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Extended abstract

Bernard Monjardet * Vololonirina Raderanirina †

Arrow's theorem has fostered a lot of work on the problem to aggregate individual preferences into a collective preference or more generally into a collective choice function. Still more generally the "russian school" (see e.g. Aizerman and Aleskerov 1995, Aizerman and Malishevski 1981 or Aleskerov 1999) has considered the problem to aggregate individual choice functions into a collective choice function. In their work they emphasize the role of three axioms on choice functions, namely the heredity axiom (H), the concordance axiom (C) and the Outcast axiom (O). Indeed, the combination of these axioms gives significant classes of choice functions. For instance, a choice function is "classically" rational (respectively rationalizable by a partial order, or path-independent) if and only if it satisfies axioms (H) and (C) (respectively axioms (H),(C) and (O), or axioms (H) and (O)). Moreover, one can describe rational -in an extended sense- choice mechanisms, inducing choice functions statisfying each of these three axioms.

We present the order structure of the sets of choice functions satisfying each of these axioms. Indeed, these sets are always partially ordered by the point-wise order between functions. Moreover, we shall see that they are always lattices. The lattice of choice functions satisfying axiom (H) has a nice structure since it is a distributive lattice with intersection and union as meet and join operations. The lattice of choice functions satisying axiom (C) is atomistic and the study of the dependence relation δ of this lattice allows us to prove that it is lower bounded. Since it is also atomistic, it has many other properties;

*CERMSEM, Université Paris 1, 106–112 boulevard de l'Hopital, 75647 Paris Cedex 13, France, email : monjarde@univ-paris1.fr

†email : raderani@aol.com

for instance, it is lower locally distributive and then lower semi-modular and ranked. The much more complex lattice of choice functions satisying axiom (O) is coatomistic but it is not even ranked.
The interest to specify such structures is in particular linked to consensus problems. Consider for instance individuals using choice mechanisms which induce choice functions satisfying axiom (O). Assume that one takes as consensus the unanimity rule (x is chosen in a set if it is chosen by all the individuals). This amounts to considering the choice function as the intersection of the individual choice functions. But this choice function does not necessarily satisfy axiom (O). In fact, according to a result in Aizerman and Aleskerov 1995, it can be a completely arbitrary choice function. Nevertheless, since the set of choice functions is a lattice, one can use the unanimity rule in this lattice (the intersection operation is replaced by the meet operation) and so to get a collective choice function having the same level of rationality as that the individual choice functions have.
Moreover, there exist latticial theories of consensus (see e.g. Barthélemy and Janowitz 1991, Leclerc 1990, Leclerc and Monjardet 1995, McMorris and Powers 1995, or Monjardet 1990) which can be applied whenever one aggregates elements of a lattice L. The consensus problem on a lattice L consists of summarizing arbitrary profiles Π (a n-profile or simply profile is a n-tuple $\Pi = (x_1, ..., x_i, ...x_n)$ of elements) of L by one or several elements of L.
In fact, there are two main kinds of approaches and results. The first one is the classical axiomatic approach. The consensus functions (i.e. the functions $L^n \mapsto L$) must satisfy some "reasonable" axioms and the theory determines the functions satisfying such conditions. The results depend not only on the axioms considered but also on the properties of the lattice and in particular on the properties of dependence relations defined on the sets of its irreducible elements. The second one is the metric approach (a generalization of Kemeny's rule). The consensus element is taken as a "closest" element (relatively to a metric defined on the lattice) to the elements to aggregate. This approach has been particularly developed in the case where the consensus element is a "median" element, a case where one can characterize axiomatically the corresponding "median procedure".
Since our sets of choice functions are lattices one can use the results obtained on the structure of these lattices and those of the latticial theories of consensus to get results on the aggregation problem for such choice functions. We present a sample of such results. Observe that the original part of this process consists to determine the properties of our lattices of choice functions, since once time these properties are known one has just to apply the relevant general results of the latticial theories. According to the structure of the corresponding lattice the same type of axioms asked for the consensus functions can determine a broad class of such functions, those associated to the families of sets called simple games (or federations) or only a restricted class like the class

of "oligarchic" or "co-oligarchic" consensus functions. Concerning the metric approach, the median choice function(s) of a profile can be easy or difficult to get according to the structure of the corresponding lattice.
Our results can be compared with those obtained by the "russian school". In fact there are some identical results, some results leading to the same consensus functions but with different axioms and some different results.

Main references

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