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# Two-Person and N-Person Red-and-Black Games


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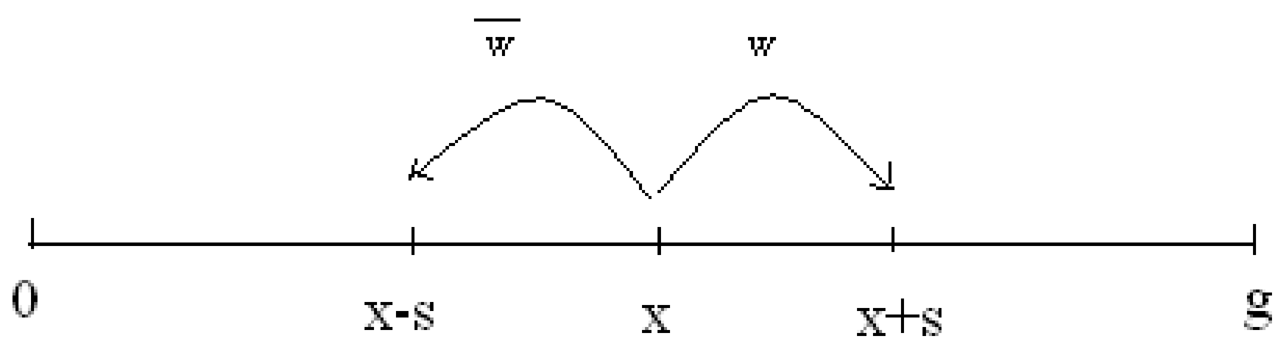
# Outline

- Original Red-and-Black gambling problem
- Two-person Red-and-Black games:
  - Proportional Two-Person Red-and-Black
  - Weighted Two-Person Red-and-Black
- Non-constant sum Red-and-Black
- Bayesian Red-and-Black

# Original Red-and-Black

Introduced by Dubins and Savage (1965):

- A player starts with an **initial fortune** $x \geq 0$
- He wants to reach the **goal** of **g** units
- At each stage he **bids s** units, not greater than his current fortune
- With **probability** w the player **wins** the s units
- With **probability** 1- w the player **loses** the s units



Law of motion:

$$x^1 = \begin{cases} x+s & \text{w.p.} \quad w \\ x-s & \text{w.p.} \quad \overline{w} = 1-w \end{cases}$$

The solution of the game depends on the value of w:

- **Subfair case: w < 1/2**. An optimal strategy is bold play, which corresponds to always staking the entire fortune or just what is needed to reach the target, whichever is smaller (Dubins and Savage, 1976);
- **Superfair case: w > 1/2**. An optimal strategy is timid play, which consists of always staking 1 unit until he reaches the goal (Ross, 1974; Maitra and Sudderth, 1996).
- **Fair case: w = 1/2**. Any strategy with positive bets is optimal.

# Two-Person Red-and-Black

- Two **players I and II** play against each other
- Player I starts with an **initial fortune** $x \geq 0$; player II starts with an **initial fortune** $y \geq 0$
- They both want to reach the **target fortune g** $(g = x+y)$
- At each stage I bids an integral amount **a** and II bids an integral amount **b**
- With some positive probability **player I wins b** and with the remaining probability **player II wins a**
- The players have **probabilities of winning** that are not fixed, but are **functions** of their bets

# The **stochastic game**:

- Set of players: $N = \{\mathrm{I}, \mathrm{II}\}$
- State space: S = {0,1, … , g} = fortunes held by player I
- Sets of actions:

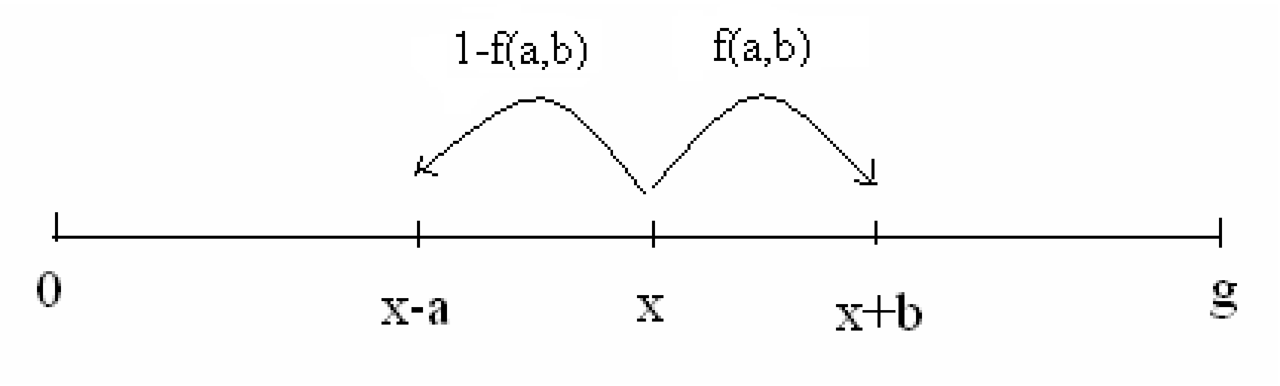
$$A(x) = \begin{cases} \{1,2,\dots,x\} & \text{if } x \in \{1,2,\dots,g-1\} \\ \{0\} & \text{if } x \in \{0,g\} \end{cases} \qquad B(x) = \begin{cases} \{1,2,\dots,g-x\} & \text{if } x \in \{1,2,\dots,g-1\} \\ \{0\} & \text{if } x \in \{0,g\} \end{cases}$$

- Payoff function:

$$\varphi_j = \begin{cases} 1 & \text{if} \quad x_j = g \\ 0 & \text{if} \quad x_j = 0 \end{cases} \qquad \text{for} \quad j = \mathrm{I}, \mathrm{II}$$

- Law of motion for player I:

$$x^1 = \begin{cases} x + b & \text{w.p.} \quad f(a,b) \\ x - a & \text{w.p.} \quad 1 - f(a,b) \end{cases}$$



- Proportional two-person red-and-black:
  - Law of motion for player I:

$$x^{m+1} = \begin{cases} x^m + b & w.p. \quad \dfrac{wa}{wa + \overline{w}b} \\[2ex] x^m - a & w.p. \quad \dfrac{\overline{w}b}{wa + \overline{w}b} \end{cases}$$

  - Theorem 4.1 Pontiggia (2005):

    In a proportional two-person red-and-black game that is subfair for player I and superfair for player II, i.e. $0<w<\frac{1}{2}$, it is Nash for player I to play a bold strategy and for player II to play a timid strategy. Vice-versa, if the game is superfair for player I and subfair for player II, then the profile (Timid, Bold) is a Nash Equilibrium. These Nash Equilibria are unique for the corresponding game.

- Weighted two-person red-and-black:
  - Law of motion for player I:

$$x^{m+1} = \begin{cases} x^m + b & w.p. \quad w\dfrac{a}{a+b} \\ \\ x^m - a & w.p. \quad 1 - w\dfrac{a}{a+b} \end{cases}$$

  - Theorem 3.2 Pontiggia (2005):

    In a weighted two-person red-and-black game that is subfair for player I, i.e. 0 < w < 1, it is Nash for player I to play a bold strategy and for player II to play a timid strategy. This NE is unique.

- Sketch of **proof** for Theorems 3.2 and 4.1:

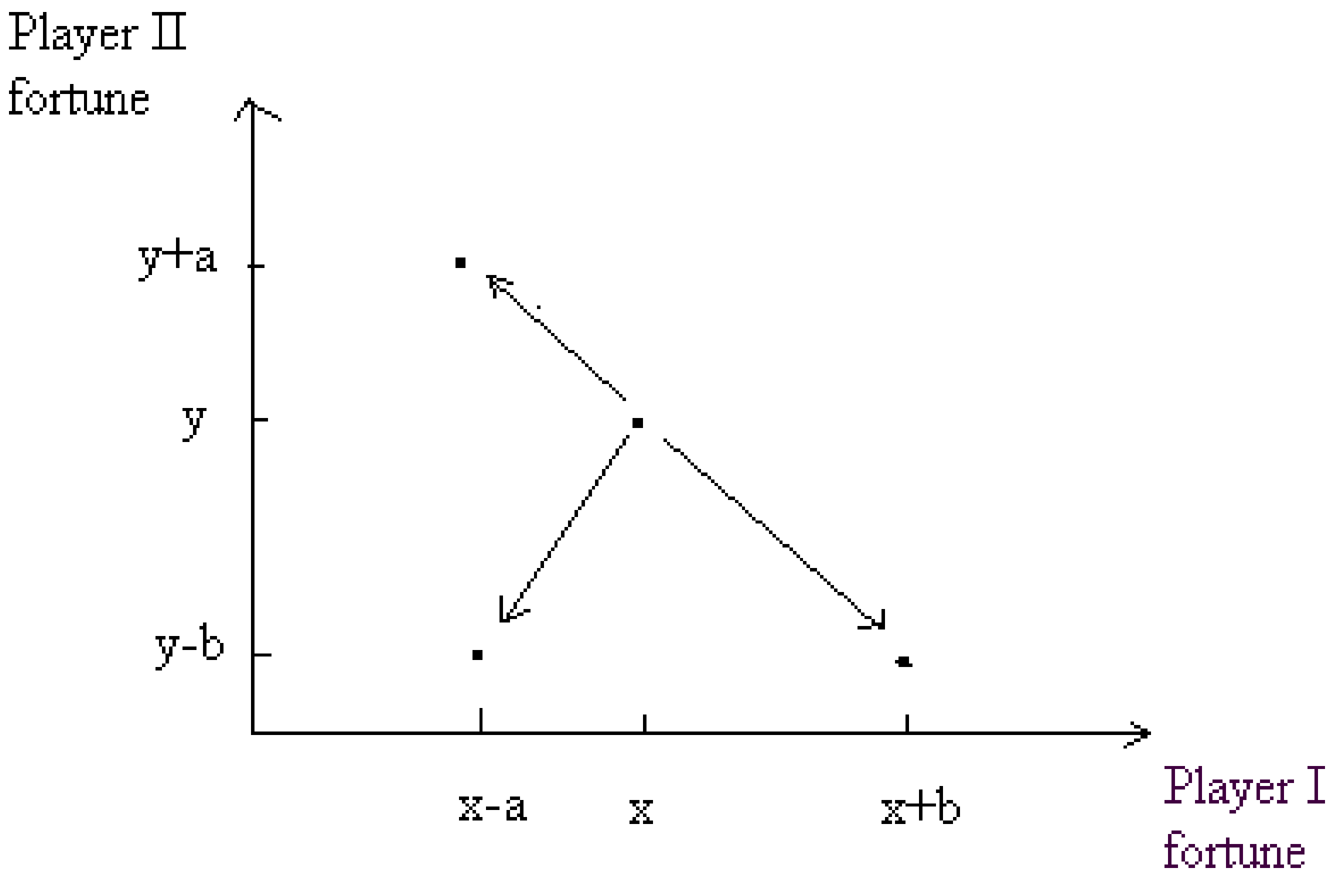
  *Calculate the function which gives the probability of reaching the goal with the suggested strategies and show that this function is excessive for one of the players given the strategy for the other.*

  *[Dubins and Savage (1965), Maitra and Sudderth (1996)]*

# Non-Constant Sum Red-and-Black

- There are two **players I and II** and a **gambling house**
- Player I starts with an **initial fortune** $x \geq 0$; player II starts with an **initial fortune** $y \geq 0$
- They both want to reach the **target fortune g** ($g \leq x+y$)
- At each stage I bids an integral amount **a** and II bids an integral amount **b**
- With some probability **player I wins b**, with another probability **player II wins a**, and with the remaining probability the players lose and the **gambling house wins a+b**
- The players have **probability of winning** that are not fixed, but are **functions** of their bets

# Non-Constant Sum



- N-person non-constant sum red-and-black:
  - Set of players: $\mathcal{N} = \{1, 2, \ldots, N\}$
  - State space:

    $$S = \{(x_1, \ldots , x_N):\ 0 \le x_j \le g \quad \text{for } j = 1, \ldots, N\}$$

  - Sets of actions:

    $$A_j(x_1,\ldots,x_N) = \begin{cases} \{1,\ldots,x_j\} & \text{if} \quad x_j \in \{1,\ldots,g-1\} \\ \{0\} & \text{if} \quad x_j \in \{0,g\} \end{cases}$$

  - Payoff function:

    $$\varphi_j = \begin{cases} 1 & \text{if} \quad x_j = g \\ 0 & \text{if} \quad x_j = 0 \end{cases} \qquad \text{for} \quad j = 1,\ldots,N$$

– Law of motion:

$$(x_1^{m+1},\dots,x_N^{m+1}) = \begin{cases} (x_1^m - a_1,\dots,x_N^m - a_N) & w.p. \quad 1-\sum_{j=1}^N f\left(\frac{a_j}{\sum_{i=1}^N a_i}\right), \\ (x_1^m + \sum_{i=2}^N a_i,\dots,x_N^m - a_N) & w.p. \quad f\left(\frac{a_1}{\sum_{i=1}^N a_i}\right), \\ \dots & \dots \quad \dots \\ (x_1^m - a_1,\dots,x_j^m + \sum_{k\neq j} a_k,\dots,x_N^m - a_N) & w.p. \quad f\left(\frac{a_j}{\sum_{i=1}^N a_i}\right), \\ \dots & \dots \quad \dots \\ (x_1^m - a_1,\dots,x_N^m + \sum_{i=1}^{N-1} a_i) & w.p. \quad f\left(\frac{a_N}{\sum_{i=1}^N a_i}\right). \end{cases}$$

- Theorem 3.1 Pontiggia (2007):

  In a non-constant sum N-person red-and-black game, assume that the win probability function f is super-additive and satisfies the condition $f(s)f(t) \leq f(st)$, for all $0 < s, t < 1$. A Nash Equilibrium is for all players to play a bold strategy.

$$\left[ Super-additivity: \quad \sum_{j=1}^{N} f(s_j) \leq f(\sum_{j=1}^{N} s_j) \right]$$

# So far…

- Up to this point we have assumed that players know all relevant information about each other and have correct beliefs about the rivals' actions.

**This is not always true!!**

- Players may be uncertain about the characteristics of other players (i.e. incomplete information).
- Players have initial beliefs (prior) about the "type" of each player (i.e. how they will act) and they can update their beliefs, as play takes place, on the basis of their actions.

# Bayesian Games

- A Bayesian game can be modeled by introducing Nature as a player in the game.
- Nature randomly choose the “type” of each player according to a probability distribution.
- At each stage of the game the players make their choices simultaneously.
- At the end of each stage the players receive information about the actions of the other players and their type.
- At each stage each player chooses the “best” action based on the current information.

A Bayesian game is a tuple $(N, A, \Theta, p, u)$

- $N = \{1, \ldots, n\}$ is the set of players
- $A = \{A_1, \ldots, A_n\}$ is the set of actions
- $\Theta = \{\Theta_1, \ldots, \Theta_n\}$ is the set of types; where $\Theta_i$ is the type space of player i
- $p: \Theta \rightarrow [0,1]$ is the joint probability distribution for the types of players;
- $u = \{u_1, \ldots, u_n\}$ where $u_i: A \times \Theta \rightarrow R$ is the utility function for player i

# Bayesian Red-and-Black Games

A Bayesian Red-and-Black game is a tuple (N, A, Θ, p, u)

- N = {I, II}
- Set of actions:

$$A(x) = \begin{cases} \{1,2,\ldots,x\} & \text{if } x \in \{1,2,\ldots,g-1\} \\ \{0\} & \text{if } x \in \{0,g\} \end{cases} \qquad B(x) = \begin{cases} \{1,2,\ldots,g-x\} & \text{if } x \in \{1,2,\ldots,g-1\} \\ \{0\} & \text{if } x \in \{0,g\} \end{cases}$$

- Law of motion (Weighted Red-and-Black):

$$x^{m+1} = \begin{cases} x^m + b & w.p. \quad w\dfrac{a}{a+b} \\ \\ x^m - a & w.p. \quad 1 - w\dfrac{a}{a+b} \end{cases}$$

- $\Theta = \{\theta_1, \theta_2\} = \{0 < w < 1,\ w > 1\} = \{\text{subfair, superfair}\}$
- $P(\theta_1) = P(\theta_2) = 1/2$

# Main References


- **Chen, M. and Hsiau, S. (2006).** Two-person red-and-black games with bet-dependent win probability functions. *Journal of Applied Probability*, 43, 905-915.

- **Dubins, L. and Savage, L. (1976).** *Inequalities for Stochastic Processes: How to Gamble if You Must*, 2nd Edn. Dover, New York.

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- **Pontiggia, L. (2007).** Non-constant sum red-and-black games with bet-dependent win probability function, *Journal of Applied Probability*, 44, 547-553.

- **Secchi, P. (1997)** Two-person Red-and-Black stochastic games, *Journal of Applied Probability*, 34, 107-126.

# Q & A

Thank You!