What can we learn from the transitivity parts of a relation?

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Abstract

A transitivity part of a relation on a set X is any subset of X on which the restriction of the relation is transitive. What can be recovered of a relation from the sole knowledge of its transitivity parts? In general, the relation itself cannot be recovered, because it has the same transitivity parts as its converse. In certain situations, the unordered pair formed by the relation and its converse can be recovered. This is the case for relations known to be indecomposable tournaments. The result first appeared in Boussaïri, Ille, Lopez, and Thomassé [2004]. Our proof is simpler, and at the same time conveys some interesting insight into the structure of tournaments.

Key words : tournament, transitivity, transitively determined

In the applications of combinatorics, the problem of efficiently storing relations in the memory of a computer often arises. In the classic case of a partial order, such a storage can evidently be carried out via its Hasse diagram. The topic of this paper stems from similar concerns and uses related ideas. For instance, consider the transitivity parts of a relation, that is, those subsets on which the restriction of the relation is transitive. As is easily checked, all the transitivity parts of a relation can be recovered from the transitivity parts having two or three elements. Are there situations in which more can be deduced from such small transitivity parts? In the case of symmetric and reflexive relations, the question can be recast as a problem about simple graphs investigated by Hayward [1996]. We will come back on this case at the end of our paper.

We are mostly concerned here with tournaments, and our main result—in Theorem 21— characterizes the tournaments which can be fully recovered (up to their converse)

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from their transitivity parts. Unbeknownst to us, the same result had been established independently by Boussaïri, Ille, Lopez, and Thomassé [2004]. While the latter authors proceed, in several instances, by induction on the number of vertices, our method of proof is different and has the interest of providing some new insights on the general structure of tournaments.

Statement of the problem

Definition 1 Let R be a relation defined on a finite set X. We use the abbreviation xy to denote the (ordered) pair (x, y), and we write, as usual, $x\,R\,y$ to mean $xy \in R$ and $R^{-1} = \{xy \mid y\,R\,x\}$ to denote the converse of R. We also sometimes use abbreviations such as $x\,R\,y\,R\,z$ to mean ($x\,R\,y$ and $y\,R\,z$). All the relations mentioned in this paper are implicitly assumed to be on the same finite vertex set X (if not mentioned otherwise). We often refer to the digraph (X, R), and use the corresponding terminology.

A subset Y of X is called a *transitivity part* of a relation R on X if the restriction of R to Y is transitive. The collection of all the transitivity parts of R is denoted by $\mathcal{T}(R)$. We write $\mathcal{T}_3(R)$ for the subcollection of $\mathcal{T}(R)$ containing all the transitivity parts of size 3. Note in passing that $\mathcal{T}_3(R)$ is considerably smaller than $\mathcal{T}(R)$ in some cases. We denote by $\overline{\mathcal{T}}_3(R)$ the collection of all the 3-subsets of X which do not belong to $\mathcal{T}_3(R)$. Occasionally, when no ambiguity can arise regarding the relation R, we may use abbreviations such as $\mathcal{T}_3$ or $\overline{\mathcal{T}}_3$ to mean $\mathcal{T}_3(R)$ or $\overline{\mathcal{T}}_3(R)$, respectively.

Definition 2 A *tricycle* of a relation R is a cycle of length 3. A *trio* is a 3-subset of X consisting of the vertices of some tricycle.

Definition 3 A relation R is a *tournament* on X if R is complete and asymmetric on X; thus, either $x\,R\,y$ or $y\,R\,x$ for all distinct $x, y \in X$, and $\neg(x\,R\,x)$ for all $x \in X$. It is clear that when R is a tournament, $\mathcal{T}(R)$ can be obtained from $\mathcal{T}_3(R)$; in fact, $Y \in \mathcal{T}(R)$ if and only if each 3-subset of Y is in $\mathcal{T}_3(R)$; moreover, $Y \in \overline{\mathcal{T}}_3(R)$ if and only if Y is a trio.

Notice that distinct tournaments can have the same transitivity parts. Indeed, a tournament R and its converse R^{-1} always have the same trios. Accordingly, our aim in the sequel is the recovery of the unordered pair $\{R, R^{-1}\}$, rather than R itself. As a first step of such a recovery, we can thus arbitrarily fix $a\,R\,b$ for some initial vertices a and b. (Fixing $b\,R\,a$ rather than $a\,R\,b$ amounts to exchanging R for R^{-1}.) For other examples of tournaments with exactly the same transivity parts, take any two strict linear orders on X. Many other examples are easily manufactured.

Definition 4 A tournament R on X is (*transitivity*) *determined* whenever, for any tournament S on X, the equality $\mathcal{T}(R) = \mathcal{T}(S)$ implies $R = S$ or $R = S^{-1}$.

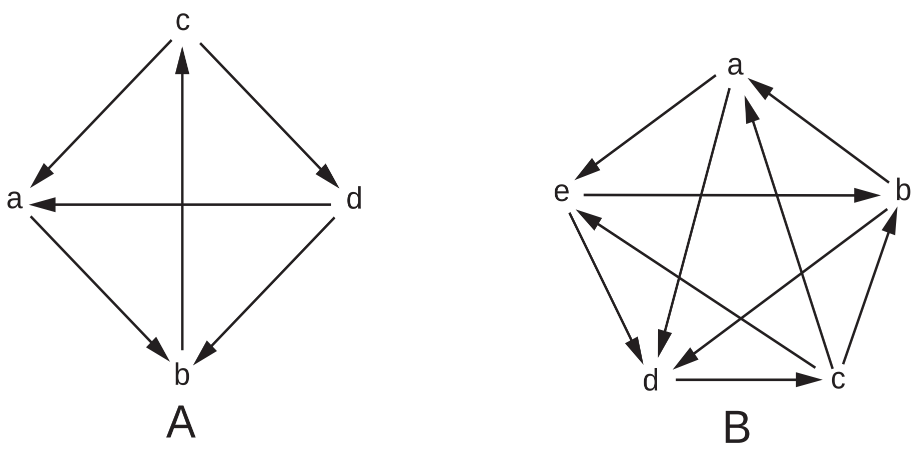

Figure 1: In the tournament represented by the graph of Figure 1A, the pair da can be reversed without altering $\mathcal{T}_3$, so this tournament is not determined. Neither is the tournament of Figure 1B, but the argument is sligthly more involved: this case fails the test derived from Theorem 21.

It is easy to find tournaments which are *not* determined. Two examples are given in Figure 1. In Figure 1A, which defines a tournament on the vertex set $\{a, b, c, d\}$, we can check that the pair da can be transposed without altering $\mathcal{T}_3 = \{\{a, c, d\}, \{a, b, d\}\}$ or $\overline{\mathcal{T}}_3 = \{\{a, b, c\}, \{b, c, d\}\}$. The tournament displayed in Figure 1B is not determined either. (It fails the test derived from Theorem 21.)

There are, however, some tournaments which are determined, such as those represented in Figures 2A and 2B. Indeed, the tournament of Figure 2A (up to its converse) is defined by fixing one initial pair, and then using the information conveyed by

$$\mathcal{T}_3 = \{\{a, b, c\}, \{a, b, e\}, \{a, c, d\}, \{b, d, e\}, \{c, d, e\}\},$$

or equivalently by

$$\overline{\mathcal{T}}_3 = \{\{a, e, d\}, \{a, b, d\}, \{a, e, c\}, \{b, d, c\}, \{b, e, c\}\}.$$

Say we fix the pair ae marked 1 in Figure 2A. The other pairs are then automatically obtained by completing the tricycles, for example in the order 2, 3, …, 10. The tournament of Figure 2B is also determined, but the verification is more complicated. Indeed, two types of inferences can be drawn from $\mathcal{T}_3(R)$ or $\overline{\mathcal{T}}_3(R)$ in the recovery of the pair $\{R, R^{-1}\}$:

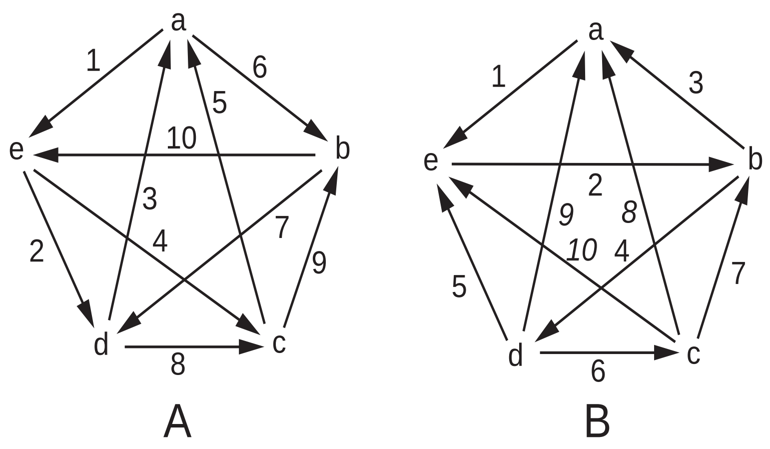

Figure 2: The tournament represented by the graph of Figure 2A is determined by the set of its trios, by applying only Type 1 inferences. By contrast, both types of inferences are required to show that the tournament in Figure 2B is determined (see text).

TYPE 1. If $\{x, y, z\} \in \overline{\mathcal{T}}_3$, then $x\,R\,y$ entails $y\,R\,z$ and $z\,R\,x$.

TYPE 2. If $\{x, y, z\} \in \mathcal{T}_3$, then $x\,R\,y$ and $y\,R\,z$ entails $x\,R\,z$.

In the tournament of Figure 2A, only Type 1 inferences were used to reconstruct the pair $\{R, R^{-1}\}$, while both types of inferences are required in the case of the tournament of Figure 2B. (The numbers 8, 9 and 10 in italics in Figure 2B refer to inferences of Type 2.) These considerations suggest the following problem:

Problem 5 *Characterize the tournaments which are determined.*

Notice an important feature of Definition 4. The quantification "for any tournament S on X" means that we suppose that the unordered pair $\{R, R^{-1}\}$ to be uncovered comes from a tournament R. Thus, the context of tournaments is assumed from the outset. A similar qualification applies to our generalization of Problem 5 in the last section, in which we assume that a family of relations is given (see Problems 23, 24, 25). In the case of Hayward [1996], for instance, the family of reflexive and symmetric relations on X forms the context.

Tournament concepts

Definition 6 A tournament R on X is *strongly connected* or *strong* if for any two vertices x and y in X, there is a directed path from x to y.

According to Moon Theorem, a tournament is strong if and only if it is Hamiltonian (cf. Bang-Jensen and Gutin [2001, Theorem 1.5.1], Gross and Yellen [2004], Laslier [1997]). The next characterization is easily checked.

Proposition 7 *A tournament R on X is strong if and only if for any proper subset Y of X, there exist $x, y \in Y$ and $u, v \in X \setminus Y$ such that $x\,R\,u$ and $v\,R\,y$ (we may have $x = y$ or $v = w$, but not both).*

Definition 8 A tournament R on X is *decomposable* if there exists some nontrivial partition $\{C_1, C_2, \ldots, C_k\}$ of X such that, for all distinct indices $i, j \in \{1, \ldots, k\}$,

$$\forall x, y \in C_i, \;\; \forall u, v \in C_j, \qquad x\,R\,u \Rightarrow y\,R\,v.$$

Any indecomposable tournament on at least three vertices is strong, but the converse is false. An example of a strong, decomposable tournament is given in Figure 3.

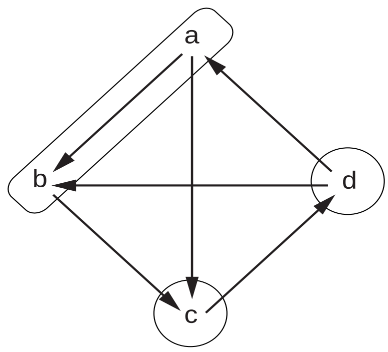

Figure 3: An example of a strong decomposable tournament. The nontrivial partition of the vertex set $\{a, b, c, d\}$ is $\{\{a, b\}, \{c\}, \{d\}\}$.

The partition of Definition 8 being nontrivial, it contains a class that is a proper subset of X with more than one vertex. Thus the following holds:

Proposition 9 *A tournament R on X is indecomposable if and only if for any proper subset Y of X with more than one vertex, there exist $x, y \in Y$ and $z \in X \setminus Y$ such that $x\,R\,z$ and $z\,R\,y$.*

The concepts of a strong and of a decomposable tournament have become classical ones (see e.g. Bang-Jensen and Gutin [2001], Gross and Yellen [2004], Laslier [1997]). We now turn to some further tools and facts that will be instrumental in the proof of our main result.

Definition 10 If V and W are two relations, we write as usual $V\,W = \{xy \mid \exists z,\ \text{with } x\,V\,z\,W\,y\}$ for their products. Let R be a tournament on X, and let Q be the set of pairs belonging to some tricycle of R; thus,

$$Q = R \cap R^{-1}R^{-1}. \tag{1}$$

We call Q the *tricyclic relation* of the tournament R. The pairs of R which do not belong to any tricycle of R form the relation

$$P = R \setminus Q = R \setminus (R^{-1}R^{-1}). \tag{2}$$

Notice that P is an irreflexive linear order if and only if $R = P$, that is, $Q = \varnothing$. In general, we refer to P as the *order* of the tournament R, a terminology justified by the next lemma.

Lemma 11 *Suppose that R is a tournament on X. Then the relation $P = R \setminus (R^{-1}R^{-1})$ is an irreflexive partial order on X.*

PROOF. Because R is irreflexive, so is P by definition. To prove the transitivity suppose that $x\,P\,y$ and $y\,P\,z$ with $x \neq z$ for three vertices x, y and z in X. Since $xy \notin R^{-1}R^{-1}$, we cannot have $z\,R\,x$; as R is a tournament, we must have $x\,R\,z$. In fact, $x\,P\,z$ is true because $x\,R^{-1}R^{-1}\,z$ cannot hold. Suppose indeed that $x\,R^{-1}R^{-1}\,z$. There must exists $w \in X$ such that $z\,R\,w$ and $w\,R\,x$. By the completeness of R, we must have either $y\,R\,w$ or $w\,R\,y$. The first case leads to $x\,R^{-1}R^{-1}\,y$ (via $y\,R\,w$ and $w\,R\,x$), and the second one to $y\,R^{-1}R^{-1}\,z$ (via $z\,R\,w$ and $w\,R\,y$), contradicting our hypothesis that $x\,P\,y$ and $y\,P\,z$. □

We now construct a partition of the tricyclic relation Q based on the fact that two pairs in Q can belong to the same tricycle.

Definition 12 Let Q be the tricyclic relation of a tournament R on X (cf. Definition 10). Let then $S \subseteq Q \times Q$ be a relation defined on Q by

$$xy\,S\,zw \quad\Longleftrightarrow\quad \begin{cases} y = z \quad \text{and} \quad x\,Q\,y\,Q\,w\,Q\,x, \\ \text{or} \\ w = x \quad \text{and} \quad x\,Q\,y\,Q\,z\,Q\,x. \end{cases}$$

Because S is symmetric, the reflexive and transitive closure $\widehat{S}$ of S is an equivalence relation on Q. The partition of Q induced by $\widehat{S}$ is the *tricyclic coloring* of Q (or of R); its classes will be called the *tricyclic colors* of Q or of R. Intuitively, pairs xy and zw of Q are in the same tricyclic color C when there is a sequence of pairs in Q, starting from xy and ending in zw, such that two successive pairs belong to some common tricycle. Such a sequence, which lies entirely in C, will be refered to as a *color sequence*. Notice that each tricylic color is a subset of Q, and thus a relation on X.

For an example of tricyclic coloring, take the tournament of Figure 1A; there is only one tricyclic color, namely $\{ab, bc, ca, cd, db\}$. In the tournament of Figure 1B, there are two tricyclic colors, namely $\{ae, eb, ba\}$ and $\{ad, dc, ca, cb, bd, ce, ed\}$.

Definition 13 Given a relation S on X, a vertex x is *covered* by S when there is $x\,S\,y$ or $y\,S\,x$ for some $y \in X$.

Lemma 14 *Let C be a tricyclic color of the tournament R on X, and let Y be the set of vertices covered by C. Then for all $x, y \in Y$ and $z \in X \setminus Y$:*

$$x\,R\,z \quad \Longrightarrow \quad y\,R\,z.$$

Intuitively, all the vertices covered by a tricyclic color C behave the same way with respect to any vertex not covered by C.

PROOF. Assume first $t\,C\,u\,C\,v\,C\,t$ for some $t, u, v \in X$ (thus $t, u, v \in Y$), and $z \in X \setminus Y$. If $t\,R\,z$, then $v\,R\,z$. Indeed, $z\,R\,v$ would give the tricycle $z\,R\,v\,R\,t\,R\,z$, and by the definition of a color, we would have $tz \in C$, contradicting $z \notin Y$. Similarly, we deduce $u\,R\,z$ from $v\,R\,z$.

Now take x, y, z as in the statement of the lemma, with $x\,R\,z$. Because x is in Y, there must be some $s \in Y$ such that $xs \in C$ or $sx \in C$. Similarly, there must be some $t \in Y$ such that $yt \in C$ or $ty \in C$. Suppose that $xs, ty \in C$. (The argument is the same in the other cases.) By the definition of a color, there is a color sequence from xs to ty. The argument of the previous paragraph, applied to each step of that sequence, yields ultimately $y\,R\,z$. □

Suppose that two tricycles of different colors jointly cover exactly one vertex w. Thus, their union cover in all five vertices (see Figure 4) and four further pairs of those vertices lie in the tournament. The crux of Lemma 15 below is that these four pairs necessarily form two tricycles sharing a pair of one of the original tricycles and so are of the same color as that tricycle.

Lemma 15 *Suppose that two trios of a tournament R on X share a single vertex w. Suppose moreover that the pairs of the two corresponding tricycles belong to distinct tricyclic colors C and D, say $\{wx, xy, yw\} \subseteq C$ and $\{wu, uv, vw\} \subseteq D$. Then, we have*

either (i) $yu, vx, yv, ux \in C$,
or (ii) $yu, vx, vy, xu \in D$.

PROOF. We must have both $y\,R\,u$ and $v\,R\,x$ because otherwise the pairs of the two tricycles $w\,R\,u\,R\,y\,R\,w$ and $w\,R\,x\,R\,v\,R\,w$ would belong to the same tricyclic color. We now have two cases: (i) $y\,R\,v$, forming the tricycle $y\,R\,v\,R\,x\,R\,y$ in C and entailing $u\,R\,x$

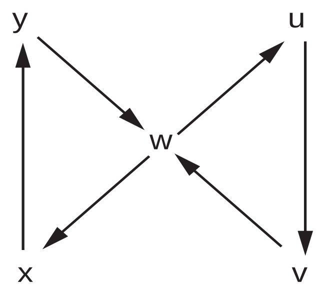

Figure 4: The hypotheses of Lemma 15 and the first step in the proof.

and the tricyle $u\,R\,x\,R\,y\,R\,u$ (because $x\,R\,u$ would yield $vx \in C \cap D$); (ii) $v\,R\,y$, forming the tricycle $v\,R\,y\,R\,u\,R\,v$ in D and entailing $x\,R\,u$ and the tricycle $x\,R\,u\,R\,v\,R\,x$ (because $u\,R\,x$ would yield $yu \in C \cap D$). These are the two cases of the lemma. ◻

Lemma 16 *Suppose that R is an indecomposable tournament on X, with Q its tricyclic relation and $\mathcal{Q}$ its tricyclic coloring. Then*

(i) *Q covers X;*

(ii) *$\mathcal{Q} = \{Q\}$, that is: there is only one tricyclic color.*

PROOF. Pick arbitrarily some tricyclic color C of $\mathcal{Q}$. Indecomposability of R, as characterized in Proposition 9, together with Lemma 14 imply that C covers X.

It remains to show $C = Q$, that is $C = D$ for any tricyclic color D. Proceeding by contradiction, we suppose $C \neq D$. Take any vertex $w \in X$. Because as just shown both C and D cover X, there must exist a tricycle $\{wx, xy, yw\} \subseteq C$ and another tricycle $\{wu, uv, vw\} \subseteq D$. This is the situation described by the hypotheses of Lemma 15. (A glance at Figure 4 may be helpful.) Thus, either Case (i) or Case (ii) of the Lemma must be true. There is no loss of generality in assuming that Case (i) holds, that is yu, vx, yv, $ux \in C$. Because D covers X, we must have $xk \in D$ or $kx \in D$ for some $k \in X \setminus \{x\}$. Thus according to Definition 12, there is some color sequence starting at wu and ending at xk or kx. Applying Lemma 15 repeatedly, we derive $x\ell \in C$ or $\ell x \in C$ for each ℓ covered by the pairs in the sequence. Then we have $xk \in C \cap D$ or $kx \in C \cap D$, giving $C = D$, a contradiction of our hypothesis $C \neq D$. Thus $C = D$ and therefore $C = Q$. ◻

Remark 17 The converse of Lemma 16 does not hold, as shown by the decomposable tournament R in Figure 3 (for which $Q = R \setminus \{ab\}$).

We now strengthen the necessary condition in Lemma 16 for a tournament to be indecomposable in order to get a necessary and sufficient condition. In view of later use in the proof of Theorem 21, we formulate the additional requirement in terms of the covering relation or Hasse diagram H of the order $P = R \setminus R^{-1}\, R^{-1}$ of R. Notice however that reformulating the quantification in Condition (iii) as “for any pair xy in P” would give an equally correct result (as seen from the next proof).

Proposition 18 *A tournament R on X, with $|X| \geq 3$, is indecomposable if and only if the order P and the tricyclic relation Q of R satisfy the following three conditions:*

(i) *Q covers X;*

(ii) *$\mathcal{Q} = \{Q\}$;*

(iii) *for any pair xy in the Hasse diagram H of P, there exists z in $X \setminus \{x, y\}$ satisfying $x\, R\, z\, R\, y$.*

PROOF. If R is indecomposable, Conditions (i) and (ii) hold by Lemma 16. For Condition (iii), the definition of indecomposability implies the existence of z in $X \setminus \{x, y\}$ such that either $x\, R\, z\, R\, y$ or $y\, R\, z\, R\, x$. The second formula cannot be true, because the pair xy, which lies in P, does not belong to any tricycle of R.

If R is decomposable, let us assume that Conditions (i) and (ii) hold, and derive that Condition (iii) fails. By assumption, there exists a proper subset Y of X with more than one element, and such that for $x, y \in Y$ and $z \in X \setminus Y$ we have $x\, R\, z$ implies $y\, R\, z$. Notice that Y cannot contain both vertices of any pair st from Q (otherwise Conditions (i) and (ii) could no be together true: some color sequence must start at st and lead to some pair covering a given vertex outside Y. At some step of the sequence, there appears a tricycle with two vertices in Y and one outside Y, contradicting the choice of Y). Thus all pairs of R formed by two vertices from Y are in P, in other words: P induces on Y a linear order. Let x be the minimum vertex for this order on Y, and let y be the next vertex in Y. Then the pair xy invalidates Condition (iii). Indeed, xy lies in the Hasse diagram H of P, otherwise there would exist $t \in X \setminus \{x, y\}$ satisfying $x\, P\, t\, P\, y$. By the choice of Y, we then have $t \in Y$, and this contradicts the choice of y. Finally, $x\, R\, z\, R\, y$ cannot hold for any $z \in X \setminus Y$ because of the choice of Y. □

Remark 19 The three conditions in Proposition 18 are independent. Three (necessarily decomposable) tournaments failing in turn exactly one of these three conditions are easily built. For instance, Figure 5 gives a counterexample for Condition (i), Figure 1B for Condition (ii) and Figure 3 for Condition (iii).

Corollary 20 *The problem of deciding whether a given tournament R on X is indecomposable is polynomial in the size of X.*

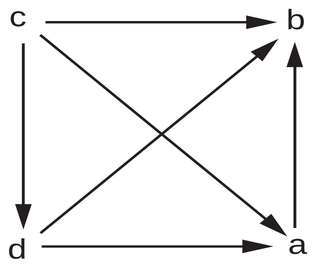

Figure 5: An example showing the independence of Condition (i) in Proposition 18.

PROOF. An algorithm directly based on Proposition 18 is outlined below. (We do not claim that this algorithm is the most efficient one.) Assume R is a tournament given on X, with $|X| \geq 3$.

MAIN STEP 1. Search for a pair uv belonging to some tricycle. If no such pair exists, output that R is decomposable and exit.

MAIN STEP 2. Build the tricyled color C of uv. If C does not cover X or does not contain all pairs belonging to some tricycle, ouptut that R is decomposable and exit.

MAIN STEP 3. Build $P = R \setminus C$. For each pair xy in the Hasse diagram of P, check that Condition (iii) from Proposition 18 holds. If it is the case, output that R is indecomposable, otherwise that R is decomposable.

Each of the three Main Steps above can be performed in time polynomial in $|X|$. □

The main result

Theorem 21 *A tournament is determined if and only if it is indecomposable.*

PROOF. Let R be a tournament on X and suppose that R is decomposable. Consider a subset Y invalidating the condition in Proposition 9 (thus $|Y| \geq 2$). Let T be the restriction of R to Y. Then R and $(R \setminus T) \cup T^{-1}$ are tournaments on X which have exactly the same trios. However, $(R \setminus T) \cup T^{-1}$ differs from both R and R^{-1}. Thus a decomposable tournament is not determined.

Conversely, assume R is an indecomposable tournament on X, with $|X| \geq 3$ (for $|X| = 2$, the Theorem holds trivially). By Lemma 16 the tricyclic relation Q of R covers X, and consequently $\overline{\mathcal{T}}_3(R) \neq \varnothing$. Select two vertices a, b in some trio. We may fix

$a\,R\,b$. The goal then is to show that for any other unordered pair $\{x, y\}$ of vertices, we can decide which of $x\,R\,y$ and $y\,R\,x$ holds, that is, we can recover $xy \in R$ or $yx \in R$. Applying inferences of Type 1 starting from $a\,Q\,b$, we are able to recover all pairs xy which belong to Q: this is true because of Lemma 16, which tells us that there is a color sequence from the pair ab to any other pair in Q, and at each step of the sequence we can apply a Type 1 inference. There remains to show that if neither xy nor yx is in Q, it is nevertheless possible to decide for $xy \in R$ or $yx \in R$ solely on the basis of Q and $\overline{\mathcal{T}}_3(R)$. (This is the situation in Figure 2B.) Suppose that $xy \in P$, with P the order of R. (The argument in the case $yx \in P$ is similar). Denote by $\tilde{P}$ the subset of P consisting of the pairs xy of P for which it has been proved that recovering of $xy \in R$ is possible. Thus, at the start, $\tilde{P}$ is empty, and we need to show that after all possible, repeated inferences have been made, $\tilde{P} = P$. We consider three cases for a pair $xy \in P$.

Case (i). There exists some vertex w such that $x\,P\,w$ and y, w are incomparable in P. Then either $y\,Q\,w$ or $w\,Q\,y$, and moreover we have been able to decide which one holds, see paragraph above. Assuming $y\,Q\,w$, we show that we can derive $xy, xw \in \tilde{P}$ by using inferences of Types 1 and 2; the other case, that is $w\,Q\,y$, is similar.

Because R is indecomposable and $|X| \geq 3$, there exists some vertex s with $s\,R\,x$. By Lemma 16, s is covered by the tricyclic color of yw, and this color is equal to the whole of Q. So, there is a color sequence from yw to some pair covering s. The first time a vertex c outside $\{t \in X \mid x\,P\,t\}$ appears in a pair of the sequence, we find vertices a, b, c such that ab is a pair of the sequence and $x\,P\,a$, $x\,P\,b$ and $a\,Q\,b\,Q\,c\,Q\,a$ hold but not $x\,P\,c$. As $c\,R\,x$ cannot hold (because of $x\,P\,b$ and $b\,R\,c$), we have $x\,R\,c$ and thus $x\,Q\,c$. Using $\{x, a, c\} \in \mathcal{T}_3$ together with $x\,Q\,c$ and $c\,Q\,a$, we obtain $x\,R\,a$ by an inference of Type 2, and then $x\,R\,b$ also by such an inference on $x\,R\,a$ and $a\,Q\,b$. Now, following the color sequence backwards from ab to yw and repeatedly applying inferences of Type 2, we deduce $xy, yw \in R$, thus $xy, yw \in \tilde{P}$.

Case (ii). There exists some vertex w such that $w\,P\,y$ and x, w are incomparable in P. This case can be settled in the same way as Case (i) was (in fact, replacing R with R^{-1} transforms Case (ii) into Case (i)).

Case (iii). We still need to establish $xy \in \tilde{P}$ in all situations not covered by Cases (i) or (ii). Let us first assume that xy is moreover in the Hasse diagram H of the order P. Because the tournament R is indecomposable and $|X| \geq 3$, Proposition 18 implies the existence of some vertex z such that $xz, zy \in R$. By the definition of the Hasse diagram H of R, we have $xz \notin P$ or $zy \notin P$. If both of these formulas hold, that is $xz, zy \in Q$, we deduce $xy \in \tilde{P}$ by an inference of Type 2 based on $\{x, z, y\} \in \mathcal{T}_3$. If $x\,P\,z$ and $z\,Q\,y$, we are in Case (i) and so we have $x\,\tilde{P}\,y$. In the last possibility, that is $x\,Q\,z$ and $z\,P\,y$, we are in Case (ii) and $x\,\tilde{P}\,y$ holds then also.

Now if $xy \in P \setminus H$, there exists a sequence $v_1 = x$, v_2, …, $v_k = y$ of vertices such that $v_i\,H\,v_{i+1}$ for $i = 1$, 2, …, $k-1$. By the preceding paragraph, $v_i v_{i+1} \in \tilde{P}$. For $i = 1$, 2, …, $k-2$, we deduce $v_i v_{i+2} \in \tilde{P}$ from $\{v_i, v_{i+1}, v_{i+2}\} \in \mathcal{T}_3$ by a Type 2 inference. Applying the same argument as many times as required, we will conclude $xy \in \tilde{P}$, which completes the proof. □

A generalization

Definition 4 can be generalized to other families of relations than tournaments (we mainly think here of a family of relations defined by first order axioms on a single relation, as for instance reflexiveness). The identity relation I on the set X consists of all loops xx. As in many cases adding or suppressing loops do not alter transitivity, we do not require that the loops of a relation be determined by its transitivity parts.

Definition 22 Let $\mathcal{C}$ be a family of relations on the set X. A relation R from $\mathcal{C}$ is *(transitivity) determined* when for any relation S from $\mathcal{C}$ the following holds: $\mathcal{T}(R) = \mathcal{T}(S)$ only if $R \Delta S \subseteq I$ or $R \Delta S^{-1} \subseteq I$ (here, Δ denotes symmetric difference). The whole family itself is *determined* if any of its relations is determined. In general, let $\mathcal{C}^*$ denote the subfamily of $\mathcal{C}$ consisting of the relations in $\mathcal{C}$ which are determined.

Notice that a relation is determined (or not) only with respect to some given family (changing the family may change the status of the relation determinateness). We now formulate a whole scheme of problems (one problem for each family of relations chosen).

Problem 23 *For a given family $\mathcal{C}$ of relations, find the subfamily $\mathcal{C}^*$ of determined relations, in the sense of Definition 22. Is the problem of deciding whether a relation from $\mathcal{C}$ is determined polynomial in the size of X?*

Particular cases of Problem 23 have been solved already. For instance, Theorem 21 settles the questions for the family $\mathcal{C}$ of tournaments (see also Boussaïri et al. [2004]); it states that $\mathcal{C}^*$ then consists of the indecomposable tournaments. Besides, Corollary 20 asserts that the corresponding decision problem is polynomial.

Next, consider the family $\mathcal{C}$ of all reflexive and symmetric relations on X. Remark that for any $R \in \mathcal{C}$, the subcollection $\mathcal{T}_3(R)$ conveys exactly the same information as $\mathcal{T}(R)$ because the restriction of R to a subset Y of X is transitive if and only if the restriction of R to any 3-element subset of Y is transitive. Moreover, any R in $\mathcal{C}$ corresponds exactly to one (simple) graph $G = (X, E)$, where $\{x, y\} \in E$ if and only if xRy. Under this recasting, finding all the relations R in $\mathcal{C}$ which are determined becomes a question discussed by Hayward [1996] under the following form. Recall that the P_3 structure of a graph G consists of the subsets of 3 vertices on which G induces a P_3 path. Thus, in this case, Problem 23 becomes: Which graphs G are recoverable from their P_3 structure? This question was partially (but elegantly) solved by Hayward [1996, Theorem 4.3 and Corollary 4.4], which in particular builds a polynomial algorithm for recognition.

For the family $\mathcal{C}$ of all relations on X, we are intrigued by the difficulty of the resulting instance of Problem 23. An example of determined relation is the full relation R with no loop (notice $\{a, b\} \notin \mathcal{T}(R)$, for any distinct vertices a, b, which in turn implies $ab, ba \in R$).

Much more ambitious problems are the following general ones.

Problem 24 *Find all families of relations which are determined in the sense of Definition 22.*

Problem 25 *Characterize those families $\mathcal{C}$ for which deciding whether a relation from $\mathcal{C}$ is determined is a polynomial problem.*

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