# The Polyhedron of all Representations of a Semiorder

Barry Balof*, Jean-Paul Doignon†, Samuel Fiorini*

**Abstract**

In many practical situations, indifference is intransitive. This led Luce (1956) to base a preference model on the following principle: an alternative is judged better than another one only if the utility value of the first alternative is significantly higher than the value of the second alternative. Here, 'significantly higher' means higher than the value augmented by some constant threshold. The resulting relations are called "semiorders" by Luce. Their axiomatic description is established by Scott and Suppes (1958)—see below for details.

Given a semiorder $P$, we form the collection of all numerical representations of $P$. This collection $\mathcal{R}$ happens to be a convex set, although not a closed or open one. It is naturally turned into (and approximated by) a polyhedral set $\mathcal{R}_\varepsilon$ consisting of all $\varepsilon$-representations. We first explain the facets of $\mathcal{R}_\varepsilon$: in general, they bijectively correspond to the "noses" and "hollows" of the semiorder $P$. Noses and hollows were introduced by Pirlot (1991) as a tool for proving the existence of the "minimal $\varepsilon$-representation" of $P$. They were further investigated by Doignon and Falmagne (1997).

Next, we impose that the $\varepsilon$-representations of the semiorder $P$ are nonnegative, and denote with $\mathcal{R}_\varepsilon^+$ the collection of such representations. Understanding the vertices and extreme rays of $\mathcal{R}_\varepsilon^+$ seems to be a more difficult problem. The minimal $\varepsilon$-representation of $P$ is always a vertex of $\mathcal{R}_\varepsilon^+$, and sometime the only one. We provide examples with other vertices, even with vertices involving another threshold than the one in the minimal $\varepsilon$-representation. We then offer some partial results on the vertices and extreme rays of $\mathcal{R}_\varepsilon^+$.

**Key words :** semiorder, numerical representation, polyhedral set

*Whitman College, Walla Walla, WA 99362, U.S.A. balofba@whitman.edu

†Université Libre de Bruxelles, Département de Mathématique c.p. 216, Bd du Triomphe, B-1050 Bruxelles. {doignon,sfiorini}@ulb.ac.be

# 1 Weak Orders and Semiorders

We denote by $X = \{a_1, a_2, \ldots, a_n\}$ a finite set of $n$ *alternatives* and by $P$ a relation on $X$. Formula $i\,P\,j$ is interpreted as “alternative $j$ is strictly prefered to alternative $i$”. Assigning utility values to alternatives consists in selecting some real-valued mapping $f$ defined on $X$. In our models, more an object is prefered, higher is its utility value. All mappings from $X$ to $\mathbb{R}$ form the real vector space $\mathbb{R}^X$, which can be identified with $\mathbb{R}^n$ when $f \in \mathbb{R}^X$ is summarized as the $n$-tuple $(f(a_1), f(a_2), \ldots, f(a_n))$.

A *weak order* $P$ on $X$ is any relation that satisfies the conditions in the following (very easy) proposition.

**Proposition 1** *There exists $f : X \to \mathbb{R}$ such that*

$$i\,P\,j \iff f(i) < f(j), \qquad \textit{for all } i, j \in X,$$

*if and only if $P$ is asymmetric and negatively transitive.*

For such a weak order, indifference between two alternatives $i$ and $j$ occurs exactly when $f(i) = f(j)$. As a consequence, indifference is a transitive relation. However, in many practical situations, it can be seen that indifference is *not* transitive. Indeed, as Luce (1956) formulated it, addding only one grain of sugar to a cup of coffee results in another cup judged as indifferent to the first one, but adding a sufficient number of grains will produce a definitively different beverage. This observation led Luce to introduce “semiorders”. Later, Scott and Suppes (1958) proved the following proposition; we call *semiorder* any relation satisfying the conditions in Proposition 2.

**Proposition 2 (Scott and Suppes, 1958)** *There exist $f : X \to \mathbb{R}$ and $r \in \mathbb{R}^+$ such that*

$$i\,P\,j \iff f(i) + r < f(j), \qquad \textit{for all } i, j \in X,$$

*if and only if $P$ is irreflexive and satisfies for all $i, j, k, \ell \in X$*

$$\begin{aligned} i\,P\,j \text{ and } k\,P\,\ell \quad &\Longrightarrow \quad i\,P\,\ell \text{ or } k\,P\,j, \\ i\,P\,j \text{ and } j\,P\,k \quad &\Longrightarrow \quad i\,P\,\ell \text{ or } \ell\,P\,k. \end{aligned}$$

Any pair $(f, r)$ as in Proposition 2 is called a *(numerical) representation* of the semiorder $P$; we denote with $\mathcal{R}$ the set of all representations of $P$. Thus $\mathcal{R} \subset \mathbb{R}^X$. Because $X$ is a finite set, there exists some strictly positive real number $\varepsilon$ such that the following holds for all $i, j \in X$:

$$\left\{ \begin{array}{rclcl} i\,P\,j & \Longrightarrow & f(i) + r + \varepsilon & \leq & f(j), \\ \neg i\,P\,j & \Longrightarrow & f(i) + r & \geq & f(j). \end{array} \right. \tag{1}$$

For a given $\varepsilon \in \mathbb{R}^{*+}$, we denote with $\mathcal{R}_\varepsilon$ the collection of all pairs $(f, r)$ satisfying Equations (1). Those pairs are the $\varepsilon$-*representations* of the semiorder $P$.

# 2 The Polyhedral Sets $\mathcal{R}_\varepsilon$ and $\mathcal{R}_\varepsilon^+$

Keeping the same notation, consider some fixed semiorder $P$ on the finite set $X$.

**Proposition 3** *The collection $\mathcal{R}$ of all representations of $P$ is a convex set in $\mathcal{R}^X$. In most cases, $\mathcal{R}$ is neither closed nor open.*

We next notice that $\mathcal{R}_\varepsilon$ is a good approximation of $\mathcal{R}$, in the following sense.

**Proposition 4** *For any strictly positive real number $\varepsilon$, the set $\mathcal{R}_\varepsilon$ of all $\varepsilon$-representations is a polyhedral set in $\mathbb{R}^X$. Moreover,*

$$\begin{aligned} \frac{\varepsilon'}{\varepsilon}\mathcal{R}_\varepsilon &= \mathcal{R}_{\varepsilon'} \qquad \textit{when} \qquad \varepsilon, \varepsilon' > 0; \\ \mathcal{R}_\varepsilon &\subseteq \mathcal{R}_{\varepsilon'} \qquad \textit{when} \qquad \varepsilon \geq \varepsilon' > 0; \\ \mathcal{R} &= \cup\!\uparrow\{\mathcal{R}_\varepsilon \mid \varepsilon > 0\}. \end{aligned}$$

Our main question now is to better understand the polyhedral set $\mathcal{R}_\varepsilon$. This question is of interest for real-life applications of semiorders. As a matter of fact, Pirlot (1990) proposed to investigate the collection of all representations of a semiorder $P$, although his paper is centered on a particular representation, called the "minimal $\varepsilon$-representation". Here is his result, a particularly remarkable one.

**Proposition 5 (Pirlot, 1990)** *Take $\varepsilon > 0$. There is a (unique) $\varepsilon$-representation $(f_0, r_0)$ of the semiorder $P$ such that for any $\varepsilon$-representation $(f, r)$ of $P$:*

$$\forall i \in X : \quad f_0(i) \ \leq \ f(i).$$

*Moreover $r_0 \ \leq \ r$.*

In order to prove the existence of the *minimal representation* $(f_0, r_0)$, Pirlot introduced (in fact, only for "reduced" semiorders) the notions of "noses" and "hollows" of a semiorder. In a further paper, he showed that noses and hollows determine completely the "reduced" semiorder $P$ (see Pirlot, 1991). The notions of noses and hollows were generalized, and also characterized, by Doignon and Falmagne (1997).

**Proposition 6 (Doignon and Falmagne, 1997)** *The noses of the semiorder $P$ are the pairs $(i, j)$ satisfying any of the two equivalent conditions:*

$$(i, j) \ \in \ R \setminus \big(R\,R\,\bar{R}^{-1} \cup R\,\bar{R}^{-1}\,R \cup \bar{R}^{-1}\,R\,R\big);$$

$(i, j) \in R$ *and* $R \setminus \{(i, j)\}$ *is again a semiorder.*

*The hollows of* $P$ *are the pairs* $(i, j)$ *satisfying any of the two equivalent conditions:*

$(i, j) \ \in \ \bar{R} \setminus \big(I \cup \bar{R}\,\bar{R}\,R^{-1} \cup \bar{R}\,R^{-1}\,\bar{R} \cup R^{-1}\,\bar{R}\,\bar{R}\big)$*;*

$(i, j) \in \bar{R}$ *and* $R \cup \{(i, j)\}$ *is again a semiorder.*

We now present the results we have obtained so far on the polyhedral set $\mathcal{R}_\varepsilon$, beginning with a complete description of the facets.

**Proposition 7** *Assume* $P$ *is a semiorder which is not a weak order. Then the facets of* $\mathcal{R}_\varepsilon$ *are in one-to-one correspondence with the noses and hollows of* $(X, P)$*. More precisely, they are all obtained as follows:*

*An equation* $f(i) + r + \varepsilon \ \leq \ f(j)$ *defines a facet iff* $(i, j)$ *is a nose.*

*An equation* $f(i) + r \ \geq \ f(j)$ *defines a facet iff* $(i, j)$ *is a hollow.*

Next, we remark that the polyhedral set $\mathcal{R}_\varepsilon$ contains lines: if $(f, r)$ is any $\varepsilon$-representation of $P$, then all points $(f + \lambda, r)$ for $\lambda \in \mathbb{R}$ also belong to $\mathcal{R}_\varepsilon$. It is thus better to project $\mathcal{R}_\varepsilon$ along the direction of these lines, or else to consider only nonnegative points of $\mathcal{R}_\varepsilon$; these are the nonnegative $\varepsilon$-representations of the semiorder $P$. We denote with $\mathcal{R}_\varepsilon^+$ their collection. The current problem now is to understand the vertices and extreme rays of $\mathcal{R}_\varepsilon^+$, in particular to relate them to the structure of $P$. We have only partial results here.

**Proposition 8** *The minimal* $\varepsilon$*-representation* $(f_0, r_0)$ *of* $P$ *is always a vertex of the polyhedral set* $\mathcal{R}_\varepsilon^+$*. Depending on the semiorder* $P$*, there can be other vertices, even vertices* $(f, r)$ *with* $r \neq r_0$*.*

More insight on the structure of the polyhedral set $\mathcal{R}_\varepsilon^+$ will be provided during the presentation.

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