k})\}} p_B(b) \tag{50}$$

According to Barry [1], the following relation between Success, Luck, and Decisiveness holds:

$$Success = Decisiveness + Luck,$$

and in our case, we have for each $k \in N$, p_B, and $gd \in \mathcal{G}$

$$SUC_k(gd, p_B) = DEC_k(gd, p_B) + LUCK_k(gd, p_B) \tag{51}$$

$$SUC_k(gd, p_B) = 1 - FAIL_k(gd, p_B). \tag{52}$$

An index analogous to the Hoede-Bakker index looks now as follows:

Definition 10 *Given $B \in \mathcal{B}$, $gd \in \mathcal{G}$, $p : I \to [0,1]$, we define for each $k \in N$*

$$\psi_k(B, gd, p) := \sum_{\{i|(Bi)_k=+1\}} p(i)gd(Bi) - \sum_{\{i|(Bi)_k=-1\}} p(i)gd(Bi). \tag{53}$$

Fact 3 *Given $B \in \mathcal{B}$, $gd \in \mathcal{G}$, and $p : I \to [0,1]$, for each $k \in N$*

$$\psi_k(B, gd, p) = SUC_k(gd, p_B) - FAIL_k(gd, p_B) \tag{54}$$

where $p_B : I \to [0,1]$ is defined by (44).

Let us notice that while ψ is equal to '*Success* − *Failure*' under the new definition of being successful (based on decisions, not as before on inclinations), in general it is not equal to '*Decisiveness*' anymore.

6 The global index - from influence to power

There are some trivial relations between the 'influence part' and the 'power part' of the global index if all inclination vectors are equally probable. Let $p^* : I \to [0,1]$ be such that

$$\forall i \in I \ [p^*(i) = \frac{1}{2^n}]. \tag{55}$$

The following fact holds:

Fact 4 *If $\overline{d}(B, k \to j) = 0$ for each $k \in N$, then $\psi_j(B, gd, p^*) = GHB_j(B, gd)$.*

Example 1 In order to illustrate the notions introduced in the paper, let us consider the following example. We have a three-actor family network in which player 1 (child) is influenced by his mother and his father (players 2 and 3, respectively). If the parents have the same inclination, the child will follow them, but if their inclinations differ from each other, player 1 will decide according to his own inclination. The family has to decide for a long Sunday bicycle trip, but since the weather happens to be quite risky, the actors are not that enthusiastic to decide for the trip. Moreover, a new attractive computer game, a romance just bought and looking very interesting, and a telecast of an important football match are of importance when making the decision. The inclinations of the players to say 'yes' are independent of each other and their probabilities are equal to $\frac{1}{2}$, $\frac{1}{3}$, and 0, for the child, the mother and the father, respectively. The parents try not to discriminate their child in family decision-making, and it is agreed that the family decides for the trip if at least two family members say 'yes'. Table 1 presents the probability distribution over all inclination vectors, and the decision vectors, while Table 2 shows the probability distribution over all decision vectors, and the group decisions.

Table 1. The inclination and decision vectors

$i \in I$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(-1,1,1)$	$(1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$p(i)$	0	$\frac{1}{6}$	0	0	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{1}{3}$
$B(i)$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(1,1,1)$	$(-1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$

Table 2. The group decision

$b \in B(I)$	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(-1,1,-1)$	$(-1,-1,1)$	$(-1,-1,-1)$
$p_B(b)$	0	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{2}{3}$
$gd(b)$	$+1$	$+1$	$+1$	-1	-1	-1

Moreover, $gd(-1,1,1) = +1$, and $gd(1,-1,-1) = -1$.
Based on the given information, we get the following:

$$\forall k, j \in \{1,2,3\}\ [\underline{d}(B, k \to j) = 0]$$

$$\forall d \in \{\overline{d}, d^*\}\ [d(B, 1 \to 2) = d(B, 1 \to 3) = d(B, 2 \to 3) = d(B, 3 \to 2) = 0]$$

$$\overline{d}(B, 2 \to 1) = \overline{d}(B, 3 \to 1) = \frac{1}{2}, \quad d^*(B, 2 \to 1) = d^*(B, 3 \to 1) = \frac{1}{6}$$

$$\forall d \in \{\overline{d}, \underline{d}, d^*\}\ [d(B, 12 \to 3) = d(B, 13 \to 2) = 0]$$

$$\overline{d}(B, 23 \to 1) = \underline{d}(B, 23 \to 1) = 1, \quad d^*(B, 23 \to 1) = \frac{1}{3}$$

$$SUC_1(gd, p_B) = 1, \quad SUC_2(gd, p_B) = SUC_3(gd, p_B) = \frac{5}{6}$$

$$FAIL_1(gd, p_B) = 0, \quad FAIL_2(gd, p_B) = FAIL_3(gd, p_B) = \frac{1}{6}$$

$$DEC_1(gd, p_B) = \frac{1}{3}, \quad DEC_2(gd, p_B) = DEC_3(gd, p_B) = \frac{1}{6}$$

$$LUCK_1(gd, p_B) = LUCK_2(gd, p_B) = LUCK_3(gd, p_B) = \frac{2}{3}$$

$$\psi_1(B, gd, p) = 1, \quad \psi_2(B, gd, p) = \psi_3(B, gd, p) = \frac{2}{3}$$

Lastly, it can be checked that B is the canonical purely influential function $B_{23 \to 1}$.

7 Conclusions

The improvement brought in this paper emphasizes the role of the influence function in the Hoede-Bakker index. The global form of the index proposed here fully takes into account the mutual influence among players. In particular, we define an upper influence index which takes into account any possibility of influence, and a lower influence index which expresses certainty of influence. To the best of our knowledge, such influence indices have not been proposed before, and seem to be a very useful tool, in particular, in the theory of coalition and alliance formation, negotiations, and more generally multi-agent systems.

There are still several other improvements we would like to bring to this framework in the future research. One of the new research lines may be to introduce dynamic aspects. The framework analyzed here is, in fact, a decision process after a single step of mutual influence. In reality, the mutual influence does not stop necessarily after one step but may iterate. We propose to study the behavior of the series $Bi, B^2i, ..., B^ni$; to find convergence conditions, and to study the corresponding decisional power index.

Another natural improvement would be to enlarge the set of possible decisions. The original framework considers only a yes-no decision in a voting situation. One may enlarge this to a yes-no-abstention scheme (ternary voting games, [2]) or, if one escapes from voting situations, to multi-choice games [5], where each player has a totally ordered set of possible actions, and more generally to games on product lattices [3].

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