On the consensus of closure systems

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1. Introduction

We consider the problem of aggregating a profile (k-tuple) $\mathcal{F}^* = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k)$ of closure systems on a given S into a consensus closure systems $\mathcal{F} = c(\mathcal{F}^*)$. The aim is, for instance, to find a structure on a set S described by variables of different types. Structural information (order, tree structure) provided by these variables may be totally or partially retained by a derived closure system (see examples in Section 2). Moreover, several consensus problems already studied in the literature are particular cases of the consensus of closure systems. A basic example is provided by hierarchical classification, where many works have followed those of Adams [Ada72] and Margush and McMorris [MM81] (see the survey [Lec98]).

Closure systems and their uses are presented in Section 2.1. Several equivalent structures are recalled in Section 2.2. Section 2.3 give elements about the involved lattice structures. Section 3 presents results provided by the particularization of general results on the consensus problem in lattices. An original approach based on implications is initiated in Section 4.

2. Closure systems

Type of data	***S* endowed with a**	**Subsets of *S***	**Type of closure system**
Numerical, ordinal variable	Weak order W	Down-sets of W	Nested
Transitive preference relation	Preorder P	Down-sets of P	Distributive
Nominal variable	Partition π	S, $\emptyset$, and classes of π	Tree of subsets of length 2
Taxonomy	Hierarchy $\mathcal{H}$	$\emptyset$ and classes of $\mathcal{H}$	Tree of subsets

Table 1. Types of data and related closure systems

2.1. Definitions and uses

A *closure system* (abbreviated as CS) on a finite given set S is a set $\mathcal{F} \subseteq \mathcal{P}(S)$ of subsets of S satisfying the following two conditions:

(C1) $S \in \mathcal{F}$;

(C2) $C, C' \in \mathcal{F} \Rightarrow C \cap C' \in \mathcal{F}$.

When considering classical types of preference or classification data describing a given set S of objects, one observes that, frequently, they naturally correspond to closure systems. A list, of course not limitative, of such situations is given in Table 1. A CS $\mathcal{F}$ is *nested* if it is linearly ordered by set inclusion: for all $F, F' \in \mathcal{F} \Rightarrow F \cap F' \in \{F, F'\}$; it is a *tree of subsets* if, for all $F, F' \in \mathcal{F} \Rightarrow F \cap F' \in \{\varnothing, F, F'\}$; it is *distributive* if, for all $F, F' \in \mathcal{F} \Rightarrow F \cup F' \in \mathcal{F}$ and $F \cap F' \in \mathcal{F}$.

2.2. Equivalent structures

Three notions are defined in this section. Together with CS's, they turn to be equivalent to each other. A *closure operator* φ is a mapping onto $\mathcal{P}(S)$ satisfying the properties of isotony (for all $A, B \subseteq S$, $A \subseteq B$ implies $\varphi(A) \subseteq \varphi(B)$), extensivity ((for all $A \subseteq S$, $A \subseteq \varphi(A)$) and idempotence (for all $A \subseteq S$, $\varphi(\varphi(A)) = \varphi(A)$. Then, the elements of the image $\mathcal{F}_\varphi = \varphi(\mathcal{P}(S))$ of $\mathcal{P}(S)$ by φ are the *closed* (by φ) sets, and $\mathcal{F}_\varphi$ is a closure system on S. Conversely, the closure operator $\varphi_\mathcal{F}$ on $\mathcal{P}(S)$ is given by $\varphi_\mathcal{F}(A) = \cap\{F \in \mathcal{F}: A \subseteq F\}$.

A *full implicational system* (FIS), denoted hereafter by I, $\rightarrow_I$ or simply $\rightarrow$, is a binary relation on $\mathcal{P}(S)$ satisfying the following conditions:

(I1) $B \subseteq A$ implies $A \rightarrow B$;

(I2) for any $A, B, C \subseteq S$, $A \rightarrow B$ and $B \rightarrow C$ imply $A \rightarrow B$;

(I3) for any $A, B, C, D \subseteq S$, $A \rightarrow B$ and $C \rightarrow D$ imply $A \cup C \rightarrow B \cup D$.

An *overhanging order* (2O) on S is also a binary relation Œ on $\mathcal{P}(S)$, now satisfying:

(O1) $A \text{ Œ } B \Rightarrow A \subset B$;

(O2) $A \subset B \subset C \Rightarrow [A \text{ Œ } C \Leftrightarrow A \text{ Œ } B$ ou $B \text{ Œ } C]$;

(O3) $A \text{ Œ } A \cup B \Rightarrow A \cap B \text{ Œ } B$.

It follows from (01) and (02) that the relation Œ is a strict order on $\mathcal{P}(S)$. The sets of, respectively, closure systems, closure operators, full implicational systems and overhanging orders on S are denoted, respectively, as **M**, **C**, **I** and **O**. They correspond to each other. The equivalence between closure systems and operators has been recalled above. For a closure operator φ and its associated FIS $\rightarrow$ and 2O Œ, the first of the equivalences below is due to Armstrong [Arm74], and the second is given in [DL04] :

$$A \rightarrow B \Leftrightarrow B \subseteq \varphi(A)$$
$$A \text{ Œ } B \Leftrightarrow A \subset B \text{ and } \varphi(A) \subset \varphi(B)$$

There is an important literature, with meaningful results, on implications, due to their importance in domains such as logic, lattice theory, relational databases, knowledge representation, or latticial data analysis (see the survey [CM03]). Overhanging orders take their origin in Adams ([Ada86]), where, named *nestings*, they were characterized in the particular case of hierarchies. Their generalization to all closure systems [DL04] make them a further tool for the study of closure systems.

2.3. Lattices

The results of this section may be found in [CM03] and, for overhangings, in [DL03] and [DL04]. First, each of the sets **M**, **C**, **I** and **O** is naturally ordered: **M**, **I** and **O** by inclusion, and **C** by the pointwise order: for $\varphi, \varphi' \in$ **C**, $\varphi \leq \varphi'$ means that $\varphi(A) \subseteq \varphi'(A)$ for any $A \subseteq S$. These orders are isomorphic or dually isomorphic:

$$\mathcal{F} \subseteq \mathcal{F}' \Leftrightarrow \varphi' \leq \varphi \Leftrightarrow I' \subseteq I \Leftrightarrow \text{Œ} \subseteq \text{Œ}',$$

where φ, I and Œ (resp. φ', I' and Œ') are the closure operator, full implication system and overhanging relation associated to $\mathcal{F}$ (resp. to $\mathcal{F}'$).

The sets **M** and **I** preserve set intersection, while **O** preserves set union. The maximum elements of, respectively, **M**, **I** and **O** are, respectively, $\mathcal{P}(S)$, $(\mathcal{P}(S))^2 = \{(A, B): A, B \subseteq S\}$ and the set $\{(A, B): A, B \subseteq S, A \subset B\}$; their minimums are, respectively, $\{S\}$, $\{(A, B): A, B \subseteq S, B \subseteq A\}$ and the empty relation on $\mathcal{P}(S)$.

From these observations, **M** and **I** are closure systems, respectively on $\mathcal{P}(S)$ and $(\mathcal{P}(S))^2$. The closure operator associated to **M** is denoted as φ. It is well-known that, with the inclusion order, any closure system $\mathcal{F}$ on S is a lattice $(\mathcal{F}, \vee, \wedge)$ with the meet $F \cap F'$ and the join $F \vee F' = \varphi(F \cup F')$. If $F \subset F'$, F' *covers* F (denoted as $F \prec F'$) if $F \subseteq G \subseteq F'$ implies $G = F$ or $G = F'$.
An element J of $\mathcal{F}$ is *join irreducible* if $\mathcal{G} \subseteq \mathcal{F}$ and $J = \vee\mathcal{G}$ imply $J \in \mathcal{G}$; an equivalent property is that J covers exactly one element, denoted J^-, of $\mathcal{F}$. The set of all the join irreducibles is denoted by $\mathcal{J}$. Setting $\mathcal{J}(F) = \{J \in \mathcal{J}: J \subseteq F\}$ for any $F \in \mathcal{F}$, one has $F = \vee\mathcal{J}(F)$ for all $F \in \mathcal{F}$. A join irreducible is an *atom* if it covers the minimum element of $\mathcal{F}$, and the lattice $\mathcal{F}$ is *atomistic* if all its join irreducibles are atoms.
Similarly, an element M of $\mathcal{F}$ is *meet irreducible* if $\mathcal{G} \subseteq \mathcal{F}$ and $M = \wedge\mathcal{G}$ imply $M \in \mathcal{G}$; equivalently, M is covered by exactly one element M^+ of $\mathcal{F}$. For any $F \in \mathcal{F}$, we have $F = \wedge\mathcal{M}(F)$, where $\mathcal{M}$ is the set of all the meet irreducibles of $\mathcal{F}$ and $\mathcal{M}(F) = \{M \in \mathcal{M}: F \subseteq M\}$.
The lattice $\mathcal{F}$ is *lower semimodular* if, for every $F, F' \in \mathcal{F}$, $F \prec F \vee F'$ and $F' \prec F \vee F'$ imply $F \wedge F' \prec F$ and $F \wedge F' \prec F$. The lattice $\mathcal{F}$ is *ranked* if it admits a numerical *rank function* r such that $F \prec F'$ implies $r(F') = r(F) + 1$. Lower semimodular lattices are ranked.
The lattice $\mathcal{F}$ is a *convex geometry* if it satisfies one of the following equivalent conditions (among many other characterizations [Mon90b]:
(CG1) For any $F \in \mathcal{F}$, there is a unique minimal subset $\mathcal{R}$ of $\mathcal{J}$ such that $F = \vee\mathcal{R}$;
(CG2) $\mathcal{F}$ is ranked with rank function $r(F) = |\mathcal{J}(F)|$;
(CG3) $\mathcal{F}$ is lower semimodular with a rank function as in (CG2) above;
Since it is a closure system on $\mathcal{P}(S)$, the ordered set **M** is itself a lattice. This lattice is an atomistic convex geometry. The set is of the atoms of **M** is $\mathbf{J} = \{\{F, S\}: F \subset S\}$ and, for $\mathcal{F} \in$ **M**, we have $\mathbf{J}(\mathcal{F}) = \{\{A, S\}: A \in \mathcal{F}\}$ and $|\mathbf{J}(\mathcal{F})| = |\mathcal{F}|-1$.

3. Lattice consensus for closure systems

In this section, we consider the main consequences of the lattice structure of **M** for the consensus problem on closure systems, that is aggregation of a profile $\mathcal{F}^* = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k)$ (of length k) of CSs into a CS $\mathcal{F} = c(\mathcal{F}^*)$. General results on the consensus problem in lattices may be found, among others, in [BM90b], [BJ91] and [Lec94]. Concerning closure systems, the results obtained in an axiomatic approach by Raderanirina [Rad01] (see also [MR04] about the related case of choice functions) are described in another contribution and not recalled here.

3.1. A property of quota rules

A *federation* on K is a family $\mathcal{K}$ of subsets of K satisfying the monotonicity property: $[L \in \mathcal{K}, L' \supseteq L] \Rightarrow [L' \in \mathcal{K}]$. Then, the federation consensus function $c_{\mathcal{K}}$ on **M** is associated to $\mathcal{K}$ by

$c_{\mathcal{K}}(\mathcal{F}^*) = \bigvee_{L\in\mathcal{K}} (\bigwedge_{i\in L} \mathcal{F}_i)$. Such consensus function includes the *quota rules*, where $\mathcal{K} = \{L\subseteq K\colon |L| \geq q\}$, for a fixed number q, $0 \leq q \leq k$. The quota rule c_q is equivalently defined as:

$$c_q(\mathcal{F}^*) = \varphi(\mathcal{A}_q),$$

where $\mathcal{A}_q = \{A \subseteq S\colon |\{i \in K\colon A \in \mathcal{F}_i\}| \geq q\}$, the set of all subsets appearing in at least q elements of $\mathcal{F}^*$, and φ is the operator mentioned in Section 2.3: $\varphi(\mathcal{A}_q)$ is the smallest CS including $\mathcal{A}_q$. For $q = k/2$, $c_q(\mathcal{F}^*) = m(\mathcal{F}^*)$ is the so-called (weak) *majority rule* and, for $q = k$, it is the *unanimity rule* $u(\mathcal{F}^*)$.

Quota rules have good properties in any lattice structure, for instance :
Unanimity : for any $F \in \mathbf{M}$, $c_q(F, F,\ldots, F) = F$;
Isotony : for any $\mathcal{F}^* = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k)$,$\mathcal{F}'^* = (\mathcal{F}'_1, \mathcal{F}'_2, \ldots, \mathcal{F}'_k)$, profiles of $\mathbf{M}$, $\mathcal{F}_i \subseteq \mathcal{F}'_i$: for all $i = 1, \ldots, k$ implies $c_q(\mathcal{F}^*) \subseteq c_q(\mathcal{F}'^*)$.

The next property of consistency type (see Section 3.2) is not general (for instance it is not true in partition lattices [BL95]) but holds in the so-called LLD lattices [Lec03], which include convex geometries; it implies unanimity. In what follows, the profile $\mathcal{F}^*\mathcal{F}'^*$ is just the concatenation of profiles $\mathcal{F}^*$ and $\mathcal{F}'^*$, which are not required to have the same length.

Proposition 3.1. *Let $\mathcal{F}^*$ and $\mathcal{F}'^*$ be two profiles of* $\mathbf{M}$. *If $c_q(\mathcal{F}^*) = c_q(\mathcal{F}'^*) = \mathcal{F}$, then $c_q(\mathcal{F}^*\mathcal{F}'^*) = \mathcal{F}$.*

3.2. Bounds on medians

For a metric approach of the consensus in $\mathbf{M}$, we first have to define metrics. For that, we just follow [BM81] and [Lec94]. A real function v on $\mathbf{M}$ such as $\mathcal{F} \subset \mathcal{F}'$ implies $v(\mathcal{F}) < v(\mathcal{F}')$ is a *lower valuation* if it satisfies one the following two equivalent properties:
(LV1) For all $s, t \in L$ such that $s \vee t$ exists, $v(s) + v(t) \leq v(s \vee t) + v(s\wedge t)$;
(LV2) The real function d_v defined on $\mathbf{M}^2$ by the following formula is a metric on $\mathbf{M}$:

$$d_v(\mathcal{F}, \mathcal{F}') = v(\mathcal{F}) + v(\mathcal{F}') - 2v(\mathcal{F}\wedge \mathcal{F}').$$

A characteristic property of lower semimodular semilattices is that their rank functions are lower valuations. So, taking property (CG2) into account, the *rank metric* is obtained taking $v(\mathcal{F}) = |\mathcal{F}|$ and, so, $d_v(\mathcal{F}, \mathcal{F}') = \partial(\mathcal{F}, \mathcal{F}') = |\mathcal{F}\Delta\mathcal{F}'|$, where Δ is the symmetric difference on subsets. The equality between the rank and the symmetric difference metric is characteristic of convex geometries or of close structures [Lec03] and is a reason to focuse on that metric.

Given the metric ∂, the *median* consensus procedure consists of searching for the *medians* of the profile $\mathcal{F}^*$, that is the elements $\mathcal{F}^\mu$ of $\mathbf{M}$ minimizing the *remoteness* $\rho(\mathcal{F}^\mu, \mathcal{F}^*) = \sum_{1\leq i\leq k}\partial(\mathcal{F}^\mu, \mathcal{F}_i)$ (see [BM81]). If $\mu(\mathcal{F}^*)$ is the set of all the medians of the profile $\mathcal{F}^*$, the median procedure has the following *consistency* property (YL78):
Let $\mathcal{F}^*$ and $\mathcal{F}'^*$ be two profiles of $\mathbf{M}$. If $\mu(\mathcal{F}^*)\cap\mu(\mathcal{F}'^*) \neq \emptyset$, then $\mu(\mathcal{F}^*\mathcal{F}'^*) = \mu(\mathcal{F}^*)\cap\mu(\mathcal{F}'^*)$.

Set $\mathbf{J}_m = \{\{A, S\}\colon A \in \mathcal{A}_{k/2}\}$ (the set of majority atoms). From results in [Lec94], every median $\mathcal{M}$ of $\mathcal{F}^*$ is the join of some subset of $\mathbf{J}_m$. It then follows that every median CS is included into the majority rule one:

Theorem 3.2. *For any profile $\mathcal{F}^*$ and for any median $\mathcal{F}^\mu$ of $\mathcal{F}^*$, the inclusion $\mathcal{F}^\mu \subseteq m(\mathcal{F}^*)$ holds.*

4. The fitting of overhangings

The results of Section 3 (and those of the same type mentioned there) only take into account the presence or absence of the same closed set in enough (oligarchies, majorities) or all (unanimity) elements of the profile. In the case of hierarchies, it was observed that this strong limitation may prevent us to recognize actual common features. This criticism remains valid for general closure systems. Moreover, consensus systems based on classes may frequently be trivial. For instance, if there does not exist any majority non-trivial subset, then, the majority rule (and unique median) is the trivial closure system reduced to $\{S\}$. Adams [Ada86] presented a consensus method able to retain common features even in such cases. It is based on overhanging orders (and, then, on implications). Here we initiate the same approach for general closure systems.

We state here a very general uniqueness result. Let □ be a binary relation on $\mathcal{P}(S)$, with the only assumption that (A, B) □ □ implies A □ B. Consider the following two properties for a closure system $\mathcal{F}$, with associated closure operator □ and overhanging relation Œ:

(A□1) □ □ Œ; (preservation of □)

(A□2) for all M □ $\mathcal{M}_{\mathcal{F}}$, (M, M^{+}) □ □. (qualified overhangings)

Theorem 4.1. *If both $\mathcal{F}$ and $\mathcal{F}'$ satisfy Conditions* (A□1) *and* (A□2)*, then $\mathcal{F} = \mathcal{F}'$.*

The following question then arises: given a binary relation □ on $\mathcal{P}(S)$ (implying strict inclusion), does it exist an overhanging relation Œ satisfying conditions (A□1) and (A□2). Adams provides a positive answer in the case of a profile of hierarchies, and with □ = □ $_{i□K}$□Œ$_i$ where Œ$_1$, Œ$_2$, …, Œ$_k$ are the overhanging orders associated to the elements of the profile. In the general case, one can consider any convenient combination of Œ$_1$, Œ$_2$, …, Œ$_k$. For instances, □ = □ $_{i□K}$ Œ$_i$ corresponds to a kind of unanimity rule on overhangings, and □ = □ $_{L□K,\, 2|L|>k}$ □ $_{i□L}$ Œ$_i$ to a majority rule.

6. Conclusion

The last section provides a framework for the consensus of closure systems. One of the main questions is to recognize the binary relations □ on $\mathcal{P}(S)$ for which an overhanging order Œ satisfying (A□1) and (A□2) exists. For instance, setting □ = □ $_{L□K,\, 2|L|>k}$ □ $_{i□L}$ Œ$_i$ accounts for the fact that a CS appears several times in a profile, contrary to intersection rules. Algorithmic issues are very important, since overhanging relations are very big objects.

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