Differential approximation of MIN SAT, MAX SAT and related problems (Extended abstract)

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28th June 2004

Abstract

We present differential approximation results (both positive and negative) for optimal satisfiability, optimal constraint satisfaction, and some of the most popular restrictive versions of them. As an important corollary, we exhibit an interesting structural difference between the landscapes of approximability classes in standard and differential paradigms.

1 Introduction and preliminaries

In this paper we deal with the approximation of some of the most famous and classical problems in the domain of the polynomial time approximation theory, the MIN and MAX SAT as well as the MIN and MAX DNF and some of their restricted versions, namely MAX and MIN k and EkSAT and MAX and MIN k and EkDNF. We study their approximability using the so-called *differential approximation ratio* which, informally, for an instance x of a combinatorial optimization problem Π, *measures the relative position of the value of an approximated solution in the interval between the worst-value of x, i.e., the value of a worst feasible solution of x, and optimal-value of x, i.e., the value of a best solution of x.*

Given a set of clauses (i.e., disjunctions) $C_1, \ldots, C_m$ on n variables $x_1, \ldots, x_n$, MAX SAT (resp., MIN SAT) consists of determining a truth assignment to the variables that maximizes (minimizes) the number of clauses satisfied. On the other hand, given a set of cubes (i.e., conjunctions) $C_1, \ldots, C_m$ on n variables $x_1, \ldots, x_n$, MAX DNF (resp., MIN DNF) consists of determining a truth assignment to the variables that maximizes (minimizes) the number of conjunctions satisfied. For an integer $k \geqslant 2$, MAX kSAT, MAX kDNF, MIN kSAT, MIN kDNF (resp., MAX EkSAT, MAX EkDNF, MIN EkSAT, MIN EkDNF) are the versions of MAX SAT, MAX DNF, MIN SAT, MIN DNF where each clause or conjunction has size at most (resp., exactly) k. Finally, let us quote two particular weighted satisfiability versions, namely, MAX WSAT and MIN WSAT. In the former, given a set of clauses $C_1, \ldots, C_m$ on n variables $x_1, \ldots, x_n$, with non-negative integer weights $w(x)$ on any variable x, we wish to compute a truth assignment to the variables that both satisfies all the clauses and maximizes the sum of the weights of the variables set to 1. We consider that the assignment setting all the variables to 0 (even if it does not satisfy all the clauses) is feasible and represents the worst-value solution for the problem. The latter problem is similar to the former one, up to the fact that we wish to minimize the sum of the weights of the variables set to 1 and that feasible is now considered the assignment setting all the variables to 1.

A problem Π in **NPO** is a quadruple $(\mathcal{I}_\Pi, \mathrm{Sol}_\Pi, m_\Pi, \mathrm{opt}(\Pi))$ where: $\mathcal{I}_\Pi$ is the set of instances (and can be recognized in polynomial time); given $x \in \mathcal{I}_\Pi$, $\mathrm{Sol}_\Pi(x)$ is the set of feasible solutions of x; the size of a feasible solution of x is polynomial in the size $|x|$ of the instance; moreover, one can determine in polynomial time if a solution is feasible or not; given $x \in \mathcal{I}_\Pi$ and $y \in \mathrm{Sol}_\Pi(x)$, $m_\Pi(x, y)$ denotes the value of the solution y of the instance x; m_Π is called the objective function, and is computable in polynomial time; we suppose here that $m_\Pi(x, y) \in \mathbb{N}$; $\mathrm{opt}(\Pi) \in \{\min, \max\}$.

Given an instance x of an optimization problem Π and a feasible solution $y \in \mathrm{Sol}_\Pi(x)$, we denote by $\mathrm{opt}_\Pi(x)$ the value of an optimal solution of x, and by $\omega_\Pi(x)$ the value of a worst solution of x. The *standard approximation ratio* of y is defined as $r_\Pi(x, y) = m_\Pi(x, y)/\mathrm{opt}_\Pi(x)$, while the *differential approximation ratio* of y is defined as $\delta_\Pi(x, y) = |m_\Pi(x, y) - \omega_\Pi(x)|/|\mathrm{opt}_\Pi(x) - \omega_\Pi(x)|$.

For a function f of $|x|$, an algorithm is a *standard f-approximation algorithm* (resp., *differential f-approximation algorithm*) for a problem Π if, for any instance x of Π, it returns a solution y such that $r(x, y) \leqslant f(|x|)$, if $\mathrm{opt}(\Pi) = \min$, or $r(x, y) \geqslant f(|x|)$, if $\mathrm{opt}(\Pi) = \max$ (resp., $\delta(x, y) \geqslant f(|x|)$).

With respect to the best approximation ratios known for them, **NPO** problems can be classified into approximability classes. One of the most notorious such classes is the class **APX** (or **DAPX** when dealing with the differential paradigm) including the problems for which there exists a polynomial algorithm achieving standard or differential approximation ratio $f(|x|)$ where function f is constant (it does not depend on any parameter of the instance).

We now define a kind of reduction, called *affine reduction* and denoted by AF, which, as we will see, is very natural in the differential approximation paradigm.

Definition 1. Let Π and Π' be two **NPO** problems. Then, Π AF-reduces to Π' ($\Pi \leq_{\mathsf{AF}} \Pi'$), if there exist two functions f and g such that:

1. for any $x \in \mathcal{I}_\Pi$, $f(x) \in \mathcal{I}_{\Pi}$;
2. for any $y \in \mathrm{Sol}_{\Pi'}(x)$, $g(x, y) \in \mathrm{Sol}_\Pi(x)$; moreover, $\mathrm{Sol}_\Pi(x) = g(x, \mathrm{Sol}_{\Pi'}(f(x)))$;
3. for any $x \in \mathcal{I}_\Pi$, there exist $K \in \mathbb{R}$ and $k \in \mathbb{R}^\star$ ($k > 0$ if $\mathrm{opt}(\Pi) = \mathrm{opt}(\Pi')$, $k < 0$, otherwise) such that, for any $y \in \mathrm{Sol}_{\Pi'}(f(x))$, $m_{\Pi'}(f(x), y) = k m_\Pi(x, g(x, y)) + K$.

If $\Pi \leq_{\mathsf{AF}} \Pi'$ and $\Pi' \leq_{\mathsf{AF}} \Pi$, then Π and Π' are called *affine equivalent*. This equivalence will be denoted by $\Pi \equiv_{\mathsf{AF}} \Pi'$. ▮

It is easy to see that differential approximation ratio is stable under affine reduction. Formally, *if, for $\Pi, \Pi' \in$ **NPO**, $\mathsf{R} = (f, g)$ is an AF-reduction from Π to Π', then for any $x \in \mathcal{I}_\Pi$ and for any $y \in \mathrm{Sol}_{\Pi'}(f(x))$, $\delta_\Pi(x, g(x, y)) = \delta_{\Pi'}(f(x), y)$.* Indeed, by Condition 2 of Definition 1, worst and optimal solutions in x and $f(x)$ coincide. Since the value of any feasible solution of Π' is an affine transformation of the same solution seen as a solution of Π, the differential ratios for y and $g(x, y)$ coincide also. Hence, the following holds.

Proposition 1. *If $\Pi \equiv_{\mathsf{AF}} \Pi'$, then, for any constant r, any r-differential approximation algorithm for one of them is an r-differential approximation algorithm for the other one.*

Optimization satisfiability problems as MIN SAT and MAX SAT are of great interest from both theoretical and practical points of view. On the one hand, the satisfiability problem (SAT) is the first complete problem for **NP** and MAX SAT, MIN SAT have generalizations or restrictions that are the first problems proved complete for numerous approximation classes under various approximability preserving reductions ([1, 12]). For instance, MAX 3SAT is **APX**-complete under the AP-reduction and **Max-SNP**-complete under the L-reduction ([11]), MAX WSAT and

MIN WSAT are **NPO**-complete under the AP-reduction ([4]), etc. In general, many optimal satisfiability problems have for the polynomial approximation theory the same status as SAT for **NP**-completeness theory. On the other hand, many problems in mathematical logic and in artificial intelligence can be expressed in terms of versions of SAT; constraints satisfaction is one such version. Also problems in database integrity constraints, query optimization, or in knowledge bases can be seen as optimization satisfiability problems. Finally, some approaches to inductive inference can be modeled as MAX SAT problems ([8, 9]).

Let us note that differential approximability of the problems dealt here, has already been studied in [2]. There, among other results, it was shown that MAX SAT and MIN DNF, as well as MIN SAT and MAX DNF are equivalent for the differential approximation, that all these problems are not solvable by polynomial time differential approximation schemata, unless $\mathbf{P} = \mathbf{NP}$, and, finally, that MIN SAT cannot be approximately solved within differential approximation ratio $1/m^{1-\epsilon}$, for any $\epsilon > 0$ (where m is the number of the clauses in its instance), unless $\mathbf{NP} =$ **co-RP**. Finally, let us mention here that both MAX WSAT and MIN WSAT belong to **0-DAPX**, the class of the problems for which no algorithm can guarantee differential approximation ratio strictly greater than 0, unless $\mathbf{P} = \mathbf{NP}$ ([10]). This class has been also introduced in [2].

	Approximation ratios	Inapproximability bounds
MAX SAT	$4.34/(m+4.34)$	$\notin$ **DAPX**
MAX E2SAT	$17.9/(m+19.3)$	$11/12$
MAX 3SAT	$4.57/(m+5.73)$	$1/2$
MAX E3SAT	$8/(m+8)$	$1/2$
MAX EkSAT	$2^k/(m+2^k)$	$1/p$, p the largest prime such that $3(p-1) \leqslant k$
MIN SAT	$2/(m+2)$	
MIN (E)kSAT	$2^k/((2^{k-1}-1)m+2^k)$	$1/p$, p the largest prime such that $3(p-1) \leqslant k$
MIN 2SAT	$4/(m+4)$	$11/12$

Table 1: Summary of the main results of the paper.

In this paper, we further study differential approximability of MAX SAT, MIN SAT, MIN DNF and MAX DNF, and give approximation results and inapproximability bounds for several versions of these problems. A summary of the main results obtained is presented in Table 1. As one can see from the second column of the first line of this table, MAX SAT is not approximable within a constant approximation ratio, unless $\mathbf{P} = \mathbf{NP}$. This result is very interesting since it indicates that **Max-NP** ([11]) is not included in **DAPX**. This is an important difference with the standard approximability classes landscape where **Max-NP** $\subset$ **APX**. Another assessment with respect to our results is that the gap between lower and upper approximation bounds for the problems dealt is still large. However, this paper undertakes a systematic study of satisfiability problems in the differential paradigm, it extends the results of [2] and shows that none of the most classical satisfiability problems is in **0-DAPX**. This approximability class has been introduced in [2] and represents the worst possible configuration for differential approximation since it includes the problems for which no polynomial time approximation algorithm can guarantee differential ratio greater than 0. Inclusion of the problems dealt here in **0-DAPX** or not, was a major question we handled since [2].

Results here are given without detailed proofs that can be found in [5] and in appendix.

2 Affine reductions between optimal satisfiability problems

Let us first note that there does not exist general technique in order to transfer approximation results from differential (resp., standard) paradigm to standard (resp., differential) one, except for the case of maximization problems and for transfers between differential and standard paradigms. Proposition 2 just below deals with this last case.

Proposition 2. *If a maximization problem* Π *can be solved within differential approximation ratio* δ*, then it can be solved within standard approximation ratio* δ*, also.*

Corollary 1. *Any standard inapproximability bound for a maximization problem* Π *is also a differential inapproximability bound for* Π.

We give in this section some affine reductions and equivalences between the problems dealt in the paper. These results will allow us to focus ourselves only in the study of MAX SAT, MIN SAT and their restrictions without studying explicitly MAX and MIN DNF. We first recall a result already proved in [2].

Proposition 3. *([2])* MAX SAT $\equiv_{\mathsf{AF}}$ MIN DNF *and* MIN SAT $\equiv_{\mathsf{AF}}$ MAX DNF.

The following proposition shows that one can affinely pass from MAX EkSAT to MAX E$(k+1)$SAT. This, allows us to transfer inapproximability bounds from MAX E3SAT to MAX EkSAT, for any $k \geqslant 4$.

Proposition 4. MAX EkSAT $\leq_{\mathsf{AF}}$ MAX E$(k+1)$SAT.

Sketch of proof. For an instance φ of MAX EkSAT on n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$, consider a new variable y and build formula φ', instance of MAX E$(k+1)$SAT as follows: for any clause $C_i = (\ell_{i_1}, \ldots, \ell_{i_k})$ of φ, where, for $j = 1, \ldots, k$, ℓ_{i_j} is a literal associated with x_{i_j}, φ' contains two new clauses $(\ell_{i_1}, \ldots, \ell_{i_k}, y)$ and $(\ell_{i_1}, \ldots, \ell_{i_k}, \bar{y})$. ∎

We now show that, for k fixed, problems kSAT and kDNF are affine equivalent.

Proposition 5. *For any fixed* k*,* MAX kSAT*,* MIN kSAT*,* MAX kDNF*,* MIN kDNF*,* MAX EkSAT*,* MIN EkSAT*,* MAX EkDNF *and* MIN EkDNF *are all affine equivalent.*

Sketch of proof. For affine equivalence between MAX kSAT and MIN kSAT, given n variables $x_1, \ldots, x_n$, denote by $\mathcal{C}_k$ the set of clauses of size k and by $\mathcal{C}_{\leqslant k}$ the set of clauses of size at most k on the set $\{x_1, \ldots, x_n\}$. Remark that any truth assignment verifies the same number v_k of clauses on $\mathcal{C}_k$ and the same number $v_{\leqslant k}$ of clauses on $\mathcal{C}_{\leqslant k}$. Note also that, since k is assumed fixed, sets $\mathcal{C}_k$ and $\mathcal{C}_{\leqslant k}$ are of polynomial size. If φ is an instance of MAX EkSAT (resp. MIN EkSAT) on variable-set $\{x_1, \ldots, x_n\}$ and on a set $\mathcal{C} = \{C_1, \ldots, C_m\}$ of m clauses, consider for MIN kSAT (resp., MAX kSAT) the instance φ' on the clause-set $\mathcal{C}' = \mathcal{C}_k \setminus \mathcal{C}$. For the case of MAX and MIN kSAT, $\mathcal{C}' = \mathcal{C}_{\leqslant k} \setminus \mathcal{C}$

For equivalence between versions of SAT and the corresponding versions of DNF, given a clause $C = (\ell_{i_1} \vee \ldots \vee \ell_{i_k})$ on k literals, build the cube (conjunction) $D = (\bar{\ell}_{i_1} \wedge \ldots \wedge \bar{\ell}_{i_k})$.

Finally, in order to show that MAX kSAT $\leq_{\mathsf{AF}}$ MAX EkSAT proceed as in Proposition 4. ∎

It is shown in [7] (see also [1]), that MAX E3SAT is inapproximable within standard approximation ratio $(7/8) + \epsilon$, for any $\epsilon > 0$, and MAX E2SAT is inapproximable within standard approximation ratio $(21/22) + \epsilon$, for any $\epsilon > 0$ (in what follows for such results we will use, for simplicity, expression "within better than"). Discussion above, together with these bounds leads to the following result.

Proposition 6. MAX 2SAT, MAX E2SAT, MIN 2SAT, MIN E2SAT, MAX 2DNF, MAX E2DNF, MIN 2DNF *and* MIN E2DNF *are inapproximable within differential approximation ratio better than 21/22. Furthermore, for any* $k \geqslant 3$, MAX kSAT, MAX EkSAT, MIN kSAT *and* MIN EkSAT, MAX kDNF, MAX EkDNF, MIN kDNF *and* MIN EkDNF, *are inapproximable within differential approximation ratio better than 7/8.*

Since the satisfiability problems stated in Proposition 6 are particular cases either of MAX SAT, or of MIN SAT, or of MAX DNF, or, finally, of MIN DNF, application of Proposition 6 and of Proposition 3 concludes the following corollary.

Corollary 2. MAX SAT, MIN SAT, MAX DNF *and* MIN DNF *are inapproximable within differential approximation 7/8.*

Results of Corollary 2 are not the best ones. In Section 4, we strengthen the one for MAX SAT. On the other hand, as it is proved in [2], MIN SAT is inapproximable within differential ratio better than $m^{\epsilon-1}$, for any $\epsilon > 0$. Proposition 5 has to be used with some precautions in order to yield positive or negative approximation results. Indeed, if one of the problem stated in it is approximable within constant differential approximation ratio (i.e., within ratio that does not depend on an instance parameter), then this ratio is naturally transferred to all the other problems. A contrario, one can see in the proof of Proposition 5 that in many cases the number of the clauses for the derived instance can be much larger that the one for the initial instance. In such cases, if we deal with ratios functions of m the form of these ratios is certainly preserved but not their value. For instance, assume that some problem Π among the ones stated Proposition 5 is approximable within ratio $f(|\varphi|)$, where $|\varphi|$ denotes the number of clauses, or cubes, in φ, and f decreases with $|\varphi|$. Assume also that there exists another problem Π' (among the ones stated in Proposition 5) such that $\Pi' \leq_{\mathsf{AF}} \Pi$ and, furthermore, that this affine reduction transforms a formula φ' of Π' into a formula φ for Π. Then, it transforms an approximation ratio $f(|\varphi|)$ for the latter into an approximation ratio $f(|\varphi|)$ for the former but, if the values $|\varphi|$ and $|\varphi'|$ are very different the one from the other, then $f(|\varphi|) \neq f(|\varphi'|)$.

In fact, one can easily observe that affine reductions of Proposition 5 perform the following differential ratio transformations: (i) reduction from MAX EkSAT to MIN EkSAT transforms ratios $f(m,n)$ into $f((2n)^k - m, n)$; (ii) reduction from MAX kSAT to MIN kSAT transforms ratios $f(m,n)$ into $f((2n+1)^k - m, n)$; (iii) reductions between SAT and DNF are invariant for approximation ratios; (iv) reduction from MAX kSAT to MAX EkSAT transforms ratios $f(m,n)$ into $f(2^{k-1}m, n+k-1)$.

In other words, dealing with common approximability of the problems stated in Proposition 5, the following remarks hold:

- if one of these problems is in **DAPX**, then all the other ones are so;
- problems MAX kSAT,MAX EkSAT, MIN k DNF and MIN EkDNF are approximable within differential ratios of $O(f(m))$ for a function f strictly decreasing with m if and only if one of them is $O(f(m))$ differentially approximable for $f(m) = O(m^{\alpha})$, for some $\alpha > 0$, or $f(m) = O(\log m)$; the same holds for the quadruple MIN kSAT,MIN EkSAT, MAX k DNF and MAX EkDNF.

Finally, reduction of Proposition 4 transforms ratios $f(m,n)$ into $f(2m, n+1)$.

3 Positive results

3.1 Maximum satisfiability

Consider an instance φ of an optimal satisfiability problem, defined on n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$; consider also the very classical algorithm RSAT assigning at any variable value 1 with probability 1/2 and, obviously, value 0 with probability 1/2.Then, denoting by $\mathrm{Sol}(\varphi)$, the set of the 2^n possible truth assignments for φ, and by $E(\mathtt{RSAT}(\varphi))$ the expectation of a solution computed by RSAT when running on φ, the following holds: $E(\mathtt{RSAT}(\varphi)) = \sum_{T \in \mathrm{Sol}(\varphi)} m(\varphi, T)/2^n$.

Algorithm RSAT can be derandomized by the following technique denoted by DSAT. For $i = 1, \ldots, n$:

- compute $E'_i = E(m(\varphi, T)|x_i = 1)$ and $E''_i = E(m(\varphi, T)|x_i = 0)$, where T is a random assignment and the values of the $i-1$ first variables have already been fixed in iterations $1, \ldots i-1$;

- set $x_i = 1$, if $E'_i \geqslant E''_i$; otherwise, set $x_i = 0$.

Lemma 1. $m(\varphi, \mathtt{DSAT}(\varphi)) \geqslant E(\mathtt{RSAT}(\varphi))$.

Note that DSAT is polynomial since, for any $i = 1, \ldots, n$, computation of E'_i and E''_i is performed in polynomial time. Indeed, for any such computation it suffices to determine with what probability any clause of φ is satisfied and to sum these probabilities over all the clauses of φ.

Proposition 7. *Algorithm DSAT achieves for* MAX EkSAT *differential approximation ratio* $2^k/(\mathrm{opt}(\varphi) + 2^k)$. *This ratio is bounded below by* $2^k/(m + 2^k)$.

Proof. We can assume that $\mathrm{opt}(\varphi) > \omega(\varphi)$. Then, $\omega(\varphi) < E(\mathtt{RSAT}(\varphi)) \leqslant m(\varphi, \mathtt{DSAT}(\varphi))$, and given that feasible values of MAX EkSAT are integer, we get: $m(\varphi, \mathtt{DSAT}(\varphi)) - \omega(\varphi) \geqslant 1$. Furthermore, $m(\varphi, \mathtt{DSAT}(\varphi)) \geqslant E(\mathtt{RSAT}(\varphi)) = m(1 - 2^{-k}) \geqslant \mathrm{opt}(\varphi)(1 - 2^{-k})$. We so get: $\delta(\varphi, \mathtt{DSAT}(\varphi)) \geqslant \max\{(\mathrm{opt}(\varphi) - \omega(\varphi))^{-1}, (\mathrm{opt}(\varphi)(1 - 2^{-k}) - \omega(\varphi))/(\mathrm{opt}(\varphi) - \omega(\varphi))\}$ and, after some algebra, this leads to the ratio claimed. ∎

We now propose a reduction transferring approximation results for MAX SAT problems from standard to differential paradigm. It will be used in order to achieve differential approximation results for MAX SAT, MAX 3SAT and MAX 2SAT.

Proposition 8. *If a maximum satisfiability problem* Π *is approximable on an instance* φ*, within standard approximation ratio* ρ*, then it is approximable in* φ *within differential approximation ratio* $\rho/((1-\rho)\omega(\varphi) + 1)$.

Proof. Consider an instance φ of Π, run both A and DSAT on φ and retain assignment T satisfying the maximum number of clauses between $\mathtt{A}(\varphi)$ and $\mathtt{DSAT}(\varphi)$. Obviously, $m(\varphi, T) \geqslant \rho\, \mathrm{opt}(\varphi)$. Hence, taking into account that $m(\varphi, T) \geqslant \omega(\varphi) + 1$, we get after some algebra, the result claimed. ∎

From the result of Proposition 8, we can deduce several corollaries by specifying values for $\omega(\varphi)$ and ρ. The main such corollaries are stated in the proposition that follows.

Proposition 9. MAX SAT *is approximable within differential approximation ratio* $4.34/(m + 4.34)$. MAX E2SAT *is approximable within differential approximation ratio* $17.9/(m + 19.3)$ *and* MAX 3SAT *within* $4.57/(m + 5.73)$.

3.2 Minimum satisfiability

We finish this section by studying MIN SAT and some of its versions. Before stating our results, we note that algorithm RSAT can be derandomized in an exactly symmetric way, in order to provide a solution for MIN kSAT with value smaller than expectation's value.

Proposition 10. *If a minimum satisfiability problem is approximable on an instance φ, within standard approximation ratio ρ, then it is approximable in φ within differential approximation ratio*

$$\frac{\rho}{(\rho-1)\left(1-\frac{1}{2^k}\right)m+\rho}$$

Proof. We have $\mathrm{opt}(\varphi) \leqslant m(\varphi, \mathtt{DSAT}(\varphi)) \leqslant E(\mathtt{RSAT}(\varphi)) < \omega(\varphi)$, i.e., $m(\varphi, \mathtt{DSAT}(\varphi)) - \omega(\varphi) \leqslant 1$. Considering the best among the solutions computed by DSAT and A (the ρ-standard approximation algorithm assumed for MIN kSAT in the statement of the theorem), denoting it by T, one gets:

$$\delta(\varphi, T) \geqslant \max\left\{\frac{1}{\omega(\varphi)-\mathrm{opt}(\varphi)}, \frac{\omega(\varphi)-\rho\,\mathrm{opt}(\varphi)}{\omega(\varphi)-\mathrm{opt}(\varphi)}, \frac{\omega(\varphi)-m\left(1-\frac{1}{2^k}\right)}{\omega(\varphi)-\mathrm{opt}(\varphi)}\right\}$$

Then, some tedious algebra, leads to the result claimed. ∎

The best standard approximation ratios known for MIN kSAT and MIN SAT are $2(1-2^{-k})$ and 2, respectively ([3]). With the ratio just mentioned for MIN kSAT, the result of Proposition 10 can be simplified as indicated in the following corollary.

Corollary 3. MIN kSAT *is approximable within differential ratio* $2^k/((2^{k-1}-1)m+2^k)$.

Proposition 11. MIN SAT *is approximable within differential ratio* $2/(m+2)$.

Proof. Use Proposition 10 with $\rho = 2$ ([3]). ∎

Also, using Corollary 3 with $k = 2$ and $k = 3$, the following corollary holds and concludes the section.

Corollary 4. MIN 2SAT *and* MIN 3SAT *are approximable within differential ratios* $4/(m+4)$ *and* $8/(3m+8)$*, respectively.*

4 Inapproximability

As it is proved in [7], for any $p \geqslant 2$ and for any $\epsilon > 0$, MAX E3LINp[1] cannot be approximated within standard approximation ratio $(1/p)+\epsilon$, even if coefficients in the left-hand sides of the equations are all equal to 1. Note that, due to Corollary 1, this bound is immediately transferred to the differential paradigm.

Finally, let us recall the following result of [7], that will be used in this section.

Proposition 12. *([7]) Given a problem $\Pi \in$ **NP** and a real $\delta > 0$, there exists a polynomial transformation g from any instance I of Π into an instance of* MAX E3LIN2 *such that: if I is a yes-instance of Π (we use here classical terminology from [6]), then* $\mathrm{opt}(g(I)) \geqslant (1-\delta)m$*, otherwise,* $\mathrm{opt}(g(I)) \leqslant (1+\delta)m/2$.

Proposition 12 shows, in fact, that MAX E3LIN2 is not approximable within standard ratio $1/2+\epsilon$, for any $\epsilon > 0$.

We prove a result analogous to the one of Proposition 12 from any problem $\Pi \in$ **NP** to MAX E3SAT.

Proposition 13. *Given a problem $\Pi \in$ **NP** and a real $\delta > 0$, there exists a polynomial transformation f from any instance I of Π into an instance of* MAX E3SAT *such that:*

[1] In MAX E3LINp we are given a positive prime p, n variables $x_1, \ldots, x_n$ in $\mathbb{Z}/p\mathbb{Z}$, m linear equations of type $\alpha_{i_\ell} x_{i_\ell} + \alpha_{j_\ell} x_{j_\ell} + \alpha_{k_\ell} x_{k_\ell} = \beta_\ell$ and our objective is to determine an assignment on $x_1, \ldots, x_n$, in such a way that a maximum number among the m equations is satisfied.

- *if I is a yes-instance of Π, then* $\mathrm{opt}(f(I)) - \omega(f(I)) \geqslant (1-2\delta)m/4$*;*
- *if I is a no-instance of Π, then* $\mathrm{opt}(f(I)) - \omega(f(I)) \leqslant \delta m/4$.

Sketch of proof. We first prove that the reduction of Proposition 12 can be translated into the differential paradigm also. Consider an instance $I' = g(I)$ of MAX E3LIN2 and a feasible solution $\vec{x} = (x_1, x_2, \ldots, x_n)$ for I (we will use the same notation for both variables and their assignment) verifying k among the m equations of I'. Then, vector $\vec{\bar{x}} = (1-x_1, \ldots, 1-x_n)$, verifies the $m-k$ equations not verified by $\vec{x}$. In other words, $\mathrm{opt}(I) + \omega(I) = m$; hence, function g claimed by Proposition 12 is such that: if I is a *yes*-instance of Π, then $\mathrm{opt}(I') - \omega(I') \geqslant (1-2\delta)m$, otherwise $\mathrm{opt}(I') - \omega(I') \leqslant \delta m$.

Consider now an instance I of MAX E3LIN2 on n variables x_i, $i = 1, \ldots, n$ and m equations of type $x_i + x_j + x_k = \beta$ in $\mathbb{Z}/2\mathbb{Z}$, i.e., where variables and second members equal 0, or 1. We transform I into an instance $\varphi = h(I)$ of MAX E3SAT in the following way: for any equation $x_i + x_j + x_k = 0$, we add in $h(I)$ the following four clauses: $(\bar{x}_i \vee x_j \vee x_k)$, $(x_i \vee \bar{x}_j \vee x_k)$, $(x_i \vee x_j \vee \bar{x}_k)$ and $(\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k)$; for any equation $x_i + x_j + x_k = 1$, we add in $h(I)$ the following four clauses: $(x_i \vee x_j \vee x_k)$, $(\bar{x}_i \vee \bar{x}_j \vee x_k)$, $(\bar{x}_i \vee x_j \vee \bar{x}_k)$ and $(x_i \vee \bar{x}_j \vee \bar{x}_k)$.

Given a solution y for MAX E3SAT on $h(I)$, we construct a solution y' for I by setting $x_i = 1$ if $x_i = 1$ in $h(I)$ also; otherwise, we set $x_i = 0$.

It suffices then to remark that the composition $f = h \circ g$ (where g is as in Proposition 12) verifies the statement of the proposition. ∎

Proposition 13 has a very interesting corollary, expressed in the Proposition 14 just below, that exhibits another point of dissymmetry between standard and differential paradigms.

Proposition 14. *Unless $\boldsymbol{P} = \boldsymbol{NP}$, no polynomial algorithm can compute, on an instance φ of* MAX E3SAT *a value that is a constant approximation of the quantity* $\mathrm{opt}(\varphi) - \omega(\varphi)$.

In view of Proposition 14, what is different between standard and differential paradigms with respect to the GAP-reduction is that in the former such a reduction immediately concludes the impossibility for a problem (assume that it is a maximization one) to be approximable within some ratio, by showing the impossibility for the optimal value to be approximated within this ratio. For that, it suffices that one reads the value of the solution returned by the approximation algorithm. In the latter paradigm such a conclusion is not always immediate. In fact, a reasoning similar to the one of the standard approximation is possible when computation of the worst solution can be done in polynomial time (this is, for instance, the case of maximum independent set and of many other **NP**-hard problems). In this case a simple reading of the value of the approximate solution is sufficient to give an approximation of $\mathrm{opt}(x) - \omega(x)$. A contrario, when it is **NP**-hard to compute $\omega(x)$ (this is the case of the problems dealt here –simply think that the worst solution for MAX SAT is the optimal one for MIN SAT and that both of them are **NP**-hard –, of traveling salesman, etc.), then reading the value $m(x, y)$ of the approximate solution does not provide us with knowledge about $m(x, y) - \omega(x)$ and, consequently no approximation of $\mathrm{opt}(x) - \omega(x)$ can be immediately estimated. So, use of GAP-reduction for achieving inapproximability results is different from the one paradigm to the other.

However, for the case we deal with, we will take advantage of a combination of Propositions 5 and 14 in order to achieve the inapproximability bound for MAX E3SAT given in Proposition 15 that follows.

Proposition 15. *Unless $\boldsymbol{P} = \boldsymbol{NP}$,* MAX E3SAT *is inapproximable within differential approximation ratio greater than 1/2.*

Sketch of proof. Assume that an approximation achieves differential ratio $\delta > 1/2$, for MAX E3SAT. Then, by Proposition 5, there exists an algorithm achieving the same differential ratio for MIN E3SAT. Denote by T_1 and T_2, respectively, the solutions computed by these algorithms on an instance φ of these problems. Then, one can prove that $m(\varphi, T_1) - m(\varphi, T_2) \geqslant (2\delta - 1)(\text{opt}(\varphi) - \omega(\varphi))$, and a simple reading of the values of T_1 and T_2, can provide us a constant approximation (since δ has been assumed to be a fixed constant greater than 1/2) of the quantity $\text{opt}(\varphi) - \omega(\varphi)$, impossible by Proposition 14. ∎

Corollary 5. *For any $k \geqslant 3$,* MAX EkSAT, MIN EkSAT, MAX kSAT *and* MIN kSAT *are differentially inapproximable within ratios better than 1/2.*

We now generalize the result of Proposition 13 in order to further strengthen inapproximability results of Corollary 5.

Proposition 16. *For any prime $p > 0$,* MAX E3LINp $\leq_{\mathsf{AF}}$ MAS E3$(p-1)$SAT.

The result of Proposition 16 together with the result of [7] stated in the beginning of the section and Proposition 1, lead to the following corollary.

Corollary 6. *For any prime p,* MAX E3$(p-1)$SAT *is inapproximable within differential ratio greater than $1/p$.*

Proposition 17. *For any $k \geqslant 3$, neither* MAX EkSAT, *nor* MIN EkSAT *can be approximately solved within differential ratio greater than $1/p$, where p is the largest positive prime such that $3(p-1) \leqslant k$.*

Corollary 7. *The following differential inapproximability bounds hold: 1/2 for* MAX *and* MIN 3SAT 4SAT *and* 5SAT*; 1/3 for* MAX *and* MIN 6SAT, ..., 11SAT*; 1/5 for* MAX *and* MIN 12SAT, ..., 17SAT ...

Finally, MAX SAT being harder to approximate than any MAX kSAT problem, the following result holds and concludes the section.

Proposition 18. MAX SAT $\notin$ ***DAPX***.

In [11] is defined a logical class of **NPO** maximization problems called **MAX-NP**. A maximization problem $\Pi \in$ **NPO** belongs to **Max-NP** if and only if there exist two finite structures $(U, \mathcal{I})$ and $(U, \mathcal{S})$, a quantifier-free first order formula φ and two constants k and ℓ such that, the optima of Π can be logically expressed as: $\max_{S \in \mathcal{S}} |\{x \in U^k : \exists y \in U^\ell, \varphi(\mathcal{I}, S, x, y)\}|$. The predicate-set $\mathcal{I}$ draws the set of instances of Π, set $\mathcal{S}$ the solutions on $\mathcal{I}$ and φ the feasibility conditions for the solutions of Π. In the same article is proved that MAX SAT $\in$ **Max-NP** and that **MAX-NP** $\subset$ **APX**.

It is easy to see that the above definition of **Max-NP** identically holds in both standard and differential paradigms. So, Proposition 18 draws an important structural difference in the landscape of approximation classes in the two paradigms, since an immediate corollary of this proposition is that **MAX-NP** $\not\subset$ **DAPX**. We conjecture that the same holds for the other one of the celebrated logical classes of [11], the class **MAX-SNP**, i.e., we conjecture that **MAX-SNP** $\not\subset$ **DAPX**

We conclude the paper by strengthening the 21/22-inapproximability bound for MAX E2SAT.

Proposition 19. MAX E2LIN2 $\leq_{\mathsf{AF}}$ MAX E2SAT. *Consequently,* MAX E2SAT *is differentially inapproximable within ratio greater than 11/12.*

References

[1] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi. *Complexity and approximation. Combinatorial optimization problems and their approximability properties.* Springer, Berlin, 1999.

[2] C. Bazgan and V. Th Paschos. Differential approximation for optimal satisfiability and related problems. *European J. Oper. Res.*, 147(2):397–404, 2003.

[3] D. Bertsimas, C-P. Teo, and R. Vohra. On dependent randomized rounding algorithms. In *Proc. International Conference on Integer Programming and Combinatorial Optimization, IPCO*, volume 1084 of *Lecture Notes in Computer Science*, pages 330–344. Springer-Verlag, 1996.

[4] P. Crescenzi, V. Kann, R. Silvestri, and L. Trevisan. Structure in approximation classes. *SIAM J. Comput.*, 28(5):1759–1782, 1999.

[5] B. Escoffier and V. Th. Paschos. Differential approximation of MIN SAT, MAX SAT and related problems. Cahier du LAMSADE 220, LAMSADE, Université Paris-Dauphine, 2004.

[6] M. R. Garey and D. S. Johnson. *Computers and intractability. A guide to the theory of NP-completeness.* W. H. Freeman, San Francisco, 1979.

[7] J. Håstad. Some optimal inapproximability results. *J. Assoc. Comput. Mach.*, 48:798–859, 2001.

[8] J. N. Hooker. Resolution vs. cutting plane solution of inference problems: some computational experience. *Oper. Res. Lett.*, 7(1):1–7, 1988.

[9] A. P. Kamath, N. K. Karmarkar, K. G. Ramakrishnan, and M. G. Resende. Computational experience with an interior point algorithm on the satisfiability problem. *Ann. Oper. Res.*, 25:43–58, 1990.

[10] J. Monnot, V. Th. Paschos, and S. Toulouse. *Approximation polynomiale des problèmes NP-difficiles : optima locaux et rapport différentiel.* Informatique et Systèmes d'Information. Hermès, Paris, 2003.

[11] C. H. Papadimitriou and M. Yannakakis. Optimization, approximation and complexity classes. *J. Comput. System Sci.*, 43:425–440, 1991.

[12] V. Vazirani. *Approximation algorithms.* Springer, Berlin, 2001.

A Proof of Proposition 2

Consider any differential polynomial time approximation algorithm $\mathtt{A}$ guaranteeing differential-approximation ratio δ for any instance x of a maximization problem Π. Denote by $\mathtt{A}(x)$, a solution computed by $\mathtt{A}$ when running on x. Then,

$$\frac{m(x,\mathtt{A}(x)) - \omega(x)}{\operatorname{opt}(x) - \omega(x)} \geqslant \delta \Longrightarrow m(x,\mathtt{A}(x)) \geqslant \delta \operatorname{opt}(x) + (1-\delta)\omega(x) \overset{\substack{\delta\leqslant 1\\ \omega(x)\geqslant 0}}{\Longrightarrow} \frac{m(x,\mathtt{A}(x))}{\operatorname{opt}(x)} \geqslant \delta$$

and the claim of the proposition is proved.

B Proof of Proposition 4

Consider an instance φ of MAX EkSAT on n variables $x_1,\ldots,x_n$ and m clauses $C_1,\ldots,C_m$. Consider also a new variable y and build formula φ', instance of MAX E$(k+1)$SAT as follows: for any clause $C_i = (\ell_{i_1},\ldots,\ell_{i_k})$ of φ, where, for $j = 1,\ldots,k$, ℓ_{i_j} is a literal associated with x_{i_j}, φ' contains two new clauses $(\ell_{i_1},\ldots,\ell_{i_k},y)$ and $(\ell_{i_1},\ldots,\ell_{i_k},\bar{y})$. Hence, φ' is the conjunction of $2m$ clauses of size $k+1$ on $n+1$ variables. Assume any truth assignment T on the variables of φ and denote by $(T,1)$ (resp., $(T,0)$) the extension of T on φ' by setting $y=1$ (resp., $y=0$). Then, it is easy to see that $m(\varphi',(T,1)) = m(\varphi',(T,0)) = m + m(\varphi,T)$.

In other words, reduction just described, associating to any assignment T' of φ' its restriction T on variables $x_1,\ldots,x_n$ as assignment for φ, is affine and the proof of the proposition is complete.

C Proof of Proposition 5

We first prove affine equivalence between MAX kSAT and MIN kSAT. Given n variables $x_1,\ldots,x_n$, denote by $\mathcal{C}_k$ the set of clauses of size k and by $\mathcal{C}_{\leqslant k}$ the set of clauses of size at most k on the set $\{x_1,\ldots,x_n\}$. Let us remark that any truth assignment verifies the same number v_k of clauses on $\mathcal{C}_k$ and the same number $v_{\leqslant k}$ of clauses on $\mathcal{C}_{\leqslant k}$. Note also that, since k is assumed fixed, sets $\mathcal{C}_k$ and $\mathcal{C}_{\leqslant k}$ are of polynomial size.

Let φ be an instance of MAX EkSAT on variable-set $\{x_1,\ldots,x_n\}$ and on a set $\mathcal{C} = \{C_1,\ldots,C_m\}$ of m clauses. Consider instance φ' on the clause-set $\mathcal{C}' = \mathcal{C}_k \setminus \mathcal{C}$. Then, for any truth assignment T on $\{x_1,\ldots,x_n\}$: $m(\varphi,S)+m(\varphi',S) = v_k$; in other words, reduction just described is an affine reduction from MAX EkSAT to MIN EkSAT. Considering φ as instance of MIN EkSAT this time, the above describe an affine reduction from MIN EkSAT to MAX EkSAT.

Furthermore, if $\mathcal{C}$ is an instance of MAX kSAT, then we can see the clause-set $\mathcal{C}_{\leqslant k} \setminus \mathcal{C}$ as an instance of MIN kSAT and the same arguments conclude an affine reduction from the former to the latter problem.

We now prove equivalence between versions of SAT and the corresponding versions of DNF. Given a clause $C = (\ell_{i_1} \vee \ldots \vee \ell_{i_k})$ on k literals, we build the cube (conjunction) $D = (\bar{\ell}_{i_1} \wedge \ldots \wedge \bar{\ell}_{i_k})$. Any truth assignment T on ℓ_{i_j} verifies C, if and only if it does not verify D, i.e., $m(C,T) = m - m(D,T)$. This specifies an affine reduction between MAX EkSAT and MIN EkDNF, MIN EkSAT and MAX EkDNF, MAX kSAT and MIN kDNF and between MIN kSAT and MAX kDNF.

We finally show equivalence between MAX kSAT and MAX EkSAT. We first notice that the latter problem being a sub-problem of the former one, direction MAX EkSAT $\leq_{\mathsf{AF}}$ MAX kSAT is immediate. On the other hand, as in Proposition 4, given an instance of MAX kSAT, one can construct, for any clause of size at most k, a set of clauses of size exactly k, in such a way that this reduction is affine.

Combination of equivalences shown above completes the proof of the proposition.

D Proof of Proposition 6

Concerning MAX 2SAT and associates, Corollary 1 extends the result of [7] to the differential paradigm. Then, Proposition 5 suffices to conclude the proof.

For MAX kSAT and associates, Corollary 1 extends the result of [7] to the differential paradigm, for MAX 3SAT and Proposition 5 transfers it to MAX E3SAT. Then, Proposition 4 extends it for any $k \geqslant 4$. Finally, Proposition 5 suffices to conclude the proof.

E Proof of Lemma 1

It is easy to see that $E(\mathtt{RSAT}(\varphi)) = (E'_1/2) + (E''_1/2)$; hence $\max\{E'_1, E''_1\} \geqslant E(\mathtt{RSAT}(\varphi))$. Furthermore, at any of the n steps of $\mathtt{DSAT}$, $\max\{E'_i, E''_i\} = (E'_{i+1}/2)+(E''_{i+1}/2) \leqslant \max\{E'_{i+1}, E''_{i+1}\}$. We so have $E(\mathtt{RSAT}(\varphi)) \leqslant \max\{E'_1, E''_1\} \leqslant \max\{E'_n, E''_n\} = \mathtt{DSAT}(\varphi)$, that concludes the proof of the lemma.

F Proof of Proposition 7

Note first that we can assume that $\mathrm{opt}(\varphi) > \omega(\varphi)$ (otherwise, MAX EkSAT would be polynomial on φ). Then,

$$\omega(\varphi) < E(\mathtt{RSAT}(\varphi)) \leqslant m(\varphi, \mathtt{DSAT}(\varphi)) \tag{1}$$

From (1) and given that feasible values of MAX EkSAT are integer, we get:

$$m(\varphi, \mathtt{DSAT}(\varphi)) - \omega(\varphi) \geqslant 1 \tag{2}$$

Since clauses in φ are of size k, the expectation that any of them is satisfied equals $1 - 2^{-k}$. Hence,

$$m(\varphi, \mathtt{DSAT}(\varphi)) \geqslant E(\mathtt{RSAT}(\varphi)) = m\left(1 - \frac{1}{2^k}\right) \geqslant \mathrm{opt}(\varphi)\left(1 - \frac{1}{2^k}\right) \tag{3}$$

Using (2) and (3), we get:

$$\delta(\varphi, \mathtt{DSAT}(\varphi)) \geqslant \max\left\{\frac{1}{\mathrm{opt}(\varphi) - \omega(\varphi)}, \frac{\mathrm{opt}(\varphi)\left(1 - \frac{1}{2^k}\right) - \omega(\varphi)}{\mathrm{opt}(\varphi) - \omega(\varphi)}\right\} \tag{4}$$

The first term in (4) is increasing with $\omega(\varphi)$, while the second one is decreasing. Equality holds when $\omega(\varphi) = (\mathrm{opt}(\varphi)(1 - 2^{-k})) - 1$. In this case, (4) gives

$$\delta(\varphi, \mathtt{DSAT}(\varphi)) \geqslant \frac{2^k}{\mathrm{opt}(\varphi) + 2^k} \geqslant \frac{2^k}{m + 2^k} \tag{5}$$

Last inequality in (5) holding thanks to the fact that $\mathrm{opt}(\varphi) \leqslant m$, qed.

G Proof of Proposition 8

Fix any maximum satisfiability problem Π, sharing the ones dealt until now, and assume that there exists a polynomial time algorithm achieving standard approximation ratio ρ for Π. Consider an instance φ of Π, run both $\mathtt{A}$ and $\mathtt{DSAT}$ on φ and retain assignment T satisfying the maximum number of clauses between $\mathtt{A}(\varphi)$ and $\mathtt{DSAT}(\varphi)$. Obviously, $m(\varphi, T) \geqslant \rho\, \mathrm{opt}(\varphi)$. Hence, the differential approximation ratio of T is

$$\delta(\varphi, T) \geqslant \frac{m(\varphi, T) - \omega(\varphi)}{\frac{m(\varphi,T)}{\rho} - \omega(\varphi)} \tag{6}$$

Since, as we have seen in the proof of Proposition 7, $m(\varphi, T) \geqslant \omega(\varphi) + 1$, (6) becomes

$$\delta(\varphi, T) \geqslant \frac{1}{\frac{\omega(\varphi)+1}{\rho} - \omega(\varphi)} = \frac{\rho}{(1-\rho)\omega(\varphi) + 1} \tag{7}$$

The proof of the proposition is now complete.

H Proof of Proposition 9

Let us remark that, in the case of MAX kSAT

$$E(\mathtt{RSAT}(\varphi)) \leqslant m\left(1 - \frac{1}{2^k}\right) \tag{8}$$

Then (1) and (8) yield:

$$\omega(\varphi) \leqslant m\left(1 - \frac{1}{2^k}\right) \tag{9}$$

For MAX SAT we can assume $\omega(\varphi) \leqslant m-1$, otherwise ($\omega(\varphi) = m$) all feasible solutions of φ have the same value. Since $1 - \rho \geqslant 0$, the differential ratio of (7) decreases with $\omega(I)$. So, it suffices to substitute $m-1$ for $\omega(\varphi)$, to use the fact that MAX SAT is approximable within standard ratio 1/1.2987 ([1]), and the proof of the proposition is complete.

For MAX 2SAT, remark first that, using (3), the expectation of the solution computed by the random algorithm RSAT is, using (9), less than, or equal to, $3m/4$. Consequently, $\omega(\varphi) \leqslant 3m/4$. Next, the fact that MAX SAT is approximable within standard ratio 1/1.0741 ([1]) suffices to conclude the proof.

For MAX 3SAT, $\omega(\varphi) \leqslant 7m/8$ and $\rho = 1/1.249$ ([1]).

I Proof of Proposition 10

As in the proof of Proposition 7, since we deal with a minimization problem, (1) becomes:

$$\text{opt}(\varphi) \leqslant m(\varphi, \mathtt{DSAT}(\varphi)) \leqslant E(\mathtt{RSAT}(\varphi)) < \omega(\varphi) \tag{10}$$

Consequently, (2) becomes:

$$\omega(\varphi) - m(\varphi, \mathtt{DSAT}(\varphi)) \geqslant 1 \tag{11}$$

Considering the best among the solutions computed by DSAT and A (the ρ-standard approximation algorithm assumed for MIN kSAT in the statement of the theorem), denoting it by T and using (10) and (11), we get:

$$\delta(\varphi, T) \geqslant \max\left\{\frac{1}{\omega(\varphi) - \text{opt}(\varphi)}, \frac{\omega(\varphi) - \rho\,\text{opt}(\varphi)}{\omega(\varphi) - \text{opt}(\varphi)}, \frac{\omega(\varphi) - m\left(1 - \frac{1}{2^k}\right)}{\omega(\varphi) - \text{opt}(\varphi)}\right\} \tag{12}$$

where the third term in (12) is due to the fact that T has a better value than the value of algorithm RSAT.

The first term in (12) is decreasing with $\omega(\varphi)$, while the second and third ones are increasing. We distinguish two cases depending on the relation between these terms.

If the second term is greater than the third one, i.e., if $\rho\,\text{opt}(\varphi) \leqslant m(1 - 2^{-k})$, then equality of the first two terms of (12) is achieved when $\omega(\varphi) = 1 + \rho\,\text{opt}(\varphi)$. In this case, (12) gives:

$$\delta(\varphi, T) \geqslant \frac{\rho}{(\rho - 1)m\left(1 - \frac{1}{2^k}\right) + \rho} \tag{13}$$

If, on the other hand, second term is smaller than the third one, i.e., if $\rho\,\text{opt}(\varphi) \geqslant m(1 - 2^{-k})$, then equality of the first and the third term in (12) is achieved when $\omega(\varphi) = m(1 - 2^{-k}) + 1$. In this case also, $\delta(\varphi, T)$ verifies (13). The proof of the proposition is now complete.

J Proof of Proposition 13

We first prove that the reduction of Proposition 12 can be translated into differential paradigm also. Consider an instance $I' = g(I)$ of MAX E3LIN2 and a feasible solution $\vec{x} = (x_1, x_2, \ldots, x_n)$ for I (we will use the same notation for both variables and their assignment) verifying k among the m equations of I'. Then, vector $\vec{\bar{x}} = (1 - x_1, \ldots, 1 - x_n)$, verifies the $m - k$ equations not verified by $\vec{x}$. In other words, $\mathrm{opt}(I) + \omega(I) = m$; hence, function g claimed by Proposition 12 is such that:

- if I is a *yes*-instance of Π, then $\mathrm{opt}(I') - \omega(I') \geqslant (1 - 2\delta)m$;
- if I is a *no*-instance of Π, then $\mathrm{opt}(I') - \omega(I') \leqslant \delta m$.

We are ready now to continue the proof of the proposition. Consider an instance I of MAX E3LIN2 on n variables x_i, $i = 1, \ldots, n$ and m equations of type $x_i + x_j + x_k = \beta$ in $\mathbb{Z}/2\mathbb{Z}$, i.e., where variables and second members equal 0, or 1. In the same spirit as in [7], we transform I into an instance $\varphi = h(I)$ of MAX E3SAT in the following way:

- for any equation $x_i + x_j + x_k = 0$, we add in $h(I)$ the following four clauses: $(\bar{x}_i \vee x_j \vee x_k)$, $(x_i \vee \bar{x}_j \vee x_k)$, $(x_i \vee x_j \vee \bar{x}_k)$ and $(\bar{x}_i \vee \bar{x}_j \vee \bar{x}_k)$;
- for any equation $x_i + x_j + x_k = 1$, we add in $h(I)$ the following four clauses: $(x_i \vee x_j \vee x_k)$, $(\bar{x}_i \vee \bar{x}_j \vee x_k)$, $(\bar{x}_i \vee x_j \vee \bar{x}_k)$ and $(x_i \vee \bar{x}_j \vee \bar{x}_k)$.

It can immediately be seen that $h(I)$ has n variables and $4m$ (distinct) clauses.

Given a solution y for MAX E3SAT on $h(I)$, we construct a solution y' for I by setting $x_i = 1$ if $x_i = 1$ in $h(I)$ also; otherwise, we set $x_i = 0$.

For instance, consider equation $x_i + x_j + x_k = 0$ in I. It is verified if either 0 or 2 of the variables are equal to 1. The several satisfaction possibilities for the clauses derived in $h(I)$ for this equation are the following:

- if zero, or two variables are set to 1 (true), then all the four clauses are satisfied;
- if one, or three variables are set to 1, then 3 clauses are satisfied.

As a consequence, iterating this argument for any clause set built from an equation, we conclude that solution y for MAX E3SAT on $h(I)$ verifies $m(h(I), y) = 3m + m(I, y')$. Since transformation between y' and y is bijective, we get $\omega(h(I)) = 3m + \omega(I)$ and $\mathrm{opt}(h(I)) = 3m + \mathrm{opt}(I)$. In other words, the reduction just described is an affine reduction from MAX E3LIN2 to MAX E3SAT.

It suffices now to remark that the composition $f = h \circ g$ verifies the statement of the proposition and its proof is concluded.

K Proof of Proposition 15

Assume that an approximation achieves differential ratio $\delta > 1/2$, for MAX E3SAT. Then, by Proposition 5, there exists an algorithm achieving the same differential ratio for MIN E3SAT. Denote by T_1 and T_2, respectively, the solutions computed by these algorithms on an instance φ of these problems. We have:

$$m(\varphi, T_1) - \omega(\varphi) \geqslant \delta(\mathrm{opt}(\varphi) - \omega(\varphi)) \tag{14}$$

where $\mathrm{opt}(\cdot)$ and $\omega(\cdot)$ are referred to MAX E3SAT. By the relations between all these parameters for the two problems specified in the proof of Proposition 5, we get:

$$\mathrm{opt}(\varphi) - m(\varphi, T_2) \geqslant \delta(\mathrm{opt}(\varphi) - \omega(\varphi)) \tag{15}$$

Adding (14) and (15) member-by-member, we get $m(\varphi, T_1) - m(\varphi, T_2) \geqslant (2\delta - 1)(\mathrm{opt}(\varphi) - \omega(\varphi))$. So, simple reading of the values of T_1 and T_2, can provide us a constant approximation (since δ has been assumed to be a fixed constant greater than 1/2) of the quantity $\mathrm{opt}(\varphi) - \omega(\varphi)$, impossible by Proposition 14.

L Proof of Proposition 16

Consider a positive prime p and an instance I of MAX E3LINp on n variables and m equations. Consider an equation $x_1 + x_2 + x_3 = \beta$ (in $\mathbb{Z}/p\mathbb{Z}$) of I and, for any $i = 1, 2, 3$, $p - 1$ new variables $x_i^1, \ldots, x_i^{p-1} \in \{0, 1\}$. Consider, finally, equation

$$\sum_{j=1}^{p-1} x_1^j + \sum_{j=1}^{p-1} x_2^j + \sum_{j=1}^{p-1} x_3^j = \beta \tag{16}$$

It is easy to see that (16) is verified if and only if the number of variables set to 1 is either β or $\beta + p$, or, finally, $\beta + 2p$.

Consider now the set of all the possible clauses on $3(p-1)$ literals issued from variables $x_i^1, \ldots, x_i^{p-1}$, $i = 1, 2, 3$. Any truth assignment will satisfy all but one clause. For example, if any variable is assigned with 1, the only unsatisfied clause is the one where all variables appear negative.

What is of interest for us is to specify when the number of variables set to 1 is either β or $\beta + p$, or, $\beta + 2p$. For this, denote by $\mathcal{C}_k$ the set of clauses on $3(p-1)$ literals issued from variables $x_i^1, \ldots, x_i^{p-1}$, $i = 1, 2, 3$ with exactly k negative literals. Then, a truth assignment setting k variables to 1, verifies $|\mathcal{C}_k| - 1$ clauses of $\mathcal{C}_k$, while any other truth assignment on the variables of $\mathcal{C}_k$ verifies all the $|\mathcal{C}_k|$ clauses. So, for an equation $x_1 + x_2 + x_3 = \beta$, we will add in the instance of MAX E3$(p-1)$SAT the set $\mathcal{C}_k$, for $k \in \{0, \ldots, 3(p-1)\}$ and $k \notin \{\beta, \beta + p, \beta + 2p\}$. Hence, if a truth assignment for these clauses has β, or $\beta + p$, or $\beta + 2p$ variables set to 1, it will verify all the clauses constructed, otherwise it will verify all but one of these clauses.

In all, for any of the variables $x_i^1, \ldots, x_i^{p-1}$ we will build one new variable and we will transform any of the m equations of I into an equation as in (16). Then, for any of these new equations we add in the instance of MAX E3$(p-1)$SAT the set of clauses as built just above. The instance φ of MAS E3$(p-1)$SAT so constructed has $n(p-1)$ variables and, since the number of clauses issued from any equation is no more than $2^{3(p-1)}$, φ will have at most $m_\varphi \leqslant m2^{3(p-1)}$ clauses.

Given a truth assignment T on the variables of φ, we set $x_i = |\{x_i^k : x_i^k = 1 \text{ in } T\}|$. Discussion above leads to $m(\varphi, T) = m_\varphi - m + m(I, S)$. On the other hand, it is easy to see that our reduction implies that any solution S of I is transformed into a truth assignment T on the variables of φ such that the relation between the values of S and T given just above is always satisfied. This relation confirms that the reduction specified is an affine one from MAX E3LINp to MAX E3$(p-1)$SAT.

Finally, let us remark that it is possible that formula φ contains many times the same clause. This, for instance, is the case if I simultaneously contains equations say $x_1 + x_2 + x_3 = \beta_1$ and $x_1 + x_2 + x_3 = \beta_2$, for $\beta_1 \neq \beta_2$. In this case, we can modify the construction described, by building the subset of $\mathcal{C}_k$ or $k \in \{0, \ldots, 3(p-1)\}$ and $k \notin \{\beta_1, \beta_1 + p, \beta_1 + 2p, \beta_2, \beta_2 + p, \beta_2 + 2p\}$. This concludes the proof of the proposition.

M Proof of Proposition 19

Consider an instance I of MAX E2LIN2 (on n variables and m equations) and an equation $x_1 + x_2 = 0$ in I. Add in φ (the instance of MAX E2SAT under construction) clauses $\bar{x}_1 \vee x_2$ and $x_1 \vee \bar{x}_2$. On the other hand, for an equation $x_1 + x_2 = 1$, add in φ clauses $x_1 \vee x_2$ and $\bar{x}_1 \vee \bar{x}_2$.

Performing this transformation for any equation in I, we finally build a formula φ of MAX E2SAT on n variables and $2m$ clauses. Moreover, for any truth assignment T on the variables of φ, one gets a solution S for I such that $m(\varphi, T) = m + m(I, S)$, qed.

It is shown in [7] that MAX E2LIN2 is inapproximable within standard approximation ratio better than 11/12. By Proposition 2, this bound is transferred to the differential paradigm. Then, the affine reduction just described concludes the proof.