An axiomatic characterization of the prudent order preference function

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Abstract

In this paper, we are interested in a preference function that associates to a profile of linear orders the set of its corresponding prudent orders. We will introduce axioms that will restrict the set of linear orders to the set of prudent orders. By slightly adapting these axioms, the prudent order preference function can be fully characterized.

Key words : Prudent Orders, Axiomatization

1 Introduction

Arrow and Raynaud[1] introduced a set of axioms that a ranking rule which combines a profile of linear orders into a compromise ranking should verify. Among these, axiom V' states that the compromise ranking should be a so-called prudent order. Intuitively, a prudent order is a linear order such that the strongest opposition against this solution is minimal, which is considered by the authors to be an interesting compromise ranking when working in an industrial or business-like context.

Apart from the works of Arrow and Raynaud [1] and Debord [3], prudent orders have also been analyzed by Lansdowne [5, 6] who compared their properties to other social ordering rules. However, the particular question of characterizing the set of prudent orders has not been addressed yet. This will be the topic of this paper.

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A characterization will be useful to highlight the particularities of prudent orders with respect to other common social ordering rules. The results presented in this paper can also be seen as a first step toward characterizing other prudent ordering rules, such as for instance the ranked pairs rule proposed by Tideman [9, 11].

Although working in very different contexts, min-based ranking rules have been characterized among others by Barbara [2] and Pirlot [7]. Let us however emphasize that, in our setting, the type of solution which we will characterize is neither a ranking, nor a choice subset, but a *set* of rankings. This has also been the case in Young's [10] axiomatization of the set of Kemeny orders.

Let us also mention that the size of the set of prudent orders can be rather large in comparison to other common social ordering rules. This has been pointed out by Debord [3], who performed simulations to estimate the number of prudent orders for small profiles. The usefulness of prudent orders as an aggregation mechanism can thus be questioned.

However, from a progressive decision aid perspective, the use of prudent orders as possible compromise rankings does make sense. Sometimes, we do not necessarily aim at finding directly one compromise ranking, but we can also be interested in depicting a whole range of possible compromise rankings. On the one hand, we want to keep the set of possible compromise rankings as large as possible in order to leave enough room for a progressive refinement. On the other hand, we want to restrict the whole set of linear orders to those which can be reasonably considered as potential compromise solutions.

This paper is organized as follows. First, we are going to recall the concept of a prudent order. We will introduce the axioms used in our characterization results in section 3. In section 4, we will present results related to the set of prudent orders. Finally, we will end the paper with a conclusion.

2 Prudent orders

We denote by $\mathcal{O}$ the set of all the linear orders on a finite set of n alternatives $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$. Let $u = (O_1, O_2, \ldots, O_q) \in \mathcal{O}^q$ be a profile of q linear orders. We define majority margins $\forall i, j\ B_{ij} = |\{k : (a_i, a_j) \in O_k\}| - |\{k : (a_j, a_i) \in O_k\}|$. It is easy to see that $\forall i, j\ B_{ij} + B_{ji} = 0$. Furthermore, a linear extension O of a relation R is a linear order that contains R: $R \subseteq O$. We will denote by $\mathcal{E}(R)$ the set of all the linear extensions of relation R.

Let $\lambda \in \{-q, \ldots 0, \ldots, q\}$ and let us define the strict cut-relation $R_{>\lambda}$ as follows: $B_{ij} > \lambda \iff (a_i, a_j) \in R_{>\lambda}$. When λ is large, then $R_{>\lambda}$ is empty and consequently does not contain any cycle. By gradually decreasing the cut value, some ordered pairs will be added to the corresponding strict cut-relation. Let β be the smallest value such that the corresponding strict cut relation is acyclic:

$$\beta = \min\{\lambda \in \{-q, \ldots, O, \ldots, q\} : R_{>\lambda} \text{ is acyclic } \}$$

A prudent order $O_P \in \mathcal{O}$ is defined as a linear order that extends the relation $R_{>\beta}$.

$$R_{>\beta} \subseteq O_P \tag{1}$$

We will characterize a function $\mathcal{PO}$, called prudent order preference function, that associates to every profile u the set of all the linear extensions of $R_{>\beta}$:

$$\begin{aligned} \mathcal{PO}(u) &= \{O_P \in \mathcal{O} : R_{>\beta} \subseteq O_P\} \\ &= \mathcal{E}(R_{>\beta}) \end{aligned}$$

Since it is always possible to extend an acyclic relation into a linear order (see Szpilrajn[8]), the set of prudent orders will newer be empty. Arrow and Raynaud justified such a compromise ranking O_P to be prudent by the fact that ordered pairs that belong to the relation $R_{>\beta}$ are pairs with no contradiction and a high support. If these ordered pairs would not belong to the final compromise ranking, there would be a large and non-divided coalition against such a ranking.

It can be shown that equation 1 is equivalent to stating that O_P is a linear order that, in a way, minimizes the strongest opposition against this ranking, the value of this strongest opposition being exactly equal to β.

$$\max_{(a_i,a_j)\notin O_P} B_{ij} = \beta \leq \max_{(a_i,a_j)\notin O} B_{ij} \quad \forall O \in \mathcal{O} \tag{2}$$

Equivalently, a prudent order is a linear order that maximizes the weakest link. Since $\forall i, j,\ B_{ij} + B_{ji} = 0$, equation 3 can in fact be rewritten as follows:

$$\min_{(a_i,a_j)\in O_P} B_{ij} \geq \min_{(a_i,a_j)\in O} B_{ij} \quad \forall O \in \mathcal{O} \tag{3}$$

3 Axioms

In this section, we are going to introduce the axioms that we will need to characterize the prudent order preference function. More generally, a preference function f is a procedure that combines a profile u into a non-empty set of linear orders $f(u)$.

$$\begin{array}{rccl} f: & \mathcal{O}^q & \rightarrow & P(\mathcal{O}) \setminus \emptyset \\ & u & \mapsto & f(u) \end{array}$$

We will denote by $(a_i a_j x)$ a linear order where a_i is followed by a_j and then by the alternatives x, where x is an arbitrary permutation of the alternatives $\mathcal{A} \setminus \{a_i, a_j\}$. Furthermore, we denote by $-x$ the reverse permutation of x.

If $u = (O_1, O_2, \ldots, O_q)$ is a first profile and $v = (\hat{O}_1, \hat{O}_2, \ldots, \hat{O}_q)$ is a second profile, then we will denote by $u + v$ the profile $(O_1, O_2, \ldots, O_q, \hat{O}_1, \hat{O}_2, \ldots, \hat{O}_q)$.

Furthermore, the strict majority relation M is defined as follows:

$$\forall i, j: \qquad [(a_i, a_j) \in M \iff B_{ij} > 0]$$

In general, M contains cycles, which is commonly referred to as Condorcet's paradox. However, in case the strict majority relation is acyclic, then the first axiom says that this information must be contained in the set of solutions.

Axiom 1 *Condorcet Consistency (CC):*
If M is acyclic, then:

$$\forall i, j: \quad [(a_i, a_j) \in M \Rightarrow \forall O \in f(u): \ (a_i, a_j) \in O]$$

In other words, this means that, if M is acyclic, then any solution $O \in f(u)$ will be a linear extension of M and consequently $f(u) \subseteq \mathcal{E}(M)$. This axiom implies that, if M is a linear order, then this linear order is the unique solution of the preference function.

Lemma 1 *If f verifies Condorcet Consistency and if M is a linear order, then* $f(u) = \{M\}$.

A stronger version of axiom CC says that, if M is acyclic, then $f(u)$ corresponds exactly to all the linear extensions of this relation M.

Axiom 2 *Strong Condorcet Consistency:*
If M is acyclic, then:

$$f(u) = \mathcal{E}(M)$$

It is easy to see that Strong Condorcet Consistency implies Condorcet Consistency.

The next axioms says that if we add to the initial profile u a new profile v that is compatible in a certain sense with the strict majority relation M of profile u, then the set of compromise solutions either stays the same or shrinks.

Axiom 3 *Majority Profile Convergence (MPC):*
Let B be the majority margins of profile u. For every pair $\{i, j\}$, let us consider two linear orders $V^1_{\{i,j\}}$ and $V^2_{\{i,j\}}$ as follows

- *If $B_{ij} > 0$ and $B_{ji} < 0$, then:*

$$\begin{cases} V^1_{\{i,j\}} = (a_i a_j x) \\ V^2_{\{i,j\}} = (-x a_i a_j) \end{cases}$$

- *If $B_{ij} = B_{ji} = 0$, then :*

$$\begin{cases} V^1_{\{i,j\}} \in \mathcal{O} \\ V^2_{\{i,j\}} = -V^1_{\{i,j\}} \end{cases} \quad \textit{or} \quad \begin{cases} V^1_{\{i,j\}} = (a_j a_i x) \\ V^2_{\{i,j\}} = (-x a_j a_i) \end{cases} \quad \textit{or} \quad \begin{cases} V^1_{\{i,j\}} = (a_i a_j x) \\ V^2_{\{i,j\}} = (-x a_i a_j) \end{cases}$$

Let us consider a new profile v defined as follows:

$$v = (V^1_{\{1,2\}}, V^2_{\{1,2\}}, V^1_{\{1,3\}}, V^2_{\{1,3\}}, \ldots, V^1_{\{2,3\}}, V^2_{\{2,3\}}, \ldots, V^1_{\{n-1,n\}}, V^2_{\{n-1,n\}})$$

Then:

$$f(u + v) \subseteq f(u)$$

This axiom deserves some comments. Given a profile u, we are going to construct a new profile v. In fact, for every pair $\{i, j\}$, we are going to consider two linear orders depending on the values of the majority margins B_{ij} and B_{ji}:

- If $B_{ij} > 0$ and $B_{ji} < 0$, then there is a strict majority of rankings in the profile u that prefer a_i over a_j. Adding the two linear orders $V^1_{\{i,j\}} = (a_i a_j x)$ and $V^2_{\{i,j\}}(-x a_i a_j)$ clearly confirms this idea, since the two linear orders only improve the strength of the majority between a_i and a_j whereas the remaining pairs all cancel themselves.
- If $B_{ij} = B_{ji} = 0$, then there are as many rankings in the profile u that prefer a_i over a_j than there are rankings that prefer a_j over a_i. For such a pair, three possibilities can naturally be considered:

1. Adding the two opposite linear orders does not significantly change the situation since all the pairs will cancel themselves.
2. Adding the two linear orders $V^1_{\{i,j\}} = (a_j a_i x)$ and $V^2_{\{i,j\}} = (-x a_j a_i)$ will break the indifference between a_i and a_j by improving the situation of a_j with respect to a_i, whereas the remaining pairs all cancel themselves.
3. Adding the two linear orders $V^1_{\{i,j\}} = (a_i a_j x)$ and $V^2_{\{i,j\}} = (-x a_i a_j)$ will break the indifference between a_i and a_j by improving the situation of a_i with respect to a_j, whereas the remaining pairs all cancel themselves.

By breaking the indifference between a_i and a_j in a certain direction, or by leaving the indifference untouched, different profiles v can be constructed. This will eventually pull the set of compromise solutions $f(u+v)$ in possibly different directions. Whatever choice will be made, the profile v will always be compatible with the majority relation and the new set $f(u+v)$ will always be contained in the set $f(u)$.

Technically, the majority margins B' of profile $u+v$ can be obtained from the majority margins B of profile u as follows $\forall i, j$:

$$B_{ij} > 0 \;\Rightarrow\; B'_{ij} = B_{ij} + 2 \tag{4}$$

$$B_{ij} < 0 \;\Rightarrow\; B'_{ij} = B_{ij} - 2 \tag{5}$$

$$B_{ij} = B_{ji} = 0 \;\Rightarrow\; B'_{ij} \in \{-2, 0, 2\} \wedge B'_{ji} = -B'_{ij} \tag{6}$$

Proposition 1 *If f verifies Condorcet Consistency and Majority Profile Convergence, then f verifies Strong Condorcet Consistency.*

We will also use a slightly different version of the MPC axiom, namely Majority Profile Invariance:

Axiom 4 *Majority Profile Invariance (MPI):*
Let B be the majority margins of profile u. For every pair $\{i, j\}$, let us consider two linear orders $V^1_{\{i,j\}}$ and $V^2_{\{i,j\}}$ as follows

- *If $B_{ij} > 0$ and $B_{ji} < 0$, then:*

$$\begin{cases} V^1_{\{i,j\}} = (a_i a_j x) \\ V^2_{\{i,j\}} = (-x a_i a_j) \end{cases}$$

- *If* $B_{ij} = B_{ji} = 0$*, then :*

$$\left\{ \begin{array}{l} V^1_{\{i,j\}} \in \mathcal{O} \\ V^2_{\{i,j\}} = -V^1_{\{i,j\}} \end{array} \right. \quad \textit{or} \left\{ \begin{array}{l} V^1_{\{i,j\}} = (a_j a_i x) \\ V^2_{\{i,j\}} = (-x a_j a_i) \end{array} \right. \quad \textit{or} \left\{ \begin{array}{l} V^1_{\{i,j\}} = (a_i a_j x) \\ V^2_{\{i,j\}} = (-x a_i a_j) \end{array} \right.$$

Let us consider a new profile v *defined as follows:*

$$v = (V^1_{\{1,2\}}, V^2_{\{1,2\}}, V^1_{\{1,3\}}, V^2_{\{1,3\}}, \ldots, V^1_{\{2,3\}}, V^2_{\{2,3\}}, \ldots, V^1_{\{n-1,n\}}, V^2_{\{n-1,n\}})$$

If the the strict majority relation of the profile $u + v$ *is not acyclic, then:*

$$f(u + v) = f(u)$$

Axiom MPI is the same as axiom MPC, except that the inclusion is replaced by an equality, under the condition that the strict majority relation of profile $u+v$ is not acyclic. It means that if we add a majority consistent profile v to a profile u, and the new profile $u+v$ contains cycles (either existing cycles of profile u or new cycles created through the addition of profile v), then the set of compromise ranking must stay the same.

Let us note that removing the non-acyclicty condition of profile $u+v$ from this axiom will lead to a contradiction with axiom SCC. In fact, if the strict majority relation of profile $u+v$, denoted by M', is acyclic, then the strict majority relation of profile u, denoted by M, must also be acyclic, since $M \subseteq M'$. According to SCC, $f(u) = \mathcal{E}(M)$ and $f(u+v) = \mathcal{E}(M')$. If we suppose that $M \subset M'$, then it can happen that $f(u+v) \subset f(u)$.

The next axioms says that that the linear orders of a profile can be permuted without changing the result.

Axiom 5 *Anonymity (A):*
Let $u^1 = (O_1, O_2, \ldots, O_q)$ *and let* $u^2 = (O_{\sigma(1)}, O_{\sigma(2)}, \ldots, O_{\sigma(q)})$*, where* σ *is a permutation on* $\{1, 2, \ldots, q\}$*. Then* $f(u^1) = f(u^2)$*.*

Let u_E be a profile such that $\forall i, j \; B_{ij} = 0$. Adding such a profile to u_E to a given profile will not alter the result.

Axiom 6 *E-invariance (EI):*

$$f(u + u_E) = f(u)$$

In fact a preference function that verifies anonymity and E-invariance only depends on the preference margins. This result has been proved by Debord [4].

Lemma 2 *A preference function f verifies Anonymity and E-Invariance if and only if it is B-invariant: let B^1 be the preference margins of u^1 and let B^2 be the preference margins of u^2. If $\forall i, j$, $B^1_{ij} = B^2_{ij}$, then $f(u^1) = f(u^2)$.*

The next axiom finally says that if the size of the profile is odd and we create a new profile by taking twice the initial profile, then the set of compromise solutions may only increase.

Axiom 7 *Weak homogeneity (WH):*
If q is odd, then:

$$f(u) \subseteq f(u + u)$$

A stronger version of this axiom simply says that if we double an odd profile, then the result does not change at all.

Axiom 8 *Homogeneity (H):*
If q is odd, then:

$$f(u) = f(u + u)$$

4 Results

First, we are going to show that the prudent order preference function verifies the axioms introduced so far.

Proposition 2 *The prudent order preference function verifies Condorcet Consistency, Strong Condorcet Consistency, Majority Profile Convergence, Majority Profile Invariance, Anonymity, E-Invariance, Weak Homogeneity and Homogeneity.*

Let us now present our first result. In fact, we will show that if i) we want to use the axioms Condorcet Consistency, Majority Profile Convergence, Anonymity, E-Invariance and Weak Homogeneity and ii) we want to have a set of possible compromise solutions as large as possible, then we must use the prudent order preference function.

Theorem 1 *The prudent order preference function is the largest preference function (in the sense of the inclusion) that verifies Condorcet Consistency, Majority Profile Convergence, Anonymity, E-Invariance and Weak Homogeneity.*

Let us insist on the interpretation of keeping the set of compromise rankings as large as possible. In a progressive decision aid approach, it can be interesting to keep the set of compromise solutions as large as possible. Since it is useless to consider all the linear orders, the above mentioned axioms will restrict the set of possible compromise solutions to all the prudent orders.

Using similar axioms, the following theorem fully characterizes the prudent order preference function.

Theorem 2 *The prudent order preference function is the only preference function that verifies Strong Condorcet Consistency, Majority Profile Invariance, Anonymity, E-Invariance and Homogeneity.*

In comparison to theorem 1, we strengthened Condorcet Consistency by Strong Condorcet Consistency, and Weak Homogeneity by Homogeneity. Furthermore, Majority Profile Convergence was replaced by Majority Profile Invariance, although the latter does not imply the first.

5 Conclusion

In this work we presented a first axiomatic characterization of a preference function that associates to a profile of linear orders the whole set of prudent orders. Among the axioms that we introduced, the axioms of Majority Profile Convergence and Invariance are the most specific of the prudent approach.

References

[1] K. Arrow and H. Raynaud. *Social Choice and Multicriterion Decision-Making*. MIT Press, 1986.

[2] S. Barbera and M. Jackson. Maximin, leximin and the protective criterion: characterizations and comparisons. *Journal of Economic Theory*, 46:34–44, 1988.

[3] B. Debord. *Axiomatisation de procédures d'agrégation de préférences*. PhD thesis, Université Scientifique et Médicale de Grenoble, 1987.

[4] B. Debord. Caractérisation des matrices des préférences nettes et méthodes d'agrégation associées. *Mathématiques des Sciences Humaines*, 97:5–17, 1987.

[5] Z. Lansdowne. Ordinal ranking methods for multicriterion decision making. *Naval Research Logistics*, 43:613–627, 1996.

[6] Z. Lansdowne. Outranking methods for multicriterion decision-making : Arrow's and Raynaud's conjecture. *Social Choice and Welfare*, 14:125–128, 1997.

[7] M. Pirlot. A charcterization of min as a procedure for exploiting valued preference relations and related results. *Journal of Multi-Criteria Decision Analysis*, 4:37–56, 1995.

[8] E. Szpilrajn. Sur l'extension de l'ordre partiel. *Fund. Math.*, 16:386–389, 1930.

[9] T. Tideman. Independence of clones as criteria for voting rules. *Social Choice and Welfare*, 4:185–206, 1987.

[10] H.P. Young and A. Levenglick. A consistent extension of Condorcet's election principle. *SIAM Journal on Applied Mathematics*, 35:285–300, 1978.

[11] T. Zavist and T. Tideman. Complete independence of clones in the ranked pairs rule. *Social Choice and Welfare*, 6:167–173, 1989.