THE MAJORITY RULE AND COMBINATORIAL GEOMETRY (VIA THE SYMMETRIC GROUP)

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Abstract. The Marquis du Condorcet recognized 200 years ago that majority rule can produce intransitive group preferences if the domain of possible (transitive) individual preference orders is unrestricted. We present results on the cardinality and structure of those maximal sets of permutations for which majority rule produces transitive results (consistent sets). Consistent sets that contain a maximal chain in the Weak Bruhat Order inherit from it an upper semimodular sublattice structure. They are intrinsically related to a special class of hamiltonian graphs called persistent graphs. These graphs in turn have a clean geometric interpretation: they are precisely visibility graphs of staircase polygons. We highlight the main tools used to prove these connections and indicate possible social choice and computational research directions.

1. Introduction

Arrow's impossibility theorem [5], says that if a domain of voter preference profiles is sufficiently diverse and if each profile in the domain is mapped into a social order on the alternatives that satisfies a few appealing conditions, then a specific voter is a dictator in the sense that all of his or her strict preferences are preserved by the mapping. One interesting question is how to determine restrictions on sets of voters preference orders which guarantee that every non-empty finite subset of candidates S contains at least one who beats or ties all others under pairwise majority comparisons [14, 15, 17]. When voters express their preferences via linear preference orders over {1, . . . , n } (i.e. permutations in S_n) a necessary and sufficient condition is provided by the following proposition. It identifies embedded 3x3 latin squares as the main reason for intransitivity of the majority rule.

Definition 1.1. A three subset $\{\alpha, \beta, \gamma\} \subset S_n$ contains an embedded 3x3 latin square if there exist $\{i, j, k\} \subset \{1, \ldots, n\}$ such that $\alpha_i = \beta_j = \gamma_k$, $\alpha_j = \beta_k = \gamma_i$ and $\alpha_k = \beta_i = \gamma_j$. $C \subset S_n$ is called consistent if no three subset of C contains an embedded 3x3 latin square.

Proposition 1.2. [15] *For a finite set of voters P with preference orders in a subset C of S_n , denote by $|aPb|$ the number of voters that prefer a to b. For every subset S of at least three candidates,*

$$\{a \in S : \forall b \in S - a, |aPb| \geq |bPa|\} \neq \emptyset$$

if and only if C does not contain an embedded 3 by 3 latin square (i.e. Consistent sets produce transitive results under majority rule).

It has been conjectured that for every n the maximum cardinality of such consistent sets is not more than 3^{n-1} [1]. Maximal consistent sets that contain a

T_6

123456	135642	351264	531624	563421
123546	312456	351624	536124	564312
123564	312546	356124	531642	564321
132456	312564	351642	536142	653124
132546	315246	356142	536412	653142
132564	315264	356412	536421	653412
135246	315624	356421	563124	653421
135264	315642	531246	563142	654312
135624	351246	531264	563412	654321

FIGURE 1. A maximal consistent subset of S_6. It is conjectured to be maximum in [15]

maximal chain in the Weak Bruhat order of S_n are upper semimodular sublattices of cardinality bounded by the n-th Catalan number [4](Theorem 2.2). This result is the basis of an output sensitive algorithm to compute these sublattices (see Remark 2.3 and Corollary 2.4). With such sublattices we associate a class of graphs (called persistent) that offers a bridge from the combinatorics of consistent sets of permutations to non degenerate point configurations (see Section 2.3 and Theorem 2.8). Every graph in this apparently "new" class can be realized as the visibility graph of a staircase polygon(see Section 3). A colorful way to view these abstract connections is that if the aggregate collection of voters is realizable as a non-degenerate collection of points then majority rule produces transitive results. Under this interpretation point configurations represent the candidates aggregate view provided by the voters rankings (one point per candidate).

2. The Weak Bruhat Order, Balanced Tableaux and Persistent Graphs

2.1. The Weak Bruhat Order of S_n. For $n \geq 2$, let S_n denote the symmetric group of all permutations of the set $\{1, \ldots, \text{n}\}$. As a Coxeter group S_n is endowed with a natural partial order called the weak Bruhat order ([2, 4, 12]. This order is generated by considering a permutation γ an immediate successor of a permutation α if and only if γ can be obtained from α by interchanging a consecutive pair of non inverted elements of α. The partial order $\leq WB$ is the transitive closure of this relation. The unique minimum and maximum elements are the identity and the identity reverse respectively, (figure).

$(S_n, \leq WB)$ is a ranked poset where the rank of a permutation α is its inversion number $i(\alpha) = |\{(\alpha_i, \alpha_j) : i < j \text{ and } \alpha_i > \alpha_j\}|$. From now on, consider all permutations in S_n written in one line notation and let s_i denote the adjacent transposition of the letters in positions i and $i+1$. With this convention αs_i is the permutation obtained by switching the symbols α_i and α_{i+1} in α. Every permutation is then representable as a word over the alphabet $\{ s_1, \ldots, s_{n-1} \}$ where the juxtaposition express α as a left to right product of the s_i's. Among these representations, those words that involve exactly $i(\alpha)$ transpositions are called the reduced words

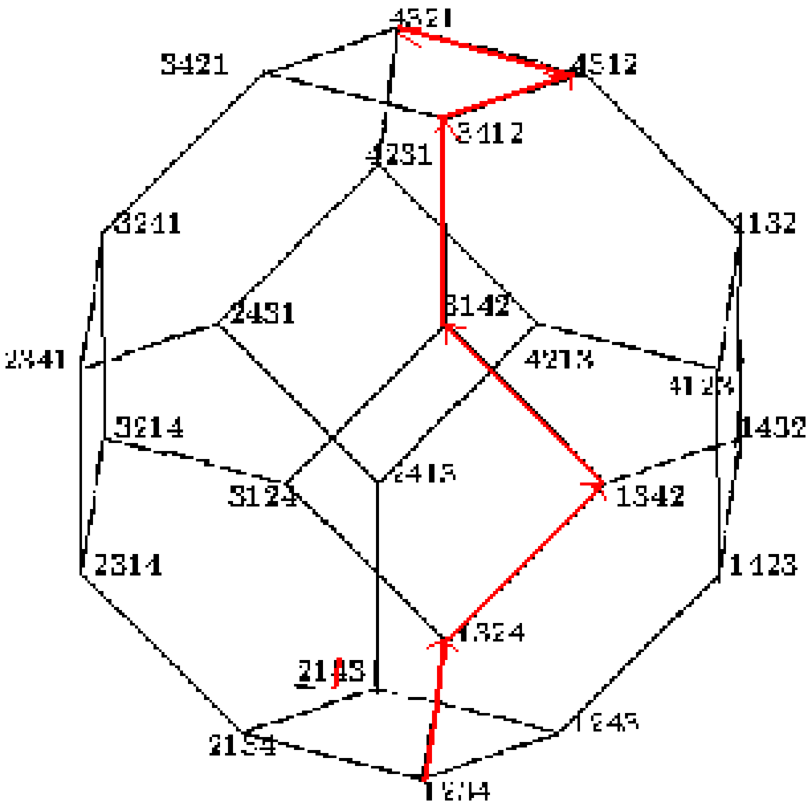

FIGURE 2. The weak Bruhat order for S_4. A maximal chain is highlighted.The identity is at the bottom and the identity reverse is at the top. By suitable relabeling we can in fact have any permutation at the top and its reverse at the bottom.

for α. Those reduced words that represent the maximum element have length $N = (n*(n-1))/2$ and they are the *maximal chains* in $(S_n, \leq WB)$ from the identity permutation to its reverse. They constitute the central combinatorial object in this work. In particular, the majority rule produces transitive results when applied to them. We define now a closure operator that allow us to characterize those maximal consistent sets of permutations that contain maximal chains.

Definition 2.1. For $\alpha \in S_n$, let $Triples(\alpha) = \{(\alpha_i, \alpha_j, \alpha_k) : i < j < k\}$ and for $C \subset S_n$, $Triples(C) = \bigcup\{Triples(\alpha) : \alpha \in C\}$. The Triples closure of a set $C \subset S_n$ is $Closure(C) = \{\alpha \in S_n : Triples(\alpha) \subset Triples(C)\}$.

It is natural to ask how to obtain $Closure(C)$ for a given set $C \subset S_n$. In particular, what is the cardinality and structure of maximal consistent sets? We provide next an answer to these questions for the case that Ch is a maximal chain in $(S_n, \leq WB)$.

2.1.1. *Maximal Connected Consistent Sets.* It is not difficult to see that any three permutations that contain an embedded 3x3 latin square can not be totally ordered in $(S_n, \leq WB)$. This means that a maximal chain Ch is a consistent set. Moreover, $|Triples(Ch)| = 4\binom{n}{3}$. Therefore $Closure(Ch)$ is a maximal consistent set. The size of $Closure(Ch)$ varies widely depending on Ch. In some cases, it is of $O(n^2)$ and in many others is of size $> 2^{n-1} + 2^{n-2} - 4$ for $n \geq 5$ ([1]). It has been conjectured (since 1985) in [2] that the maximum cardinality of a consistent set in S_n is $\leq 3^{n-1}$. The next result provides information about the structure and maximum cardinality of those consistent sets containing a maximal chain in the weak Bruhat order. It is a useful result because it furnishes an algorithm to generate the Closure of a maximal chain Ch. This allow us to have at our disposal all the possible rankings that are compatible with Ch. They represent in this case the maximum allowable set of ranking choices for the voters if we want to obtain transitivite results from the majority rule. Transitivity conditions like Inada's single peakedness [16] correspond to the choice of a particular maximal chain in $(S_n, \leq WB)$.

Theorem 2.2. [4] *The closure of any maximal chain in $(S_n, \leq WB)$ is an upper semimodular sublattice of $(S_n, \leq WB)$ that is maximally consistent. Its cardinality is $\leq$ the nth Catalan number.*

Remark 2.3. The question that comes to mind next is where a permutation $\alpha \in Closure(Ch)$ lives in the Hasse diagram of $(S_n, \leq WB)$?. The answer is that it lies close to Ch. Namely, $Closure(Ch)$ is a connected subgraph (the undirected version) in the Hasse diagram of the weak Bruhat order. To see this let $Path(Ch)$ be the labeled ordered path from the identity to the identity reverse, defined by Ch, in the Hasse diagram of $(S_n, \leq WB)$, ie. $Path(Ch) = (t_1, \ldots, t_N)$ where $t_l = (i, j)$ if the symbols i and j were interchanged by the lth transposition in Ch. Notice that this is an alternate notation referring to the actual symbols in a permutation rather than their positions but it is better suited for this portion of the paper. Let $Path_k(Ch)$ denote the set of permutations appearing in the first k steps of $Path(Ch)$, for $k = 1, \ldots, N$. It follows from the proof of the previous theorem that $Closure(Path_k(Ch))$ has a unique maximum element which is precisely the maximum element in $Path_k(Ch)$. Call this element the kth bottom element. Moreover, $Closure(Path_{k+1}(Ch)) - Closure(Path_k(Ch))$ = a projection of certain connected subset of $Closure(Path_k(Ch))$ that is determined by the adjacent transposition t_{k+1}. This is stated more precisely in the following corollary.

Corollary 2.4. *For a maximal chain Ch in the weak Bruhat order of S_n, let $Projectable_{k+1}(Ch)$ be the set of $\gamma \in Closure(Path_k(Ch))$ for which there exists a downward path from γ to the bottom element of $Closure(Path_k(Ch))$ such that all the adjacent transpositions used in the path are disjoint from t_{k+1}. $Closure(Ch)$ can be computed by an iterated application of the following property.*

$$Closure(Path_{k+1}(Ch)) - Closure(Path_k(Ch)) = \{\alpha \in S_n : \exists \gamma \in Projectable_{k+1}(Ch) \text{ for which } t_{k+1}(\gamma) = \alpha\}$$

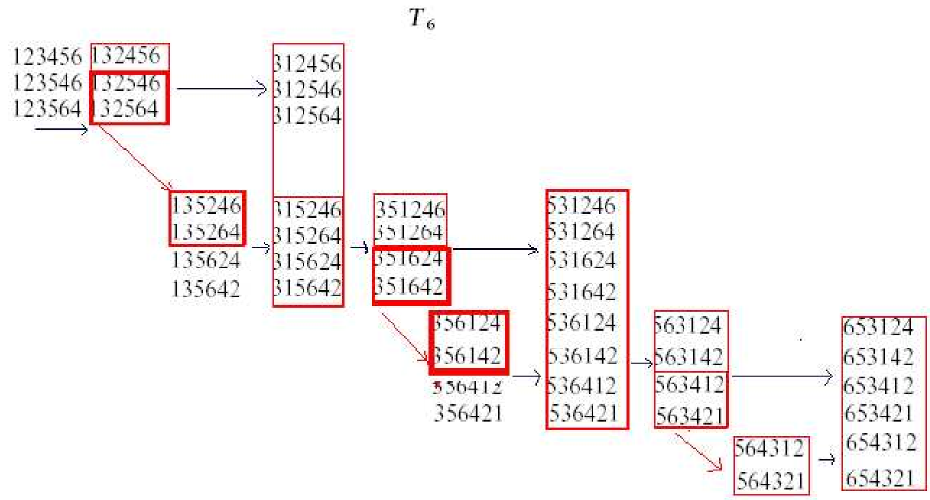

FIGURE 3. The maximal consistent subset of S_6 of Figure 1 viewed as a sublattice of the Weak Bruhat Order.The subsets enclosed in rectangles are the ones obtained by a projection. The maximal chain is the one defined by the sequence of transpositions $Path(Ch) = \{45, 46, 23, 25, 26, 24, 13, 15, 16, 14, 12, 35, 36, 34, 56\}$. Incoming arrows to a rectangle correspond to a single transposition used to project a previous subset. These transpositions are $\{23, 25, 13, 15, 16, 35, 36, 34, 56\}$.

Remark 2.5. The previous corollary can be turned into an algorithm that computes $Closure(Ch)$ in time proportional to $|Closure(Ch)|$, i.e. is an output sensitive algorithm. To our knowledge, no consistent set has been found of cardinality larger than the ones produced by this algorithm. The reason could be that maximal consistent sets that are not connected are not larger than connected ones. Figure 1 is an example of a maximal consistent subset of S_6 with 45 permutations which is conjectured in [15] to be the overall maximum in this case. It was constructed by ad hoc methods but since it contains a maximal chain it can be described succintly as $Closure(Ch)$ where $Path(Ch) = \{45, 46, 23, 25, 26, 24, 13, 15, 16, 14, 12, 35, 36, 34, 56\}$. Its overall structure is illustrated by a coarse drawing of the corresponding sublattice of $(S_6, \leq WB)$ in Figure 2. Each subset obtained by a projection is isomorphic to its pre-image. Incoming arrows into a rectangle depict the pieces that form the preimage of a projection by an adjacent transposition.

Next we present an alternative encoding of these maximal chains by special tableux of staircase shape called balanced tableaux. These tableaux provide the bridge between the weak Bruhat order and special combinatorial graphs called persistent.

2.2. **Balanced Tableaux.** A Ferrer's diagram of staircase shape is the figure obtained from $n-1$ left justified columns of squares of lengths $n-1$, $n-2$, ... , 1. A tableau T of staircase shape is a filling of the cells of the Ferrer's diagram of staircase shape with the distinct integers in the set { 1, . . . , N }. We denote by $SS(n)$ the set of tableux of staircase shape. A tableau $T \in SS(n)$ is said to be balanced if for any three entries $T(i,j)$, $T(j,k)$, $T(i,k)$ we have either $T(i,j) < T(j,k) < T(i,k)$ or $T(i,j) > T(j,k) > T(i,k)$. The key property that we exploit is a beautiful bijection due to Edelman and Greene. Namely, given a maximal chain in $(S_n, \leq WB)$, set $T(i,j) = l$ if and only if i and j are the symbols interchanged in going from the $(l-1)$th permutation to the lth permutation in the chain. It is proved in [12] that this mapping defines a one to one correspondence between balanced tableux in $SS(n)$ and maximal chains in $(S_n, \leq WB)$ (The balanced tableau associated with the maximal chain used in Figure 2 is depicted below).

1					
11	2				
7	3	3			
10	6	14	4		
8	4	12	1	5	
9	5	13	2	15	6

With each balanced tableau T we associate a graph $skeleton(T)$ with vertex set {1, . . . , n} and edge set = { $(j,i) : T(j,i) > T(j,i') \; \forall i', \; i < i' < j$ }. In other words, the edges in $skeleton(T)$ record those entries in T whose values are larger than all the entries above in its column (i.e. they are restricted local maximum in their columns). By the balanced property this is equivalent to { $(j,i) : T(j,i) < T(j,i')$, $\forall \; i < i' < j$ } (i.e. they are restricted local minimum in their rows). The skeleton corresponding to the above balanced tableau(i.e. the maximal chain used in Figure 2) is

1					
1	2				
0	1	3			
0	1	1	4		
0	0	0	1	5	
0	0	0	1	1	6

The reader may be pondering about the properties of these graphs that arise as skeletons of the balanced tableaux associated with maximal chains in the weak Bruhat order.The next section offers a graph theoretical characterization.

2.3. **Persistent Graphs.** Chordal graphs are a well studied class with a variety of applications. We introduce now an ordered version of chordality that together with an additional property called inversion completeness define what we call persistent graphs ([**?**]).

Definition 2.6. A connected graph $G = (V, E)$ with an specified linear ordering $H = (1, \ldots, n)$ on V is called chordal with respect to H if every H-ordered cycle of length ≥ 4 has a chord. G is called inversion complete with respect to H if for every 4-tuple $i < j < k < l$, it is the case that $\{(H_i, H_k), (H_j, H_l)\}$ $\subset$E(G)*impliesthat*(H$_i$, H_l) $\in$E(G). In other words, pairs of edges that interlace in the order provided by H force the existence of a third edge joining the minimum and maximum(in the order) of the involved vertices.

Definition 2.7. A graph $G = (V, E)$ with a Hamiltonian path H is called H-persistent if it is ordered chordal and inversion complete with respect to H.

The following theorem provides a graph theoretical characterization of the skeletons of balanced tableaux. Namely, they are precisely persistent graphs.

Theorem 2.8. *A graph $G = (V, E)$ is H-persistent if and only if is the skeleton of a balanced tableau $T \in SS(n)$ where $|V| = n$ and $H = (1, 2, \dots, n)$.*

Proof Sketch: That the skeleton of a balanced tableau $T \in SS(n)$ is hamiltonian with hamiltonian path $H = (1, \dots, n)$ follows from the definition of the skeleton. That the obtained graph is H-persistent is a consequence of the balanced property. The interesting direction is how to associate with a given H-persistent graph a balanced tableau. The core of the proof relies on the following facts.

(1) Any H persistent graph with at least n edges has at least an edge e such that $G - e$ is H-persistent. Call such an edge a reversible edge.
(2) The complete graph is H-persistent for $H = (1, 2, \dots, n)$ and it is the skeleton of the balanced tableau T where for $j > i$, $T(j, i) = (((j-1) * (j-2)/2) + i)$ for $i \in \{1, \dots, j-1\}$. Each row and column is sorted in increasing order.
(3) Given an H-persistent graph G, [3] presents and $O(n^5)$ algorithm that provides a sequence of persistent graphs that starts with the complete graph K_n and ends with G. The algorithm deletes successively a set $\{e_1, \dots, e_k\}$ of reversible edges and constructs for each $i = 1, \dots, k$ a maximal chain Ch_i in $(S_n, \leq WB)$ such that $skeleton(Ch_i)$ is isomorphic to the persistent graph $G_i = G - \{e_1, \dots, e_i\}$.
(4) G_k is isomorphic to a persistent graph G given as input.

Items $1, 2, 3, 4$ above allow us to conclude that any persistent graph G is the skeleton of a balanced tableau $T \in SS(n)$ where T is the encoding of the maximal chain Ch_k produced by the algorithm where k is the number of edges that have to be reversed in K_n to obtain G. •

Since balanced tableaux and maximal chains in the weak Bruhat order of S_n are just different encodings of the same objects we will abuse notation by using $Skeleton(Ch)$ to refer to the graph associated with the balanced tableau corresponding to Ch. It makes sense them to define an equivalence relation on maximal chains based on the skeletons of their corresponding balanced tableaux. Namely, two maximal chains are related if their corresponding balanced tableaux have the same graph skeleton. The reader may be wandering what this has to do with the majority rule. The answer is that if Ch' is a maximal chain $\subset Closure(Ch)$ then $Skeleton(Ch')$ is identical to $Skeleton(Ch)$, i.e. each maximal connected consistent set C in the weak Bruhat of S_n has a unique persistent graph associated with it. This graph encodes the local column maximums (and local row minimums)of the tableaux associated with any of the maximal chains appearing in C. The corresponding graph represents a global characteristic of the set of rankings which offers a "novel" approach to understanding voters profiles. As an example, the well known single peakedness condition for transitivity corresponds to a very special persistent graph. This line of thinking brings immediately the characterization question, i.e. do persistent graphs characterize maximal connected consistent sets? In other words, is the $Closure$ of a maximal chain Ch equal to the union of all maximal chains Ch' which have the same skeleton as Ch?. The answer is not always. For sure we know that $Closure(Ch)$ is contained in the set of all chains that have the same skeleton as Ch but the reverse is not true. However, we can provide a geometric characterization and this is the purpose of the next section.

3. Maximal Chains in the Weak Bruhat Order with the same Skeleton and Non-degenerate Point Configurations

Let $Conf$ be a non-degenerate configuration of n points on the plane. Without loss of generality, assume that not two points have the same x-coordinate and label the points from 1 through n in increasing order of their x-coordinates. The points in the configuration determine $N = \binom{n}{2}$ straight lines. We can construct a tableau T of shape $SS(n)$ that encodes the linear order on the slopes of these lines by setting $T(i,j) = l$ if and only if the rank of the slope of the line through i and j in this linear order is l. As the reader may suspect the obtained tableau is a balanced tableau and therefore it encodes a maximal chain in the weak Bruhat Order. This chain is precisely the first half of the Goodman and Pollack circular sequence associated with the configuration ([13]). The question is what is a gometric interpretation of the skeleton of the corresponding tableau?. In other words, what geometric property is encoded by the corresponding persistent graph?. The answer lies in the notion of visibility graphs of staircase polygons (Proposition) and that is the subject of the remaining part of this section.

Definition 3.1. Consider a configuration $Conf$ of n points $\{p_1, \ldots, p_n\}$ with coordinates (x_i, y_i) for point p_i. $Conf$ is called a staircase configuration if for every $i < j$ $x_i < x_j$ and $y_i > yj$. A staircase path consists of a staircase configuration plus the $n-1$ straight line segments joining p_i and p_{i+1}, for $i = 1, \ldots, i = n-1$. A staircase polygon P is a staircase path together with the segments from the origin to p_1 and from the origin to p_n.

Definition 3.2. Two vertices p and q of a simple polygon P are said to be visible if the open line segment (p,q) joining them is completely contained in the interior of P or if the closed segment $[p,q]$ joining them is a segment of P itself. The visibility graph of a simple polygon P, $Vis(P) = (V,E)$ where V is the set of vertices of P and E is the set of polygon vertex pairs that are visible.

Proposition 3.3. *The visibility graph of a staircase polygon P with vertexes $\{p_1, \ldots, p_n\}$ is a persistent graph with respect to the hamiltonian path $H = (p_1, \ldots, p_n)$.*

Proof Sketch: The first half period of the Goodman and Pollack circular sequence ([13]) associated with the point configuration, defined by the vertexes of a staircase polygon P, is a maximal chain in the weak Bruhat order. Therefore its associated tableau T which completely encodes the ordering of the slopes is balanced and its associated skeleton is persistent by Theorem 2.8. To see that this graph is identical to the visibility graph of P let m_{ik} denote the magnitude of the slope between points p_i and p_k where $k > i+1$. p_i is visible from p_k if and only if the open line segment joining them lies in the interior of P. For the case of staircase polygons this implies that there is no $j, k < j < i$ such that $m_{ik} \leq m_{ij}$. Therefore $m_{ik} > m_{jk}$ for $j = i-1, i-2, \ldots, k+1$. Since T encodes this ordering this means that v_i is visible from v_k iff $T(i,k)$ is larger than all entries that lie above it, i.e. $T(i,k)$ is a restricted local maximum. •

From the majority rule view point the previous proposition says that when the voters rankings have a corresponding staircase point configuration the candidates

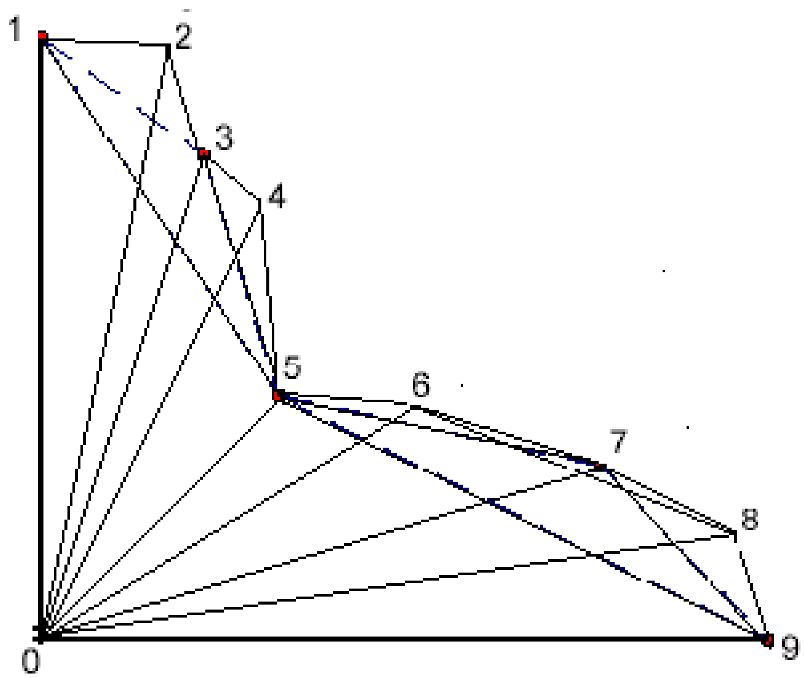

FIGURE 4. A Staircase Polygon. Since vertex 0 that is the origin sees everybody we remove it from consideration.

can be placed on a staircase path and each voter ranking correspond to its view of the candidates in the configuration when the voter is located outside the convex hull of the point set. The local maximum statistics obtained from the slopes ranking are encoded by geometric visibility among the candidates within the corresponding staircase polygon. What about a converse, i.e. Is it clear when is it that the voters rankings have a corresponding staircase configuration?. The next result states that if the set of voters rankings is the $Closure(Ch)$ for $Ch \in (S_n, \leq WB)$ then there exists a staircase polygon P on n points on the plane so that its $Visibility$ graph is isomorphic to the $Skeleton(Ch)$.

Theorem 3.4. [3] *Let M_n denote a maximal consistent set and let Ch be a maximal chain in $(S_n, \leq WB)$. $M_n = Closure(Ch)$ iff $Skeleton(Ch)$ is the visibility graph of a staircase polygon P on n points.*

Proof Sketch: ($\leftarrow$) The visibility graph of a staircase polygon P is identical to $skeleton(T)$ where T encodes the ranking of the $N = \binom{n}{2}$ slopes determined by the n polygon vertices as in the previous proposition. By letting Ch denote the corresponding maximal chain in $(S_n, \leq WB)$ and using Theorem 2.2 the result follows •

Proof Sketch: $(\rightarrow)$ $M_n = Closure(Ch)$ implies that $Skeleton(Ch)$ is H-persistent where $H = (1, 2, \ldots, n)$ by Theorem 2.8 . The difficult part is to prove that there exists a staircase polygon P such that $Vis(P)$ is identical to $Skeleton(Ch)$. The tricky aspect is that Ch may not be realizable at all as a non-degenerate configuration of points. In fact, deciding if a given Ch is realizable in the sense described in this paper is NP-hard. However, what we are able to prove constructively is that there exists a maximal chain Ch' in $(S_n, \leq WB)$ such that $Skeleton(Ch')$ is identical to $Skeleton(Ch)$ even though Ch may not be realizable. This means that there is a geometric staircase ordering of the candidates whose corresponding set of local maximum is the same as those of any chain in $Closure(Ch)$. In other words by lifting the hard question of direct realizability of maximal chains to persistent graphs we get out of a difficult mathematical stumbling block. The essential tool is an inductive geometric simulation of the main steps followed in the proof of Theorem 2.8. Namely, take $Skeleton(Ch)$ and create corresponding geometric steps that produce from a convex staircase configuration, realizing the complete graph K_n, staircase configurations whose visibility graphs are precisely the intermediate persistent graphs $G_i = G - \{e_i, e_2, \ldots, e_i\}$ where the $e_i's$ are reversible edges. In this way a staircase realization of $G_k = Skeleton(Ch)$ is eventually produced. Full details are deferred to the full paper version•

4. Conclusions

Maximal chains in the weak Bruhat order of the symmetric group are consistent sets that determine structurally maximally connected consistent sets. With each such maximal consistent set we associate a persistent graph that turns out to be a visibility graph of a simple polygon. An interpretation of these results is that these classes of voters profiles can be represented by non-degenerate staircase configuration of points(one point per candidate) where each voter ranking is hisview of the point configuration. This offer a meta generalization of conditions for transitivity of the majority rule. Many intereting questions remain to be answered. They are:

(1) Are there any maximal consistent subsets of S_n of larger cardinality than those which are characterized as $Closure(Ch)$ with Ch a maximal chain in $(S_n, \leq WB)$?
(2) Given $C \subset S_n$ what is the complexity of determining if $C \subset Closure(Ch)$ for some Ch a maximal chain in $(S_n, \leq WB)$?
(3) How to generalize the results obtained here to weak orders instead of linear orders?

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