Better Ways to Cut a Cake

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Abstract

Procedures to divide a cake among n people with n-1 cuts (the minimum number) are analyzed and compared. For 2 persons, cut-and-choose, while envy-free and efficient, limits the cutter to exactly 50% if he or she is ignorant of the chooser's preferences, whereas the chooser can generally obtain more. By comparison, a new 2-person surplus procedure (SP), which induces the players to be truthful in order to maximize their minimum allocations, leads to a *proportionally equitable* division of the surplus—the part that remains after each player receives 50%—by giving each person exactly the same proportion of the surplus as he or she values it.

For $n \geq 3$ persons, a new equitable procedure (EP) yields a *maximally equitable* division of a cake. This division gives all players the highest common value that they can achieve, but it may not be envy-free. (In the 2-person case, this division is vulnerable to manipulation.) The applicability of SP and EP to the fair division of a heterogeneous, divisible good, like land, is briefly discussed.

Better Ways to Cut a Cake

1. Introduction

In this paper, we show how mathematics can illuminate the study of cake-cutting in ways that have practical implications. Specifically, we analyze cake-cutting algorithms that use a minimal number of cuts (n-1 if there are n people), where a cake is a metaphor for a heterogeneous, divisible good, whose parts may be valued differently by different people.

These algorithms not only establish the existence of fair divisions—defined by the properties described below—but also specify a procedure for carrying them out. In addition, they give us insight into the difficulties underlying the simultaneous satisfaction of certain properties of fair division, including *strategy-proofness*, or the incentive for a person to be truthful about his or her valuation of a cake.

As is usual in the cake-cutting literature, we postulate that the goal of each person is to maximize the value of the minimum-size piece (*maximin piece*) that he or she can guarantee, regardless of what the other person does. Thus, we assume that each person is *risk-averse*: He or she will never choose a strategy that may yield a more valuable piece of cake if it entails the possibility of getting less than a maximin piece.

In section 2, we analyze the well-known 2-person, 1-cut cake-cutting procedure, "I cut, you choose," or cut-and-choose. It goes back at least to the Hebrew Bible (Brams and Taylor, 1999, p. 53) and satisfies two desirable properties:

1. *Envy-freeness*: Each person thinks that he or she receives at least a tied-for-largest piece and so does not envy the other person.

2. *Efficiency (Pareto-optimality)*: There is no other allocation that is better for one person and at least as good for the other person.

But cut-and-choose does not satisfy a third desirable property:

3. *Equitability*. Each person's subjective valuation of the piece that he or she receives is the same as the other person's subjective valuation.

To bypass this problem, we propose in section 3 a new 2-person cake-cutting procedure that, while it does not satisfy equitability in an absolute sense, does satisfy it in a relative sense, which we call *proportional equitability*: After ensuring that each person receives exactly 50% of the cake, it gives each person the same proportion of the cake that remains, called the *surplus*, as he or she values it. Thereby this procedure, which we call the *surplus procedure* (SP), gives each person at least 50% of the entire cake and generally more. By contrast, cut-and-choose limits the cutter to exactly 50% if he or she is ignorant of the chooser's preferences.

Remarkably, maximin strategies under SP require that each person be truthful about his or her preferences for different parts of the cake, rendering SP *strategy-proof*. This is because if a person is not truthful, he or she cannot guarantee at least a 50% share or, even if he or she does, may decrease the proportion of the surplus that a truthful strategy guarantees. By comparison, giving each person the same absolute amount of the surplus is *strategy-vulnerable*, because each person will have an incentive to lie about his or her preferences.

In section 4 we give a 3-person example that proves that if there are $n \geq 3$ persons, envy-freeness and equitability may be incompatible. We then describe in section 5 a new

n-person equitable procedure (EP) that gives all persons the maximal *equal* value that they all can achieve. Like SP, it is strategy-proof.

In section 6, we discuss trade-offs in cake division. Whereas SP does not limit one person to exactly 50% of the cake, as does cut-and-choose, it is more information-demanding, requiring that both persons report their value functions over the entire cake, not just indicate their 50-50 points.

While EP is equally information-demanding in the *n*-person case, it may create envy, which SP never does. We conclude by briefly discussing the applicability of both SP and EP to real-world problems and cite related literature on pie-cutting, in which radial cuts are made from the center of a disk, and on fair-division procedures applicable to multiple divisible and indivisible goods.

2. Cut-and-Choose

Assume that two players, A and B, value a cake along a line that ranges from $x = 0$ to $x = 1$. More specifically, we postulate that the players have continuous value functions, $v_A(x)$ and $v_B(x)$, where $v_A(x) \geq 0$ and $v_B(x) \geq 0$ for all x over [0, 1], and their measures are finitely additive, nonatomic probability measures. Finite additivity ensures that the value of a finite number of disjoint pieces is equal to the value of their union. It follows that no subpieces have greater value than the larger piece that contains them. Nonatomic measures imply that a single cut, which defines the border of a piece, has no area and so contains no value. In addition, we assume that the measures of the players are absolutely continuous, so no portion of cake is of positive measure for one player and zero measure for another player.

Like probability density functions (pdfs), the total valuations of the players—the areas under $v_A(x)$ and $v_B(x)$—are 1. We assume that only parallel, vertical cuts, perpendicular to the horizontal x-axis, are made, which we will illustrate later.

Under cut-and-choose, one player cuts the cake into two portions, and the other player chooses one. To illustrate, assume a cake is vanilla over [0, 1/2] and chocolate over (1/2, 1]. Suppose that the cutter, player A, values the left half (vanilla) twice as much as the right half (chocolate). This implies that $v_A(x) = 4/3$ on [0, 1/2], and $v_A(x) = 2/3$ on (1/2, 1].

To guarantee envy-freeness when the players have no information or beliefs about each other's preferences, A should cut the cake at some point x so that the value of the portion to the left of x is equal to the value of the portion to the right.[1] The two portions will be equal when A's valuation of the cake between 0 and x is equal to the sum of its valuations between x and 1/2 and between 1/2 and 1:

$$(4/3)(x - 0) = (4/3)(1/2 - x) + (2/3)(1 - 1/2),$$

which yields $x = 3/8$. In general, the only way that A, as the cutter, can ensure itself of getting half the cake is to give B the choice between two portions that A values at exactly 1/2 each.

To show that cut-and choose does not satisfy equitability, assume B values vanilla and chocolate equally. Thus, when A cuts the cake at $x = 3/8$, B will prefer the right portion, which it values at 5/8, and consequently will choose it. Leaving the left portion

[1] When the cutter does have information or beliefs about the chooser's preferences, he or she may do better with a less conservative strategy (Brams and Taylor, 1996, 1999).

to A, B does better in its eyes (5/8) than A does in its eyes (1/2), rending cut-and-choose inequitable.

If the roles of A and B as cutter and chooser are reversed, the division remains inequitable. In this case, B will cut the cake at $x = 1/2$. A, by choosing the left half (all vanilla), will get 2/3 of its valuation, whereas B, getting the right half, will receive only 1/2 of its valuation. Because cut-and-choose selects the endpoints of the interval of envy-free cuts (3/8 for A, 1/2 for B), any cut strictly between 3/8 and 1/2 will be envy-free.

3. The Surplus Procedure (SP)

The rules of SP ensure that both A and B will obtain at least 50% of the cake, as they value it, and generally give each more:

1. Independently, A and B report their value functions, $f_A(x)$ and $f_B(x)$, over [0, 1] to a referee. These functions may be different from the players' true value functions, $v_A(x)$ and $v_B(x)$.

2. The referee determines the 50-50 points, a and b, of A and B—that is, the points on [0, 1] such that each player reports that half the cake, as it values it, lies to the left and half to the right (these points are analogous to the median points of pdfs).[2]

3. If a and b coincide, the cake is cut at $a = b$. One player is randomly assigned the piece to the left of this cutpoint and the other player the piece to the right. The procedure ends.

4. Assume that a is to the left of b, as illustrated below:

0----------*a*----------*b*----------1.

[2] We could assume that the referee asks the players first to indicate their 50-50 points and then to submit their pdfs for the half of the cake that includes the 50-50 point of the other player, which the referee would identify. This procedure would be somewhat less information-demanding than asking the players to submit their pdfs for the entire cake, but it would require the extra step of the referee's informing the players, after they have indicated their 50-50 points, of which half, as each player defines it, it needs to provide information on its value function.

Then A receives the portion [0, *a*], and B the portion [*b*, 1], which each player values at 1/2 according to its reported value function.

5. Let *c* (for cutpoint) be the point in [*a*, *b*] at which the players receive the same *proportion p* of the cake in this interval, as each values it:

0----------*a*----------*c*-------*b*----------1.

Then A receives the portion [*a*, *c*], and B the portion (*c*, *b*], so the players' combined portions are piece [0, *c*] for A and piece (*c*, 1] for B.

To determine *c*, we set the proportion *p* that A receives from subinterval [*a*, *c*] equal to the proportion that B receives from subinterval (*c*, *b*]:

$$p = \frac{\int_a^c f_A(x)dx}{\int_a^b f_A(x)dx} = \frac{\int_c^b f_B(x)dx}{\int_a^b f_B(x)dx}. \qquad (1)$$

In our earlier example, in which $a = 3/8$ and $b = 1/2$ and the pdfs are as given in section 2,

$$p = \frac{\int_{3/8}^{c} (4/3)dx}{\int_{3/8}^{1/2} (4/3)dx} = \frac{\int_{c}^{1/2} dx}{\int_{3/8}^{1/2} dx}$$

$$p = \frac{(4/3)(c - 3/8)}{(4/3)(1/2 - 3/8)} = \frac{(1/2 - c)}{(1/2 - 3/8)},$$

which yields $c = 7/16$, the midpoint of the interval [3/8, 1/2] between the players' 50-50 points.

Whenever the players have uniform densities over the interval between their 50-50 points, as they do in our example, they will receive the same proportions of the interval at all points in it equidistant from *a* and *b*. In particular, at *c* = 7/16,

$$p = \frac{(1/2 - 7/16)}{(1/2 - 3/8)} = \frac{1}{2},$$

so each player obtains 1/2 the value it places on the entire interval, [*a*, *b*].

Note that giving A and B the same proportion of the interval does not ensure equitability, because A and B value the interval differently. A values it at (1/8)(4/3) = 1/6 (and obtains 1/12 at *c*), and B values it at (1/8)(1) = 1/8 (and obtains 1/16 at *c*).

To ensure that A and B obtain exactly the same value from the interval rather than the same proportion of value, we set equal the numerators of equation (1). Substituting *e* (for equitable point) for *c* in the limits of integration in our example, we have

$$\int_{3/8}^{e} (4/3)dx = \int_{e}^{1/2} dx$$

$$(4/3)(e - 3/8) = (1/2 - e),$$

which yields *e* = 3/7. At this cutpoint, A and B each obtain 1/14 from the interval.

There are conflicting arguments for cutting the cake at *c* (proportional equitability) and at *e* (equitability). An argument for cutting it at *c* is that the player that values the interval more (A in our example) should derive more value from it. The opposite argument reflects the egalitarian view that the players, in addition to the 50% portions they receive outside the interval, should get exactly the same value from the interval.

We will not try to resolve these conflicting claims for proportional equitability versus (absolute) equitability. Instead, we introduce a new property that only proportional equitability satisfies.

Define a procedure to be *strategy-vulnerable* if a maximin player can, by misrepresenting its value function, assuredly do better, whatever the value function of the other player (or, as we will discuss later, other players). A procedure that is not strategy-vulnerable is *strategy-proof*, giving maximin players always an incentive to let $f_A(x) = v_A(x)$ and $f_B(x) = v_B(x)$.

Theorem 1. *SP is strategy-proof, whereas any procedure that makes e the cut-point is strategy-vulnerable.*

Proof. To show that maximin players will be truthful when they submit their value functions to a referee, we show that A or B may do worse if they are not truthful in reporting:

1. *Their 50-50 points, a and b, based on their value functions.*

Assume B is truthful and A is not. If A misrepresents *a* and causes it to crisscross *b*, as illustrated by the location of *a*' below,

0----------*a*---*b*-- *a*'----------1.

then A will obtain [*a*', 1] and, in addition, get some less-than-complete portion of (*b*, *a*'). But this is less than 50% of the cake for A and, therefore, less than what A would obtain under SP if it was truthful.

2. *Their value functions over [a, b].*

Assume, again, B is truthful. Assume A considers misrepresenting its value function in a way that moves *c* rightward, as shown below:

0----------*a*----------*c*→-----*b*----------1.

It can do this by (i) decreasing the value of $\int_a^c f_A(x)dx$, the numerator on the left side of equation (1), or (ii) increasing the value of the denominator, $\int_a^b f_A(x)dx$. But in order for A to misrepresent in this manner, it would have to know $f_B(x)$ and therefore *b*, which *c* depends on. But A does not know $f_B(x)$ and *b* and, consequently, cannot determine *c*. Hence, it cannot assuredly reduce its value of the interval [*a*, *c*] relative to [*a*, *b*] in order to make this proportion less than its true proportion and so move *c* rightward. Indeed, A's attempted misrepresentation could backfire by moving *c* leftward rather than rightward, which would give A a smaller proportion of [*a*, *b*].

To be sure, if A knew the location of *b*, it could concentrate its value just to the left of *b*, which would move *c* rightward. But we assumed that A is ignorant of the location of *b*, and even which side of *a* (left or right) it is on. Hence, A cannot misrepresent its value function and assuredly do better, which makes SP strategy-proof.

On the other hand, assume the cake is cut at *e*, so its division is equitable rather than proportionally equitable (at *c*). When the players are truthful so $f_A(x) = v_A(x)$ and $f_B(x) = v_B(x)$, one player will receive [0, *e*] and the other player will receive (*e*, 1], which they will value equally and at least as much as 1/2; point *e* will be unique when the players' measures are absolutely continuous with respect to Lebesgue measure (Jones, 2002).

We next show that A can submit a value function different from $v_A(x)$ that moves *e* to a position favorable to it regardless of (i) player B's value function and (ii) whether or not player A receives the left or the right piece of the cake. Because A is unaware of

whether *e* is to the left or to the right of its 50-50 point, *a*, A should submit a value function that has the same 50-50 point as $v_A(x)$, as discussed in (1) above.

But to increase the value of its piece beyond 50%, A should submit a value function, $f_A(x)$, that decreases the value of $\int_a^e f_A(x)dx$ if *a* is to the left of *b*, and decreases the value of $\int_e^a f_A(x)dx$ if *a* is to the right of *b*. The former strategy will move *e* rightward, whereas the latter strategy will move *e* leftward, of its true value.

Clearly, A can effect *both* movements of *e* by decreasing the value of $\int_a^b f_A(x)dx$. Because A does not know the value of *b*, however, and even whether it is to the left or to the right of *a*, A can best minimize its value over [*a*, *b*] by concentrating almost 1/2 its value near 0 and almost 1/2 near 1—the endpoints of the cake—while ensuring that

$$\int_0^a f_A(x)dx = \int_a^1 f_A(x)dx = 1/2$$

so that its 50-50 point is truthful. Thereby A decreases its value around its 50-50 point, which will move *e* toward B's 50-50 point—whichever side of *a* that *b* is on—and so help A. (Optimally, A should let the value strictly between its 50-50 point and the edges of the cake, where its value is concentrated, approach 0 in the limit.) Thereby any procedure that makes *e* the cut-point is strategy-vulnerable. Q.E.D.

Because both A and B receive at least 50% of their valuations under SP, the resulting division is not only proportionally equitable but also envy-free. If there are more than two players, however, an envy-free allocation may be neither equitable nor proportionally equitable.

4. Three or More Players: Equitability and Envy-Freeness May Be Incompatible

To show that it is not always possible to divide a cake among three players into envy-free and equitable portions using 2 cuts, assume that A and B have (truthful) piecewise linear value functions that are symmetric and V-shaped,

$$v_A(x) = \begin{cases} -4x + 2 \text{ for } x \in [0,1/2] \\ 4x - 2 \text{ for } x \in (1/2,1] \end{cases}$$
$$v_B(x) = \begin{cases} -2x + 3/2 \text{ for } x \in [0,1/2] \\ 2x - 1/2 \text{ for } x \in (1/2,1] \end{cases}.$$

Whereas both functions have maxima at $x = 0$ and $x = 1$ and a minimum at $x = 1/2$, A's function is steeper (higher maximum, lower minimum) than B's, as illustrated in Figure 1. In addition, suppose that a third player, C, has a uniform value function, $v_C(x) = 1$, for $x \in [0, 1]$.

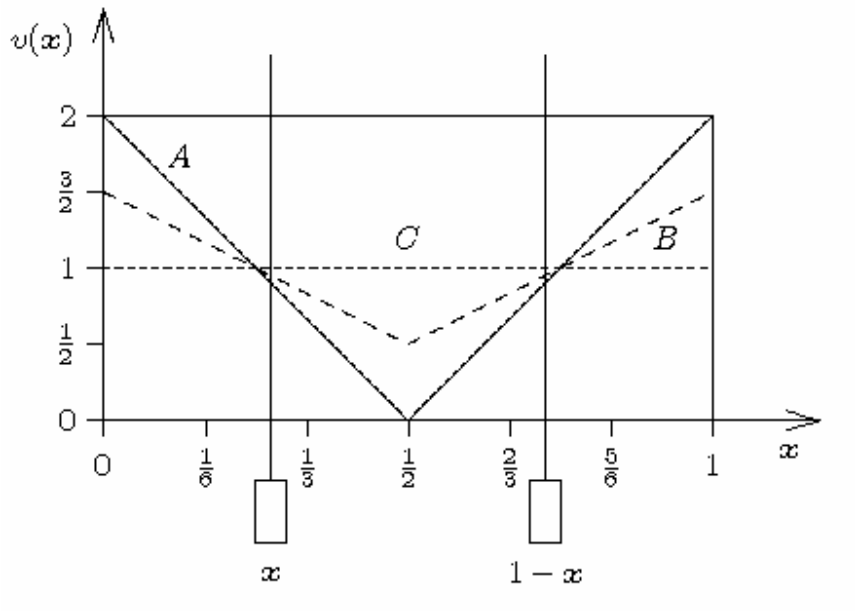

Figure 1. Impossibility of Envy-Free and Equitable Cuts for Three Players

In this example, every envy-free allocation of the cake will be one in which A gets the portion to the left of x, B the portion to the right of $1–x$ (A and B could be

interchanged), and C the portion in the middle. If the horizontal lengths of A's and B's portions are not the same (i.e., x), the player whose portion is shorter in length will envy the player whose portion is longer. But such an envy-free allocation will not be equitable, because A will receive a larger portion in its eyes than B receives in its eyes, violating equitability. Thus, an envy-free allocation is not equitable in this example, nor an equitable allocation envy-free, though both these allocations will be efficient with respect to parallel, vertical cuts.[3]

Two envy-free procedures have been found for 3-person, 2-cut cake division. Whereas one of the envy-free procedures requires four simultaneously moving knives (Stromquist, 1980), the other requires only two simultaneously moving knives (Barbanel and Brams, 2004). Although there are no known 4-person, 3-cut procedures for dividing a cake into envy-free pieces, Barbanel and Brams show that no more than 5 cuts are needed to ensure 4-person envy-freeness. Brams, Taylor, and Zwicker (1997) previously showed that a maximum of 11 cuts is needed, based on an arguably simpler 4-person, envy-free procedure (for chores, a maximum of 16 cuts may be needed; see Peterson and Su, 2002).

Beyond 4 players, no procedure is known that yields an envy-free division of a cake unless an unbounded number of cuts is allowed (Brams and Taylor, 1995, 1996; Robertson and Webb, 1998). While this number can be shown to be finite, it cannot be specified in advance—this will depend on the specific cake being divided. The complexity of what Brams and Taylor call the "trimming procedure" makes it of dubious practical value.

We next show, it is always possible to find an equitable division of a cake among three or more players that is efficient (see note 2). In fact, the equitability procedure (EP)

[3] Not all equitable divisions need by efficient. If C were given an end piece and A or B the middle piece in the example, cutpoints could be found such all the players receive, in their own eyes, the same value. However, this value would be less than what another equitable allocation, in which C gets the middle piece and A and B the end pieces, yields. By contrast, an envy-free allocation that uses n–1 parallel, vertical cuts is always efficient (Gale, 1993; Brams and Taylor, 1996, pp. 150-151).

enables $n \geq 3$ players to achieve an equitable and efficient division of a cake that is also strategy-proof.

5. The Equitability Procedure (EP)

The rules of EP are as follows:

1. Independently, A, B, C, . . . report their value functions $f_A(x), f_B(x), f_C(x), \ldots$ over [0, 1] to a referee. These functions may be different from the players' true value functions, $v_A(x), v_B(x), v_C(x) \ldots$.

2. The referee determines the cutpoints that equalize the common value that all players receive for each of the $n!$ possible assignments of pieces to the players from left to right.

3. The referee chooses the assignment that gives the players their maximum common value.

We next illustrate EP using the 3-person example in section 4. It is evident that the ordering of players that will maximize the common value to the players is to give the left piece to A (or B), the middle piece to C, and the right piece to B (or A).

Let the cutpoints be e_1 and e_2. Assume A receives the piece defined by the interval $[0, e_1]$, C the piece defined by the interval $(e_1, e_2]$, and B the piece defined by the interval $(e_2, 1]$. The players' values will be equal when

$$\int_0^{e_1} (-4x + 2)dx = \int_{e_1}^{e_2} dx$$

$$\int_{e_1}^{e_2} dx = \int_{e_2}^{1} (2x - 1/2)dx .$$

After integration and evaluation, we have

$$-2e_1^2 + 2e_1 = e_2 - e_1$$

$e_2 - e_1 = 1/2 - e_2^2 + e_2/2.$

When solved simultaneously, these equations give $e_1 \approx 0.269$ and $e_2 \approx 0.662$. Players A, C, and B (from left to right, in that order) all value their pieces at 0.393, so each thinks it receives nearly 40% of the cake.

For n players, there will be n-1 cutpoints e_i. For each assignment of pieces to the players from left to right, solving simultaneously the n-1 equations that equalize the value functions of adjacent players will give the e_i*'s*.

Choosing the assignment that gives the players a maximum common value yields a division that is efficient. This is because one player cannot get more value without another player's getting less, which would, of course, destroy equitability.

Theorem 2. *EP is strategy-proof.*

Proof. Assume that some player X is not truthful under EP but that all other players are truthful. For X to increase its allocation, it would have to know its borders in order to misrepresent its true value function and guarantee itself more. But because X is ignorant of the reported value functions of the other players, it will not be able to determine these borders, nor even where its piece lies in the left-right assignment of pieces to players if it did know these borders. Hence, X cannot ensure itself of a more valuable piece if it does not know the value functions of the other players. Q.E.D.

Assume that X, by misrepresenting its own value function, increases the value of its piece, as we showed was possible in the 2-person case of equitable division even with no information about the value functions of the other player. But then X will have no assurance that it will receive this more valuable piece, because its misrepresentation may change the left-right assignment of piece to players. This was not possible in the 2-person case as long as X was truthful about its 50-50 point: By undervaluing the cake

around its 50-50 point, X could increase its portion of the surplus while still retaining its 50% portion on the left or right side.

However, when there are additional players and there is no identifiable surplus to be divided among them—as in the 2-person case between A and B—X has no assurance that it will retain the piece that its misrepresentation might increase in size. Indeed, X may end up with a piece that it values less than 1/*n* of the cake.

Theorem 3. *If a player is truthful under EP, it will receive at least 1/n of the cake regardless of whether or not the other players are truthful; otherwise, it may not.*

Proof. Consider the moving-knife procedure, due to Dubins and Spanier (1961), in which a referee moves a knife slowly across a cake from left to right.[4] A player that has not yet received a piece calls "stop," and makes a mark, when the knife reaches a point that gives it 1/*n* of the cake rightward of the last point at which the knife was stopped by a player (or from the left edge for the first player to call stop). It is easy to show that a truthful player will be able to get a 1/*n* piece, with some cake generally remaining near the right edge.[5] By moving all players' marks rightward (Shishido and Zeng, 1999), one can give each player an equal amount greater than 1/*n*, exhausting the remainder, because the players' measures are nonatomic. If a player is not truthful, it will *appear* that it received a piece that is at least 1/*n* under EP, but its true value may be less than 1/*n*. Q.E.D.

To illustrate a misrepresentation that may give a player less than 1/*n*, assume player C in the example given in section 4 knows the value functions of players A and B, but A and B do not C's value function. We first show how C can maximize its value

[4] Moving-knife procedures are discussed in, among other places, Brams, Taylor, and Zwicker (1995), Brams and Taylor (1996), and Robertson and Webb (1998). For nonconstructive results on cake-cutting, which address the existence but not the construction of fair divisions that satisfy different properties, see Barbanel (2005).

[5] Even though the Dubins-Spanier assignment gives each player at least 1/*n*, it may not be the assignment from left to right that gives the players the maximal equitable division. Under EP, a different assignment of equal-valued pieces to players could give each more.

function when it knows the value functions of A and B, which we will assume are truthful.

Let c_1 be the cutpoint on the left that defines A's piece, starting from the left edge, and let c_2 be the cutpoint on the right that defines B's piece, starting from the right edge. Then C should undervalue the middle portion between c_1 and c_2 so that A and B receive exactly the same value from their pieces—as required by EP—

$$\int_0^{c_1} (-4x + 2)dx = \int_{c_2}^1 (2x - 1/2)dx,$$

while C receives as much of the middle portion of the cake as possible.

C can maximize the value of the middle portion by making B, which values the middle portion more than A does, indifferent between receiving this portion and obtaining the right portion:

$$\int_{c_1}^{1/2} (-2x + 3/2)dx + \int_{1/2}^{c_2} (2x - 1/2)dx = \int_{c_2}^1 (2x - 1/2)dx.$$

This "optimal" misrepresentation by C ensures that it obtains as physically large a middle piece as possible at the same time that it appears to receive the same-value pieces as A and B do on the left and right.[6]

After integration and evaluation of the two foregoing equations, we have

$$4c_1^2 - 4c_1 = 2c_2^2 - c_2 - 1$$
$$2c_1^2 - 3c_1 = -4c_2^2 + 2c_2.$$

Solving these equations simultaneously gives $c_1 = 0.230$ and $c_2 = 0.707$. A and B receive the same value of 0.354 for the left and right pieces, respectively, whereas C *appears* to receive this value for the middle piece.

[6] Just as C must lower its value of $[c_1, c_2]$ to that which gives it the same value that B attaches to this middle portion, it must also raise its values of $[0, c_1)$ and $(c_2, 1]$ to those of B as well. Because this will allow either the middle or the right pieces to be assigned to either B or C, C should slightly perturb its values so that it is appears that it values the middle portion more, ensuring that it, rather than B, will receive it.

But the true value for C of its now enlarged middle piece, $c_2 - c_1 = 0.477$, is 21 percent greater than its value when it is truthful (0.393), so C clearly benefits from this misrepresentation But had C undervalued the middle portion more, and consequently overvalued the left or right portions by a greater amount, it would have received one of the latter under EP, which would have given it a true value of less than 0.393.

Thus, without information on the value functions of the other players, a player may misrepresent in a way that lowers its value over being truthful. Indeed, such misrepresentation may give it less than $1/n$ of the cake, making truthfulness not only a maximin strategy but also one that gives a player at guarantee of at least $1/n$ (Theorem 3).

The guarantee of at least $1/n$ to the players under EP generalizes the guarantee of at least 1/2 to the two players under SP.[7] The additional players under EP create greater uncertainty about their allocations, making EP more difficult to exploit than SP. Consequently, EP is able to ensure a maximal equitable allocation that is strategy-proof, whereas SP can only ensure a proportionally equitable allocation that is strategy-proof.

6. Conclusions

We have described a new 2-person, 1-cut cake-cutting procedure, called the surplus procedure (SP). Like cut-and-choose, it is envy-free and efficient and also induces the players to be truthful when they have no information about each other's preferences, rendering it strategy-proof. But unlike cut-and-choose, SP produces a proportionally equitable division, whereas an analogous equitable procedure is strategy-vulnerable.

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[7] A minimal-cut envy-free procedure also gives this guarantee, because a player cannot receive less than $1/n$ without envying another player. However, EP maximizes the minimum amount *greater than* $1/n$ that a player receives, whereas an envy-free procedure may give one player less than this amount. In the 3-person example in section 4, for instance, an envy-free allocation will give A a larger proportion of the cake than B receives, though both players will value the two (equal) envy-free pieces each gets exactly the same. Under EP, by contrast, the players will value their pieces differently, causing A to envy B for getting a physically larger piece, though each player values its proportion of the cake exactly the same as the other player values its proportion.

SP is more information-demanding than cut-and-choose, requiring that the players submit to a referee their value functions over an entire cake, not just indicate a 50-50 point. Practically, players might sketch such functions, or choose from a variety of different-shaped functions, to indicate how they value a divisible good like land.

Thus, land bordering water might be more valuable to one person (A), whereas land bordering a forest might be more valuable to the other (B). Even if players know these basic preferences of each other, and hence that *a* will be closer to the water and *b* will be closer to the forest, uncertainty about the other player's 50-50 point makes it impossible for maximin players to exploit SP without knowledge of the other player's value function.

For three persons, there may be no envy-free division that is also equitable, so a choice may have to be made between these two properties. For four or more persons, there is no known minimal-cut, envy-free procedure, whereas the equitability procedure (EP) ensures an equitable and efficient division that is strategy-proof for any *n*.

It is pleasing to have strategy-proof procedures that yield efficient, envy-free, and proportionally equitable divisions in the case of two players, and efficient and equitable divisions in the case of more than two players. If there are multiple divisible goods that must be divided, however, 2-person procedures like "adjusted winner" (Brams and Taylor, 1996, 1999) seem more applicable than cake-cutting procedures, though Jones (2002) shows that adjusted winner can be viewed as a cake-cutting procedure.

The small literature on pie-cutting, in which radial cuts are made from the center of a disk instead of vertical cuts along a horizontal axis, raises new issues, including whether there always exists an envy-free and efficient division of a pie (Gale, 1993). Barbanel and Brams (2004) and Brams, Jones, and Klamler (2005) provide some positive as well as negative answers, but suffice it to say that several questions remain open.

The fair division of indivisible goods poses significant new challenges that lead to certain paradoxes (Brams, Edelman, and Fishburn, 2001). But recently progress has been

made in finding ways of dividing such goods (Brams and Fishburn, 2000; Edelman and Fishburn, 2001; Brams, Edelman, and Fishburn, 2003; Herreiner and Puppe, 2002; Brams and King, 2004). Procedures that work for both divisible and indivisible goods have also recently been developed that have a number of practical applications, such as determining how roommates might share the rent of a house or how students might be assigned to courses as fairly as possible (Su, 1999; Brams and Kilgour, 2001; Haake, Raith, and Su, 2002; Potthoff, 2002; Abdulkadiroglu, Sönmez, and Unver, 2004).

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