Finding your way in a graph

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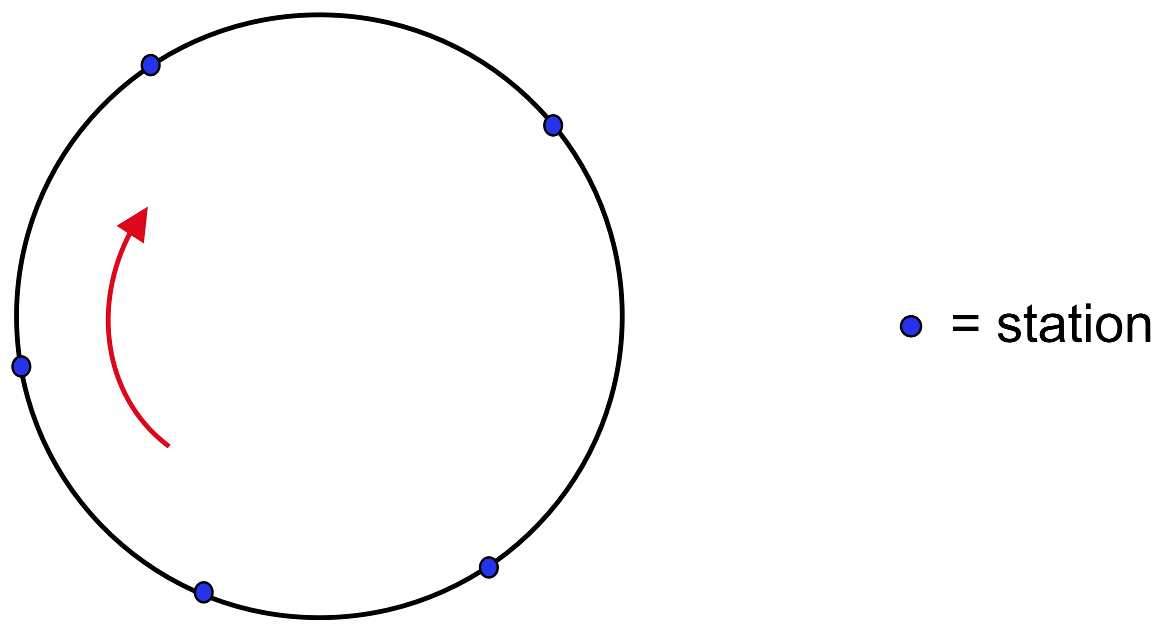
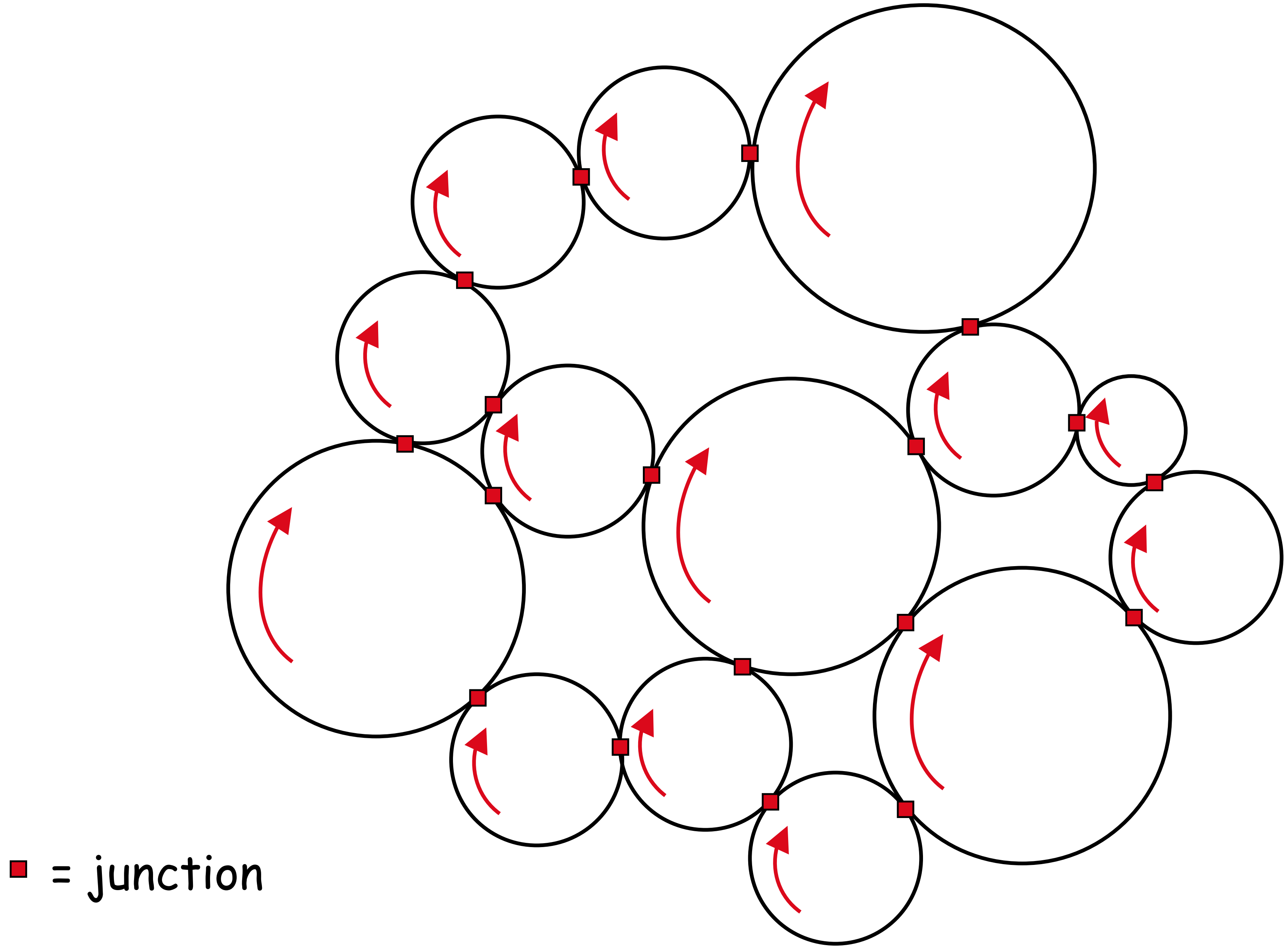
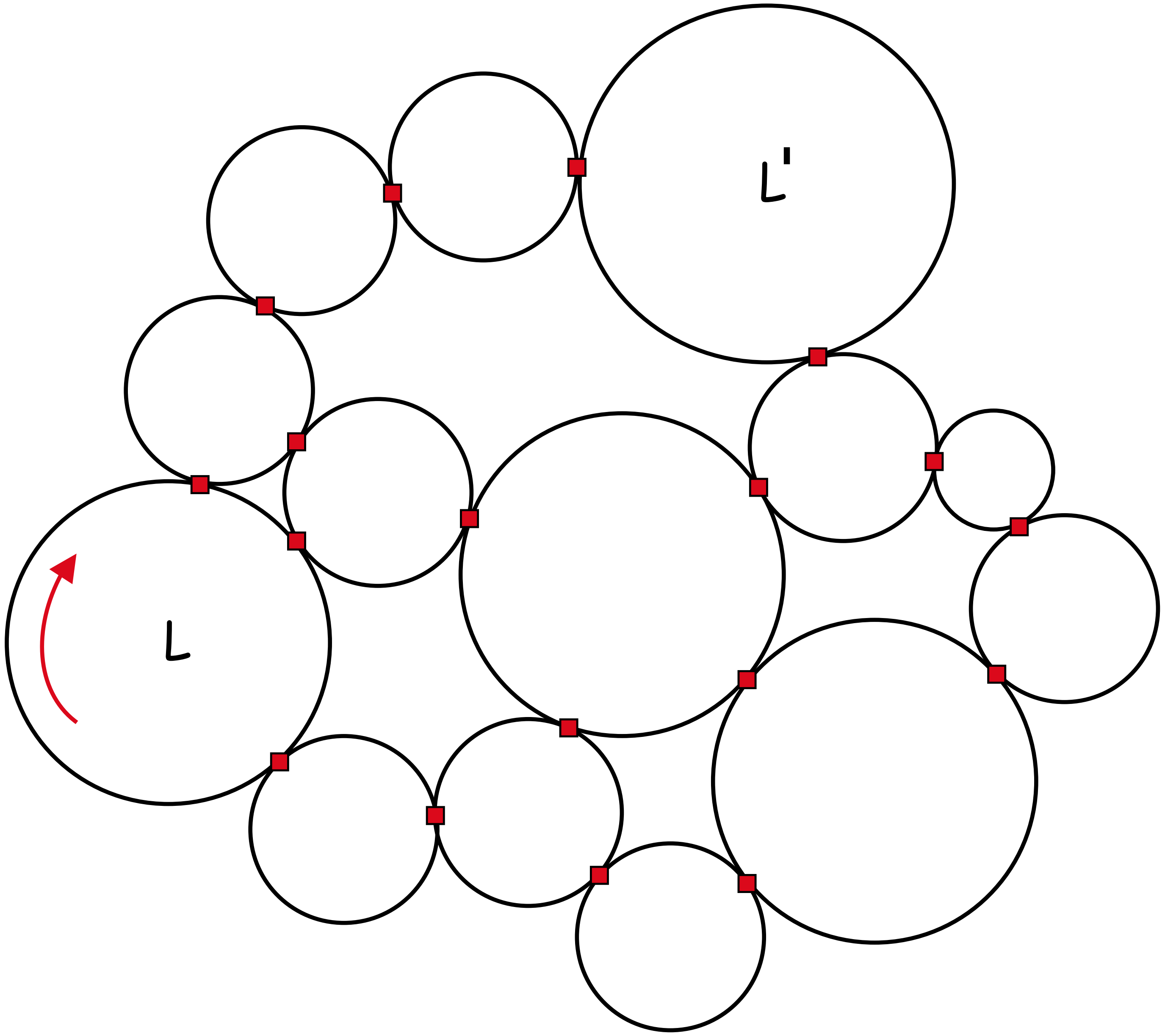


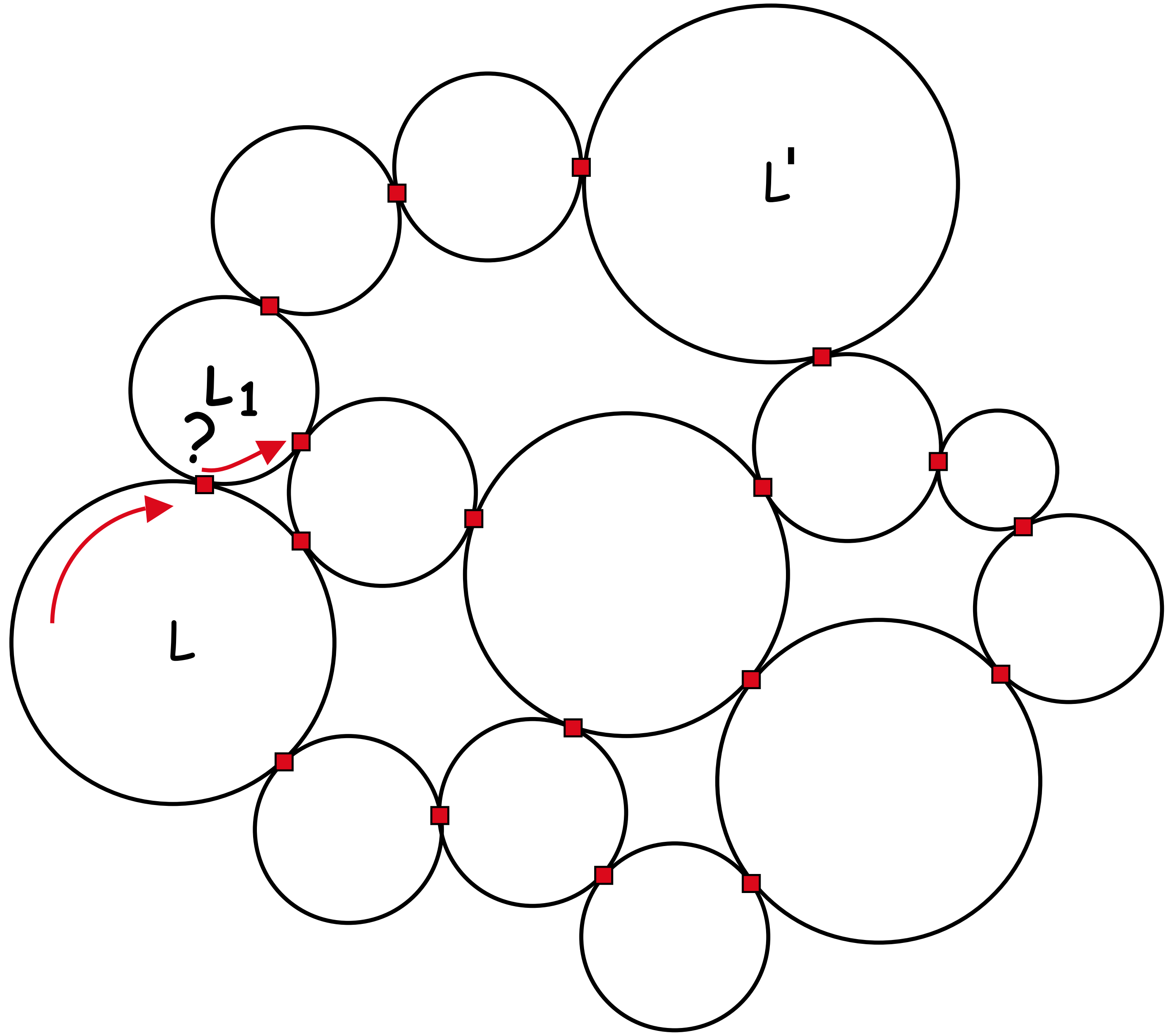
One loop



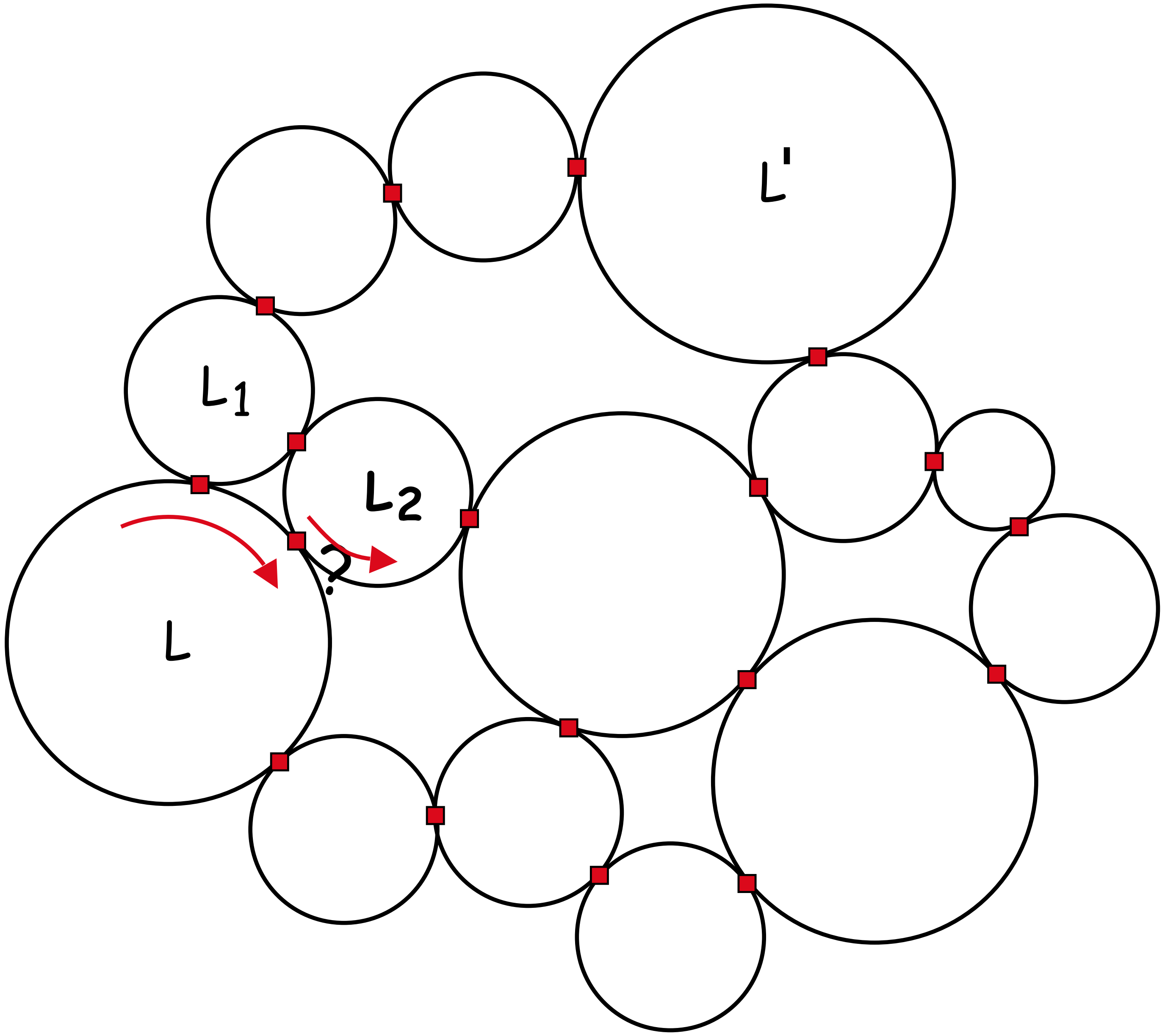
A system of interconnected loops



What is the best way to go from L to L'?



What is the best way to go from L to L' ?



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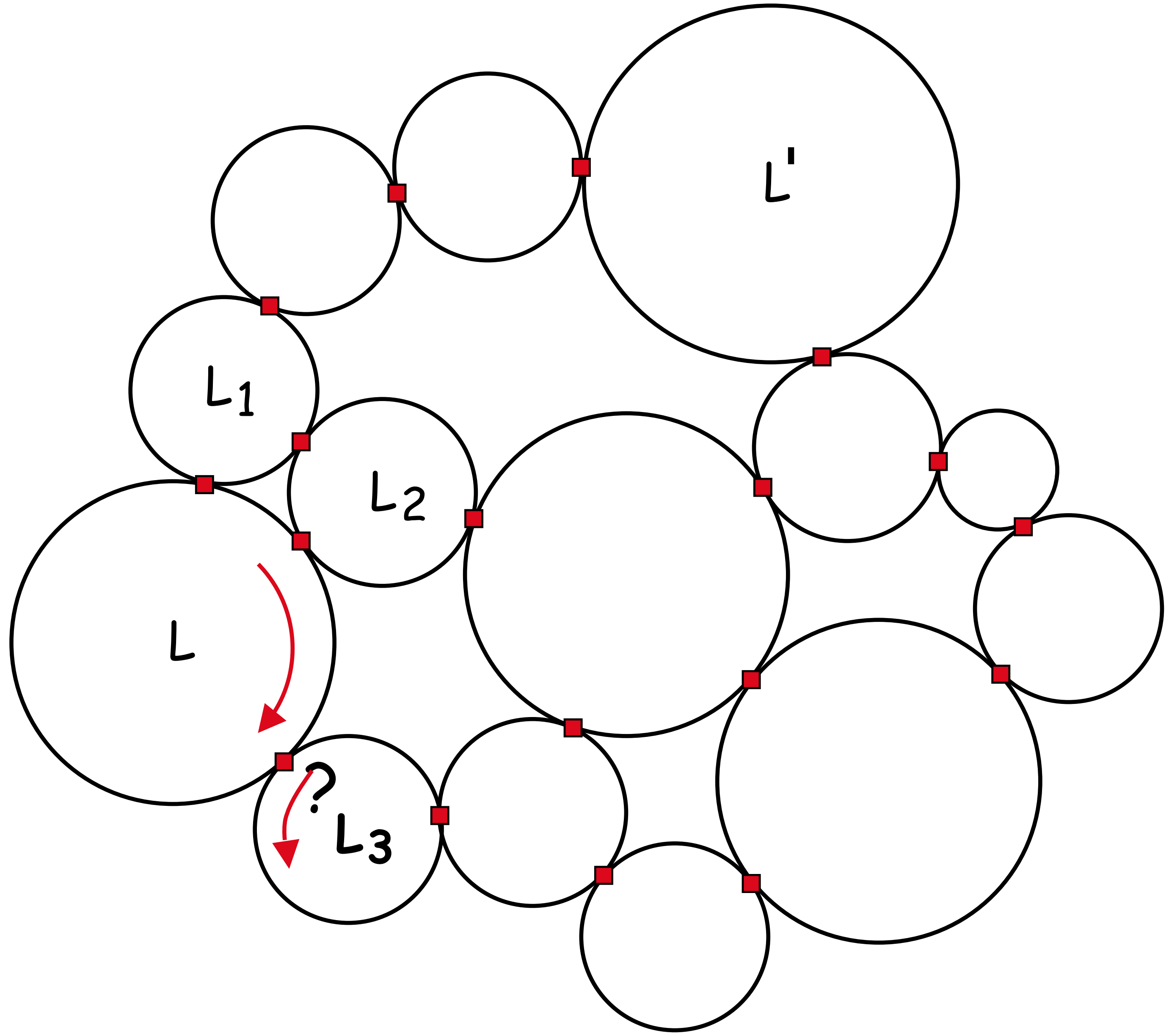

What is the best way to go from L to L' ?

This was called “**loop switching**” when it was introduced by John Pierce at Bell Labs in the late ’60’s.

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John R. Pierce

Holder of more than 90 patents, and author of 14 technical books and over 200 technical articles, he sold two poems and one painting and received royalties from Decca records ("Music from Mathematics“ and "The Voice of the Computer"). He also published 20 science fiction stories, under the pseudonym "J. J. Coupling.“ He also invented the word “transistor”.

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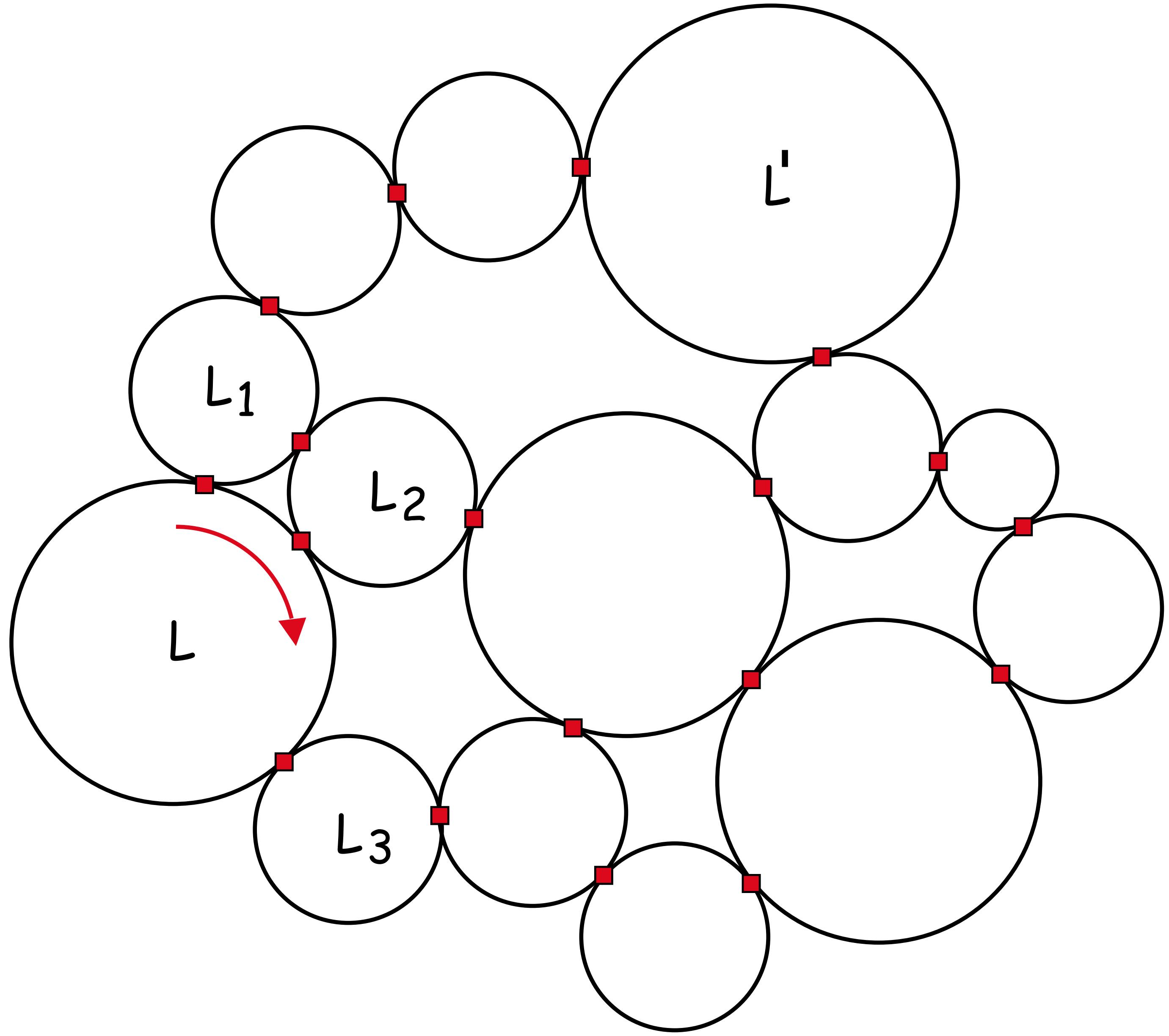
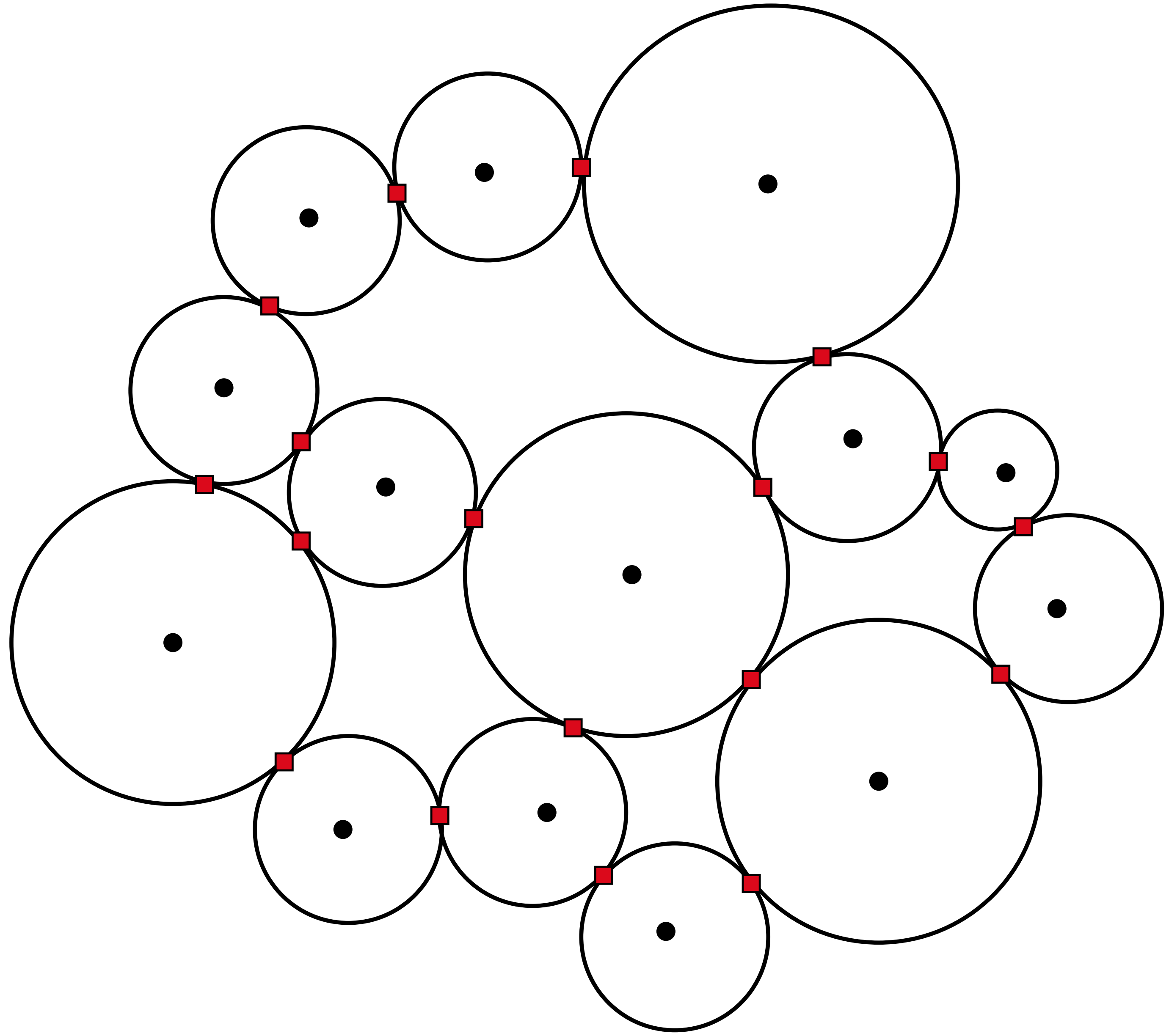
Loop switching was a precursor to what is now called **packet switching**.



A system of interconnected loops



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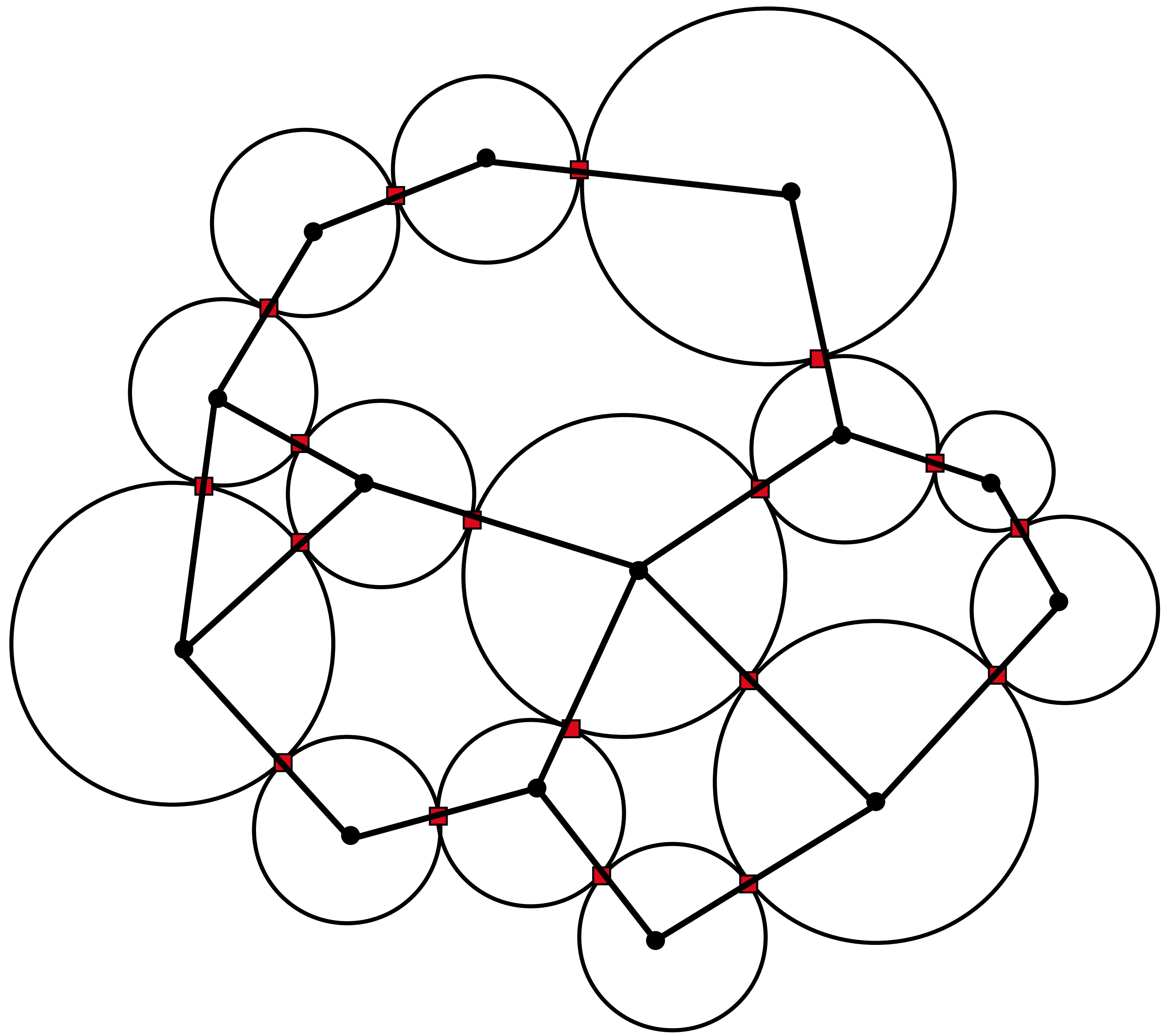
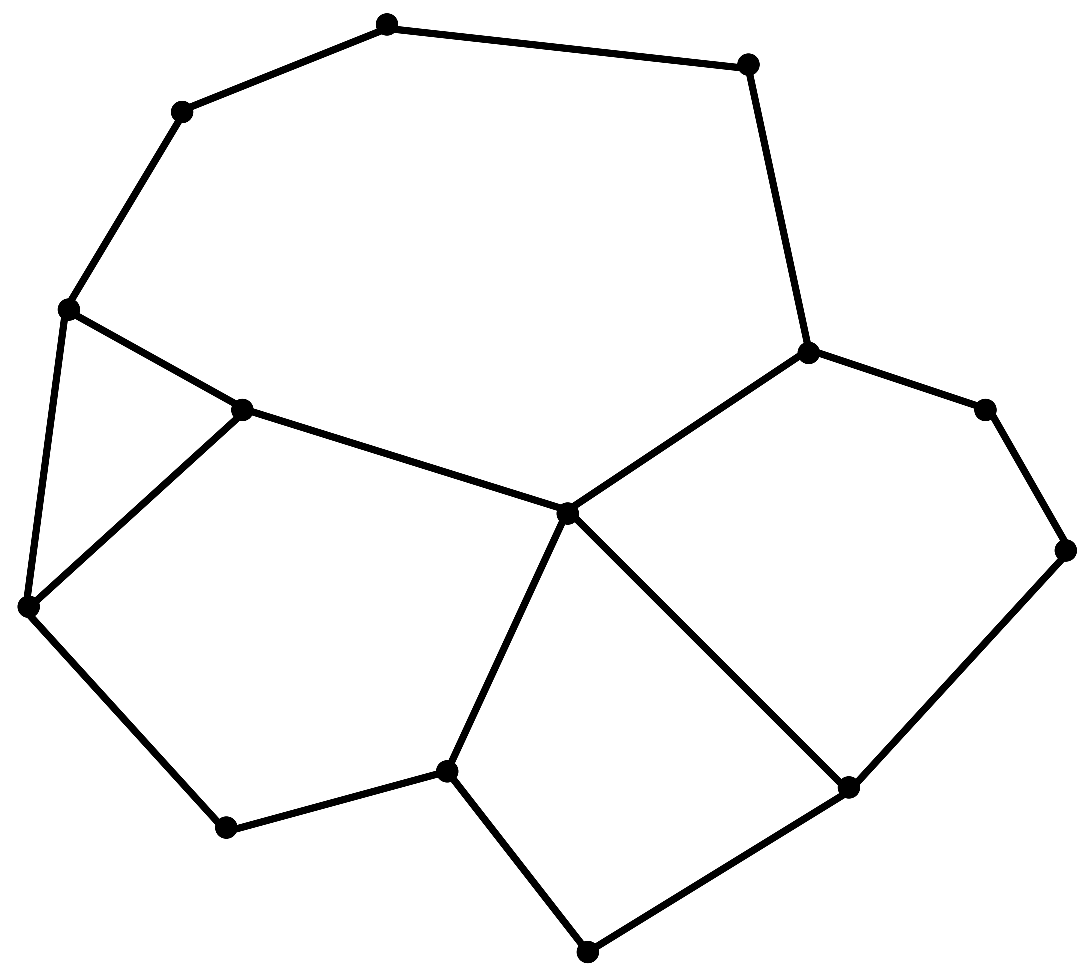
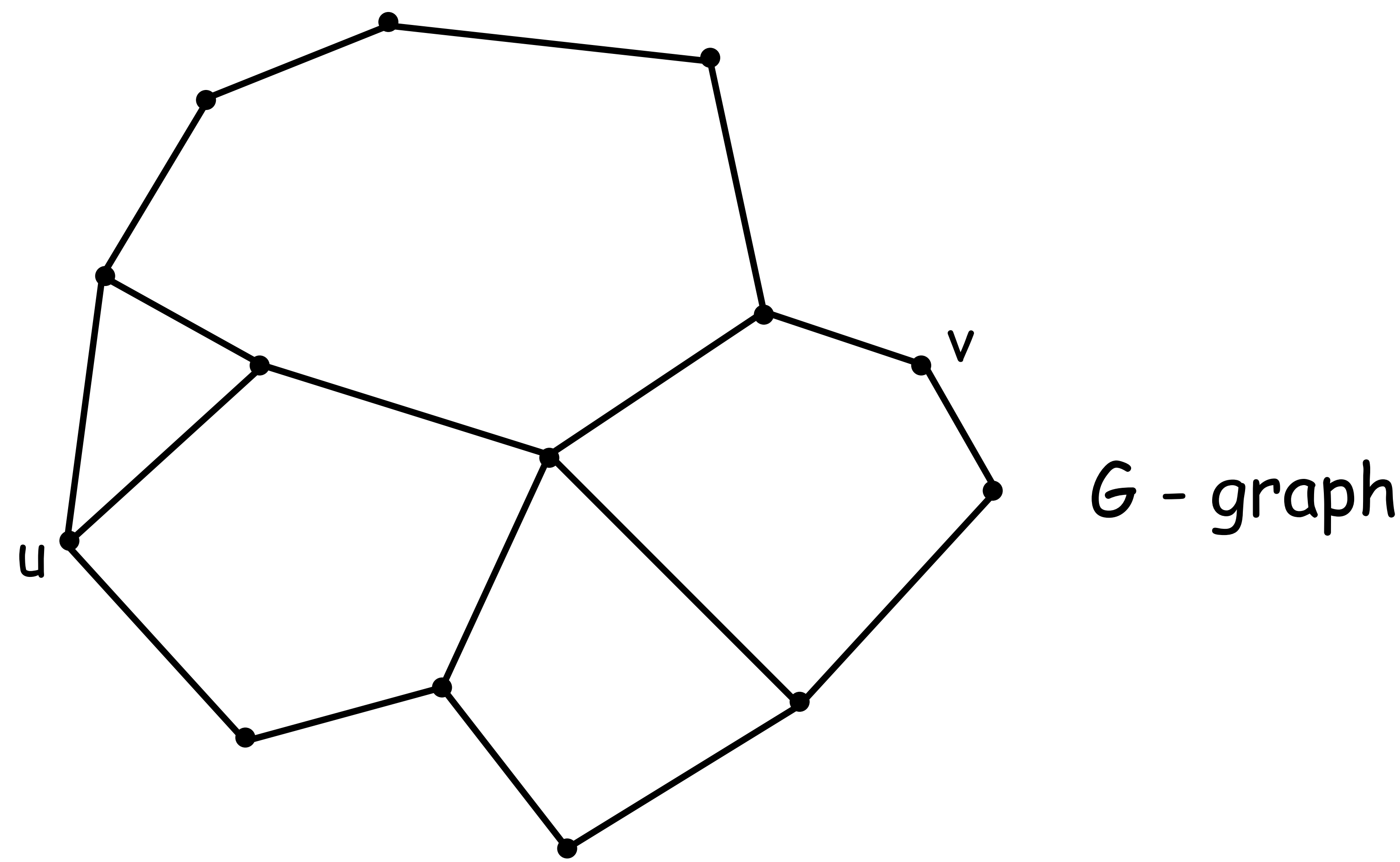


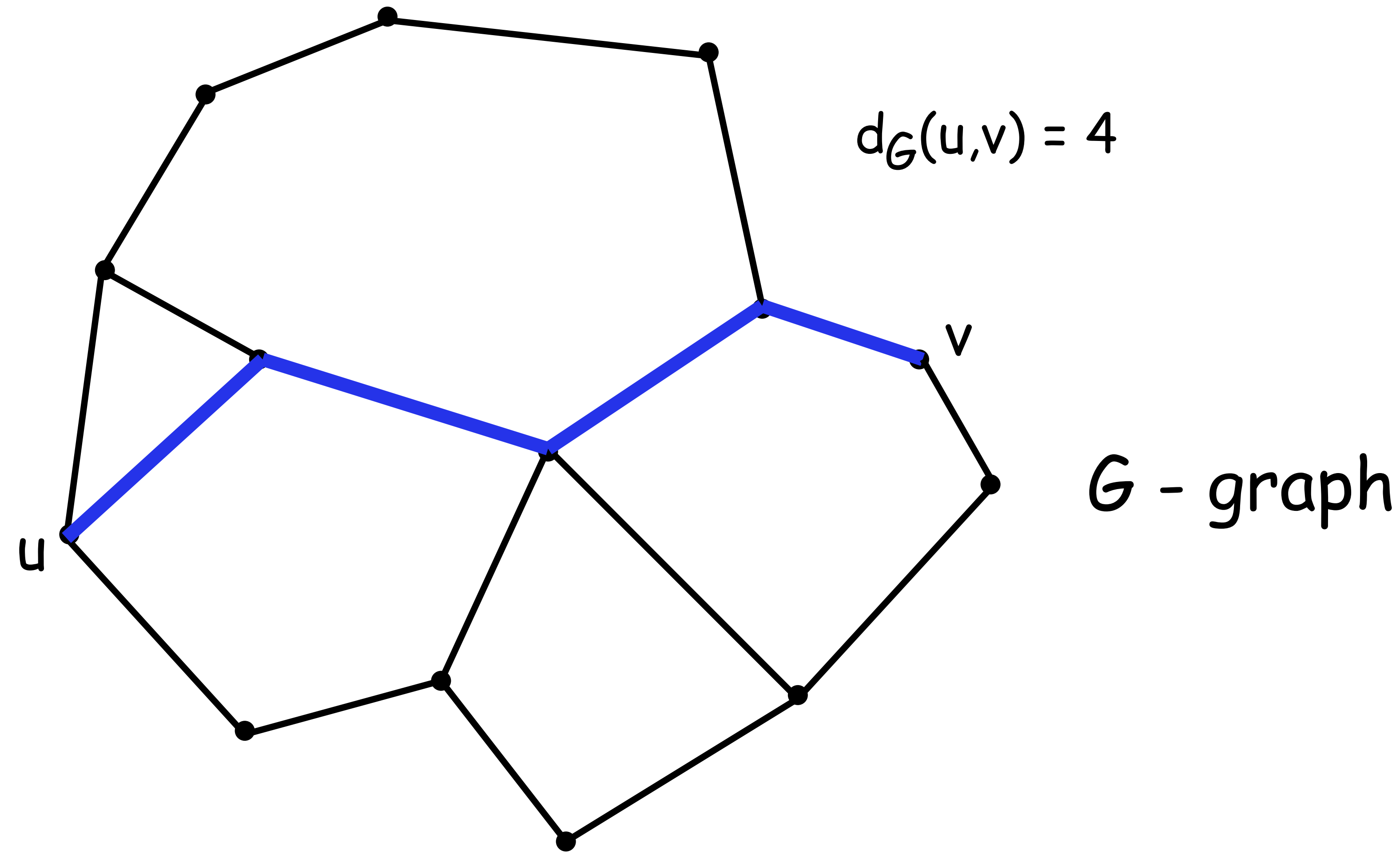
A system of interconnected loops
and the corresponding **graph** G



G - graph



The **distance** $d_G(u,v)$ between u and v is defined to be the minimum number of edges in any path joining u and v.



The **distance** $d_G(u,v)$ between u and v is defined to be the minimum number of edges in any path joining u and v.

The **Hamming distance** between two binary n-tuples is defined to be the number of positions in which they differ.

Denote Hamming distance by d_H .

For example, if s = (1,0,0,1,0,1,1,1,0) and t = (0,0,1,1,0,0,1,0,1) then d_H (s,t) = 5.

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Richard Hamming

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Richard Hamming
1915-1998

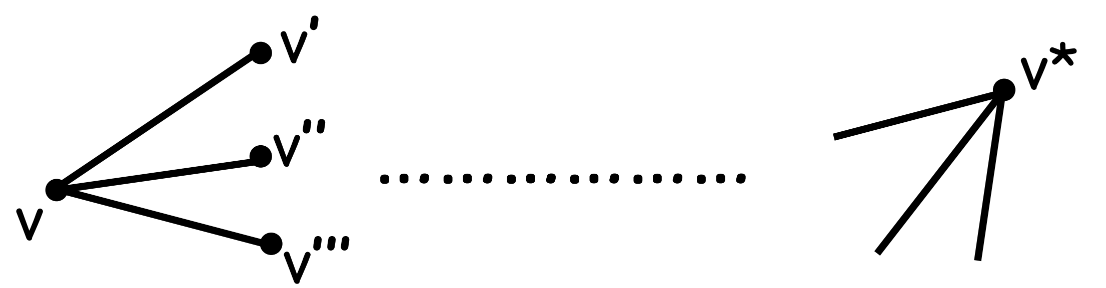
Routing messages in G

If we are currently at v and our final destination is v* then we go to v' provided that v' is closer to v* than v is, i.e.,

$$d_G(v',v^*) < d_G(v,v^*)$$

Hamming distance routing

Assign to each vertex v of G, a suitable binary N-tuple $A(v)$, called its **address**.



Hamming distance routing

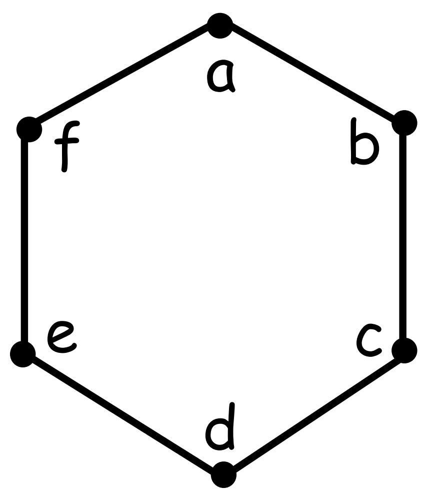
Assign to each vertex v of G, a suitable binary N-tuple A(v), called its **address**.

If we are currently at v and our final destination is v*
then we go to v' provided that

$$d_H(A(v'),A(v^*)) < d_H(A(v),A(v^*))$$

Of course, this only works if the Hamming distances between addresses accurately reflects the actual graph distances in G.

For example:



G

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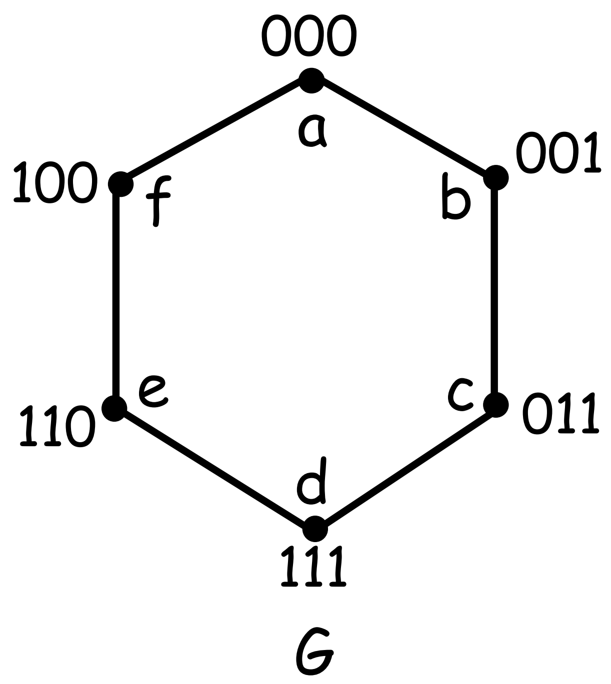

$d_G(a,c) = 2 = d_H(000,011)$

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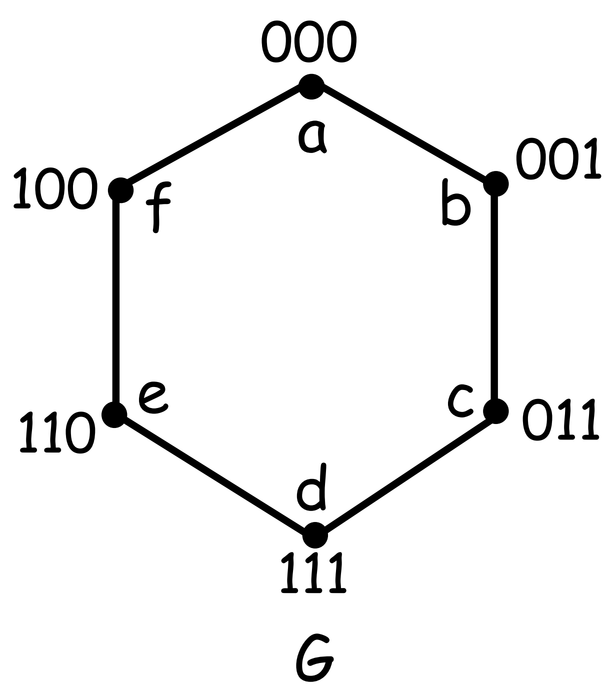

$d_G(a,c) = 2 = d_H(000,011)$

$d_G(e,b) = 3 = d_H(110,001)$, etc.

An assignment $v :\rightarrow A(v)$ of binary N-tuples to the vertices of G is called a valid addressing of G (of length N) provided we have:

$$d_G(u,v) = d_H(A(u),A(v))$$

for all vertices u and v in G.

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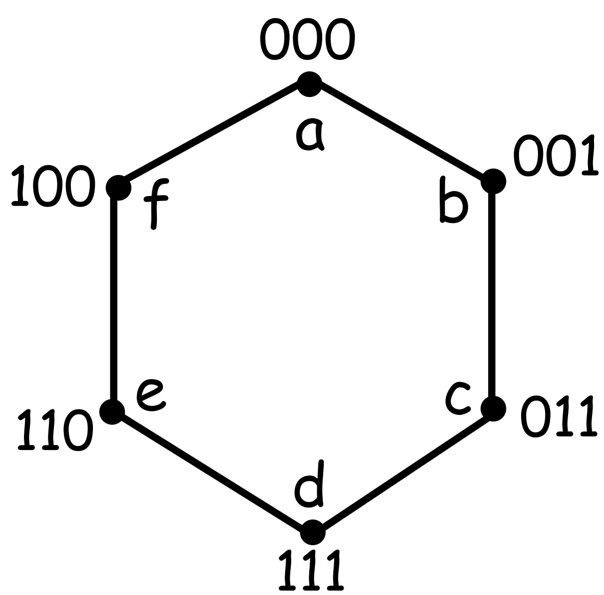
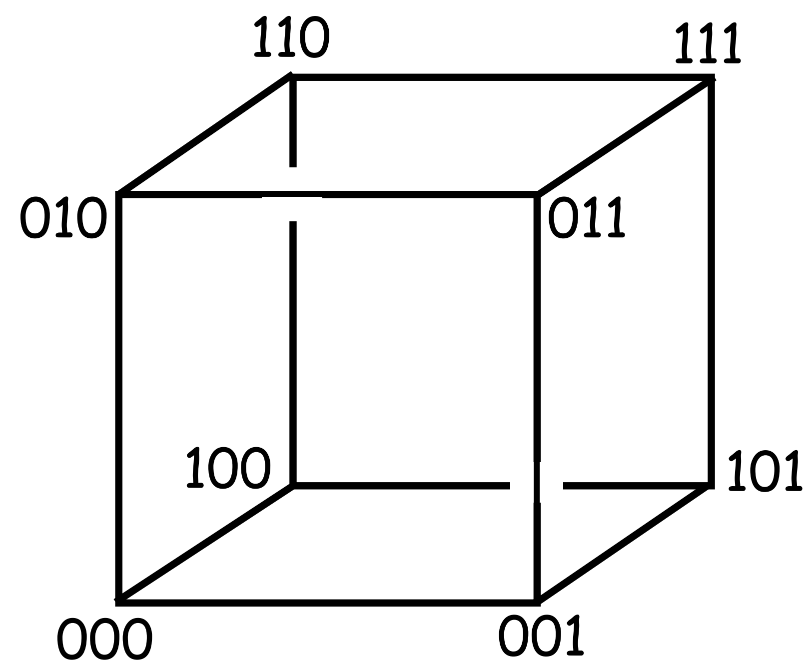
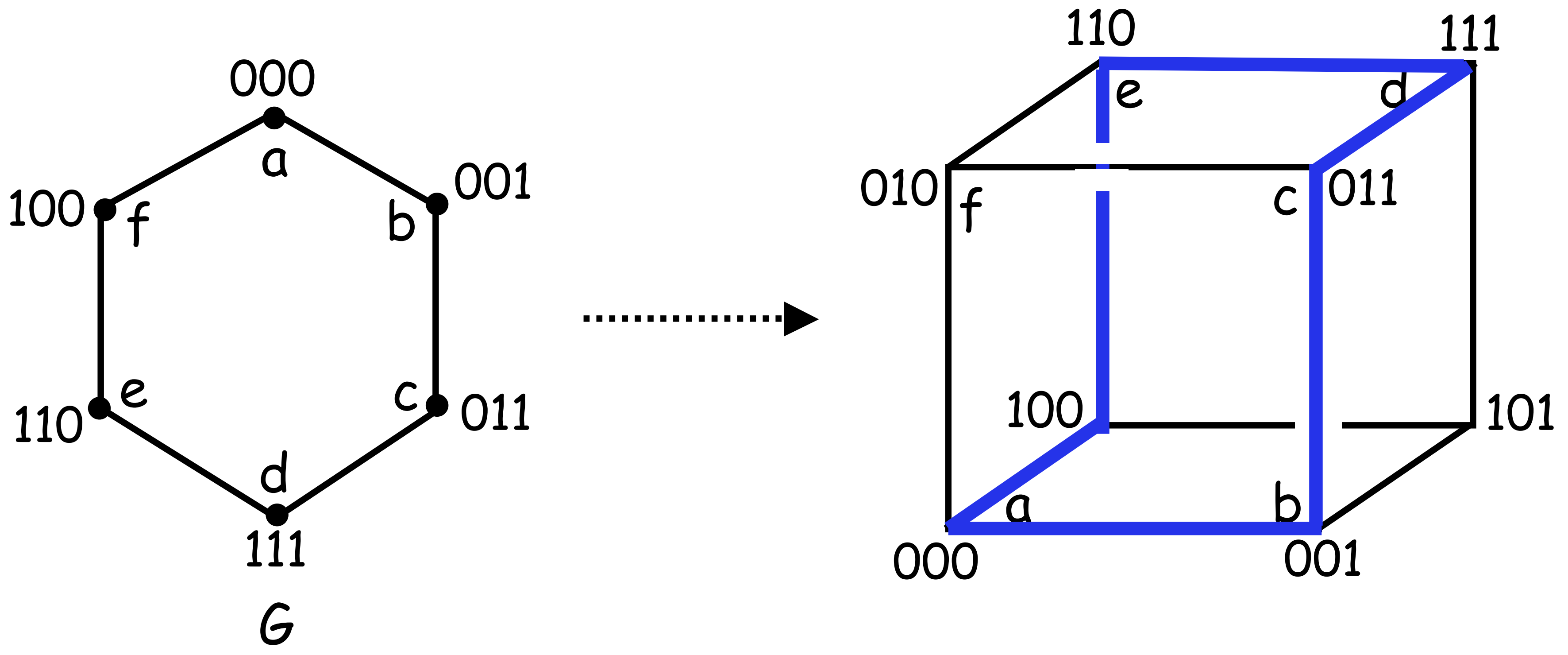
Note that a valid addressing of G is actually an isometric embedding of G into an N-cube!

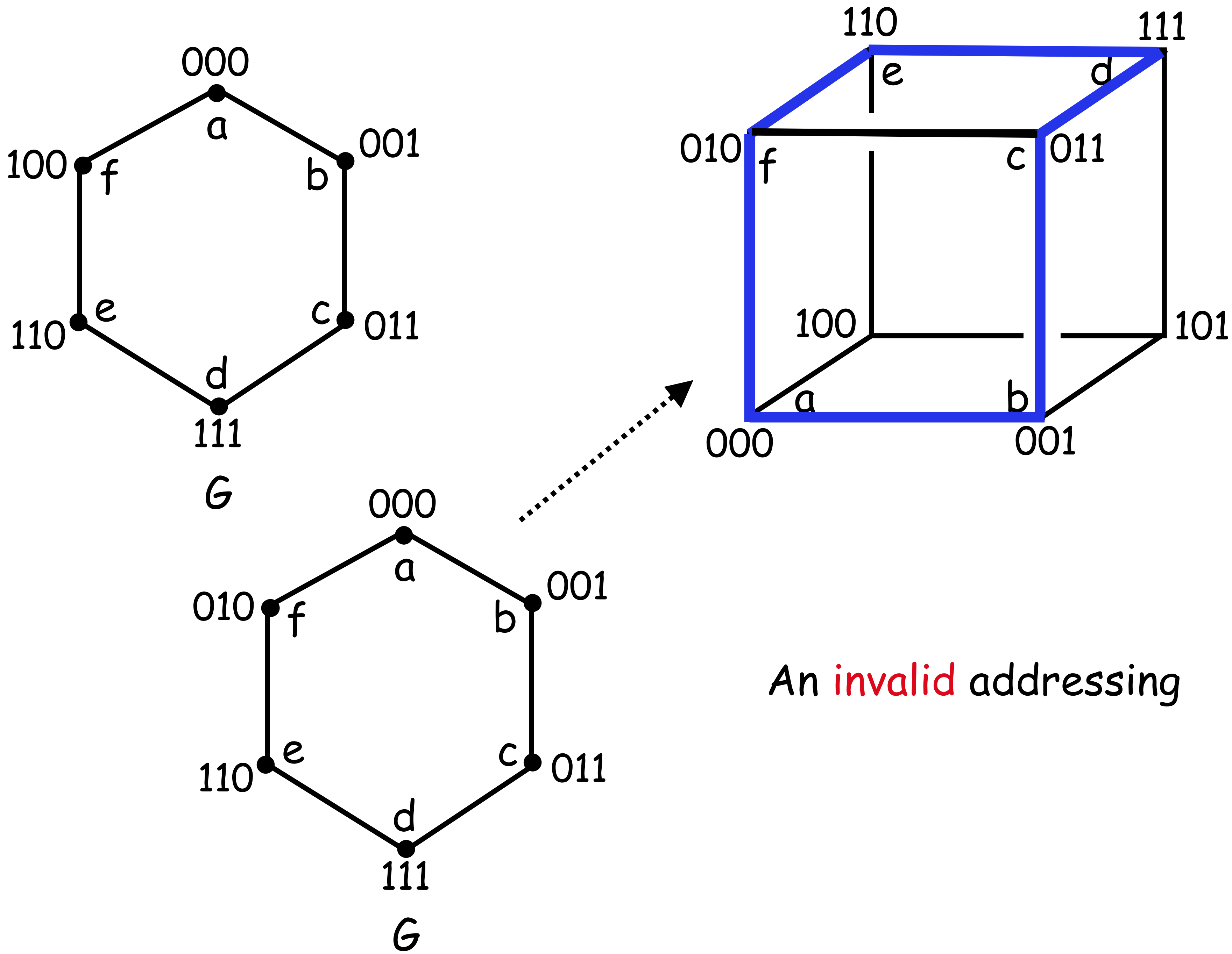


G



A valid addressing of G.




000
a
001
b
100
f
e
110
c
011
d
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G
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a
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011
100
101
a
b
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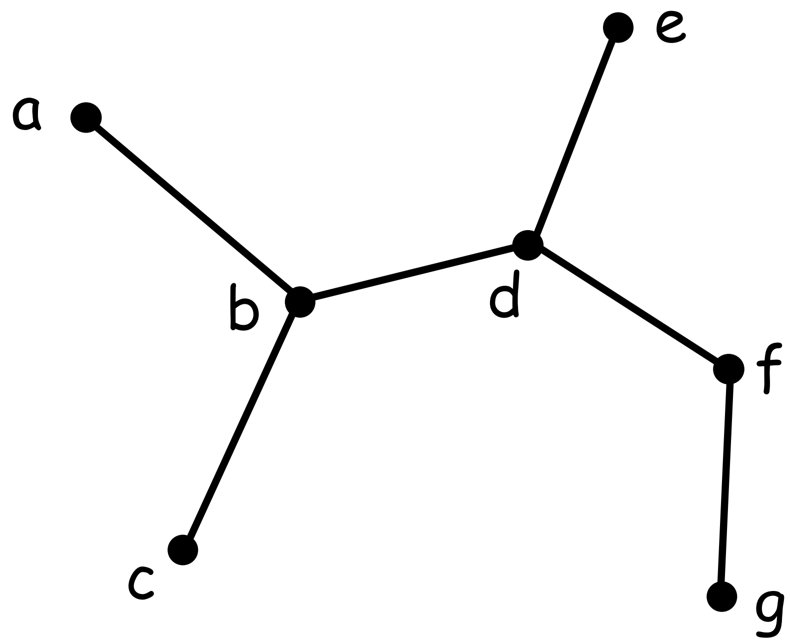
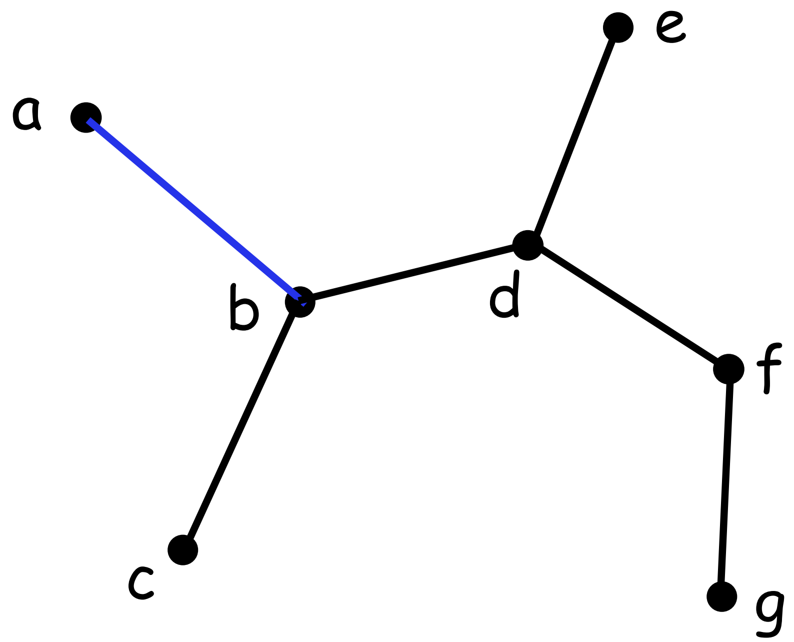
001
An invalid addressing

Trees



A tree T

Trees



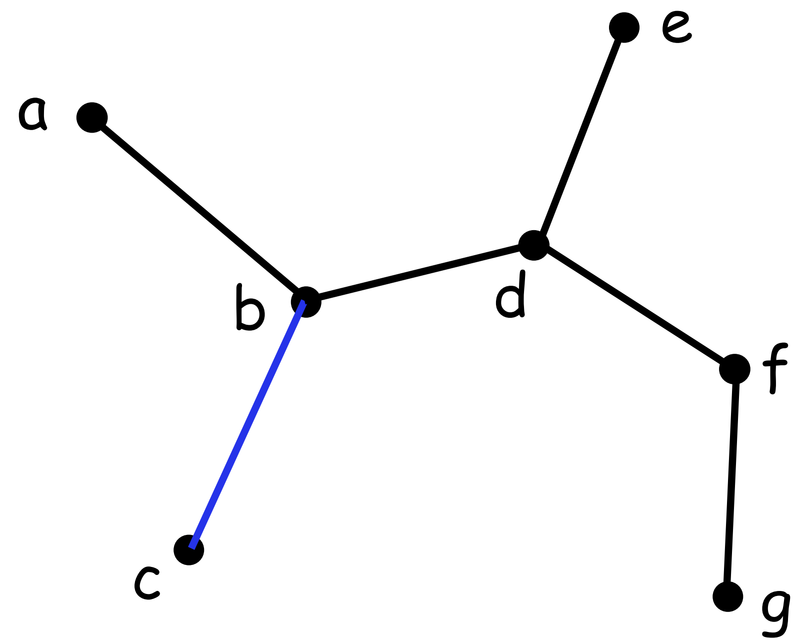
A tree T

a - 0

b - 1

So far, so good!

Trees



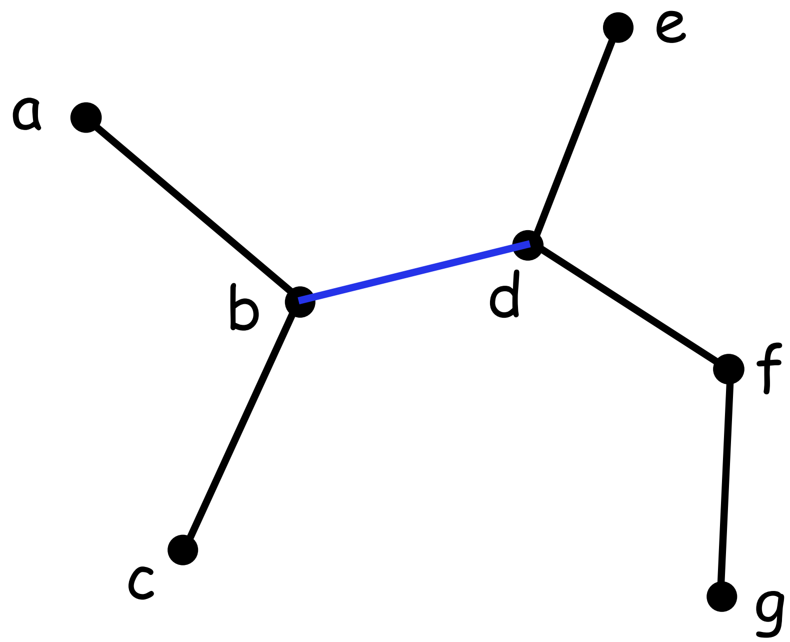
A tree T

a - 0 0

b - 1 0

c - 1 1

Trees



A tree T

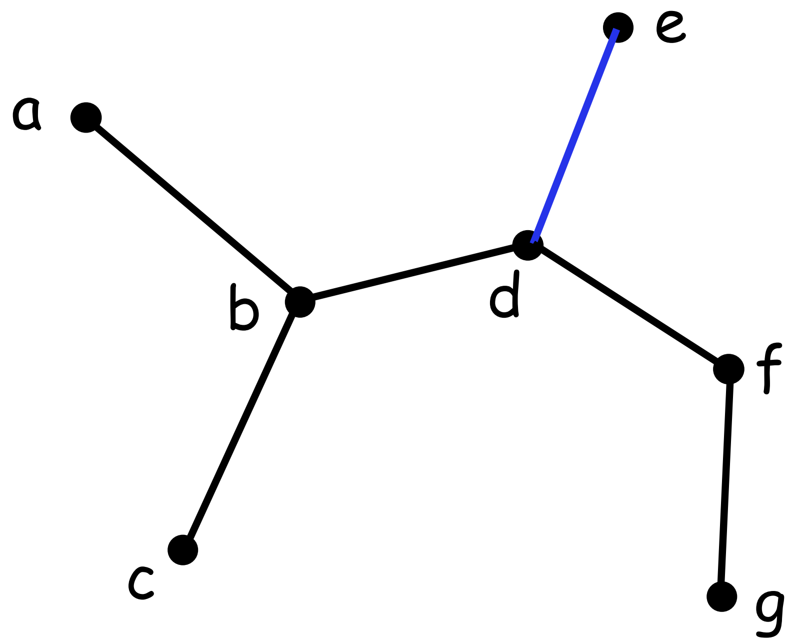
a - 0 0 0

b - 1 0 0

c - 1 1 0

d - 1 0 1

Trees



A tree T

a - 0 0 0 0

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d - 1 0 1 0

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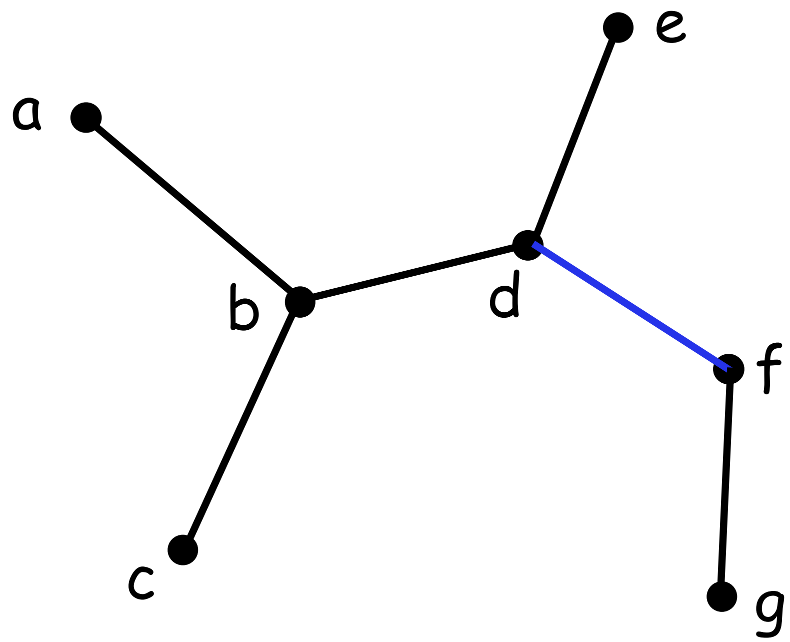
Trees



A tree T

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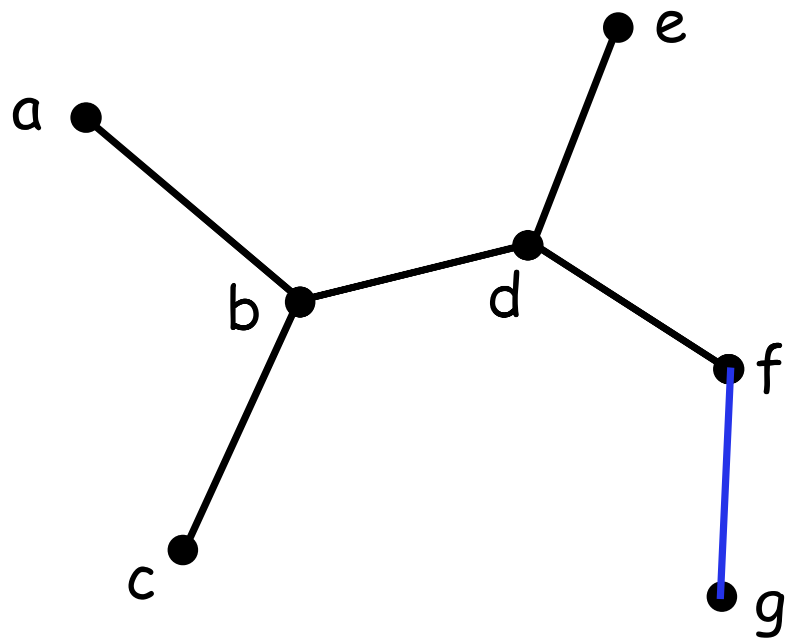

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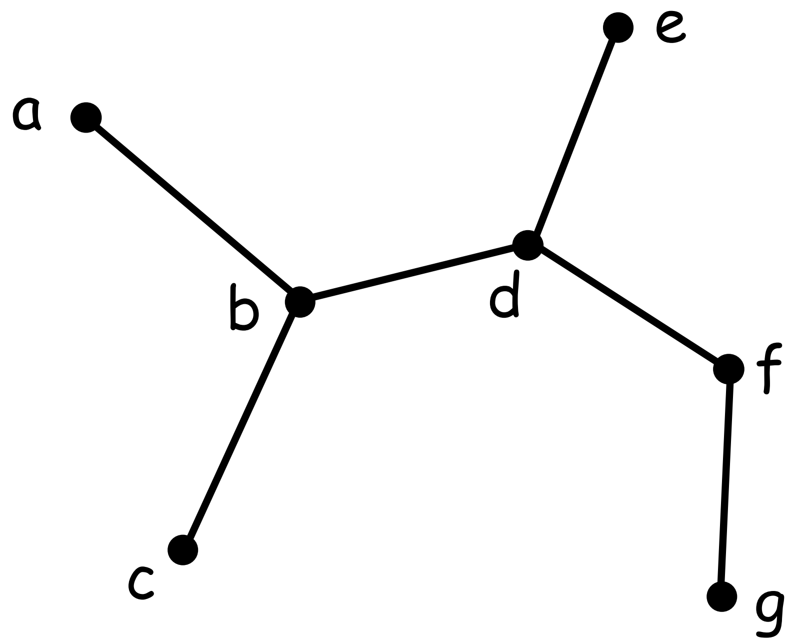
Trees



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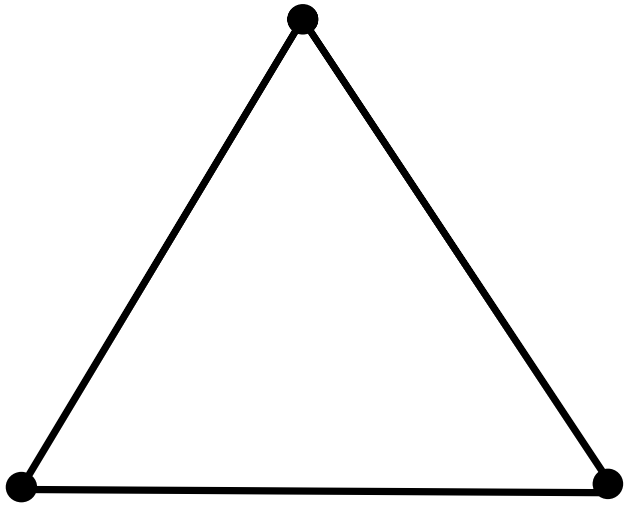
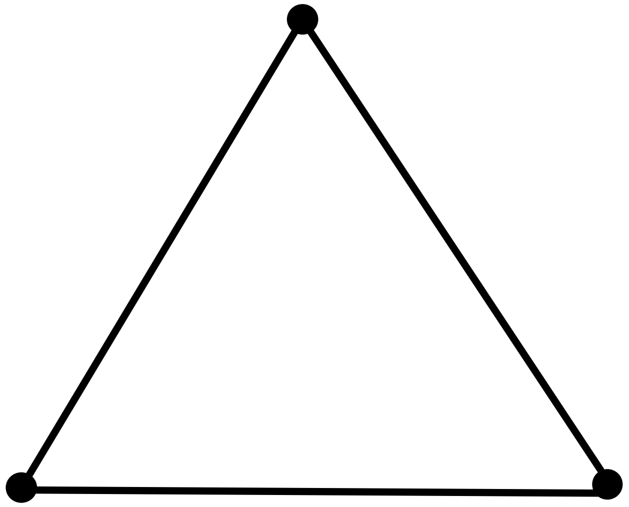
A valid addressing of T

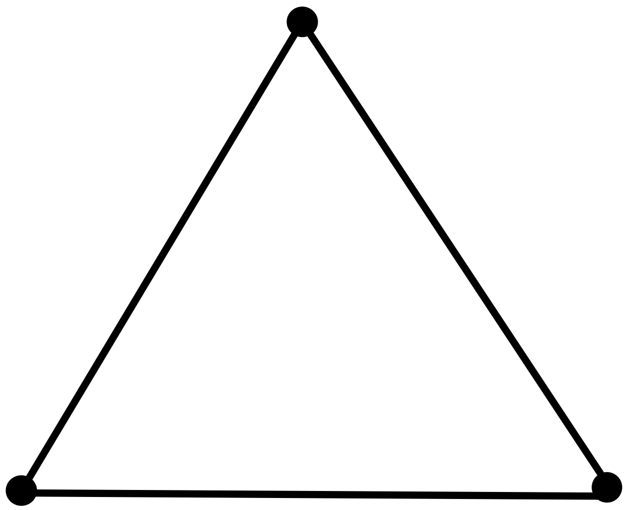


What about a triangle ? ?



What about a triangle ? ?

Introduce a new symbol * , and define $d_H(0,*) = d_H(1,*) = 0$.



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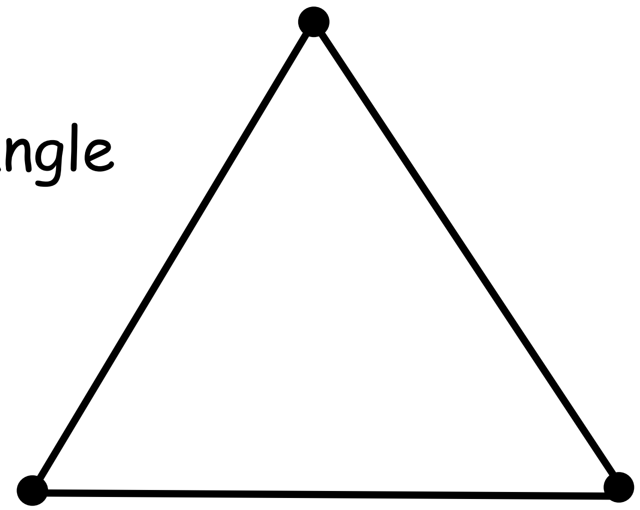
Introduce a new symbol $*$, and define $d_H(0,*) = d_H(1,*) = 0$.

For example, $d_H(0\ 1 * 0 * 1\ 0\ 1,\ * 0\ 1 * * 0\ 1\ 1) = 3$

Now we can address a triangle



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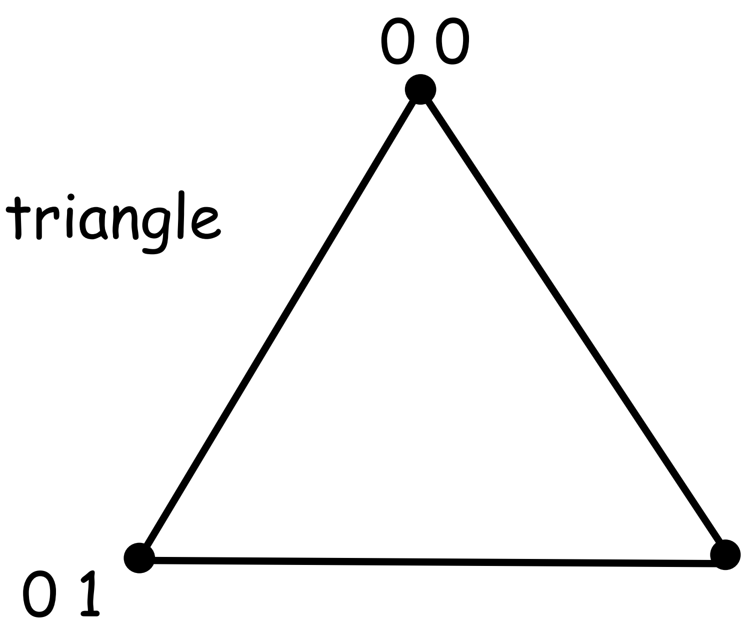

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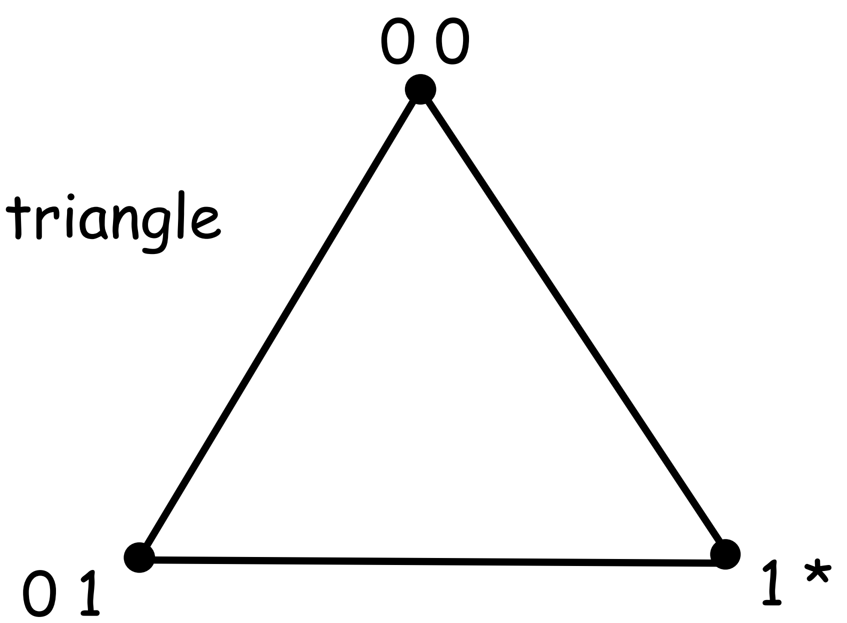

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A valid extended addressing of G is an assignment $A(v)$ to each vertex v in G an N-tuple of 0, 1, and *'s so that for all vertices u and v in G,

$$d_G(u,v) = d_H(A(u),A(v))$$

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Theorem: Valid extended addresses exist for every graph G.

Proof:

$$A(v_1) = \overbrace{0\ldots\ldots0}^{d_G(v_1,v_2)}$$

$$A(v_2) = 1\ldots\ldots1$$

Proof:

$$A(v_1) = \overbrace{0\ldots\ldots0}^{d_G(v_1,v_2)}\ \overbrace{0\ldots\ldots0}^{d_G(v_1,v_3)}$$

$$A(v_2) = 1\ldots\ldots1\ \ *\ldots\ldots*$$

$$A(v_3) = *\ldots\ldots*\ \ 1\ldots\ldots1$$

Proof:

$\overbrace{\quad}^{d_G(v_1,v_2)}$ $\overbrace{\quad}^{d_G(v_1,v_3)}$ …………………… $\overbrace{\quad}^{d_G(v_i,v_j)}$

$A(v_1)$ = 0……0 0……0……………………*……*………………

$A(v_2)$ = 1……1 *……*……………………*……*………………

$A(v_3)$ = *……* 1……1……………………*……*………………

⋮

$A(v_i)$ = *……* *……*……………………0……0………………

⋮

$A(v_j)$ = *……* *……*……………………1……1………………

⋮

Unfortunately, the length of the addresses maybe very long by using this method!

Define $N(G)$ to be the least N such that a valid (extended) addressing of G of length N exists.

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Conjecture: If G has n vertices then $N(G) \leq n - 1$.

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Theorem (Peter Winkler)

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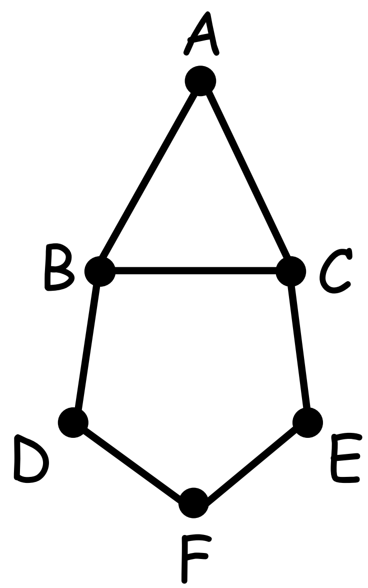
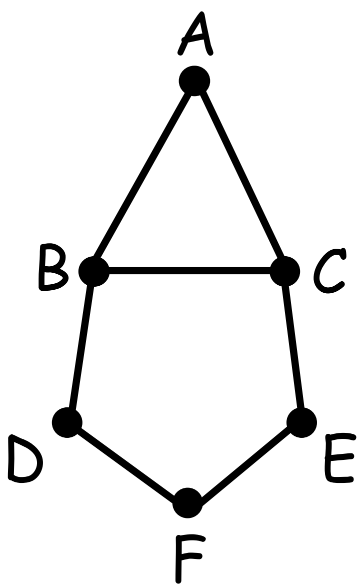


	A	B	C	D	E	F
A	0	1	1	2	2	3
B	1	0	1	1	2	2
C	1	1	0	2	1	2
D	2	1	2	0	2	1
E	2	2	1	2	0	1
F	3	2	2	1	1	0

Distance matrix $D(G) = (d_{ij})$



vertex	-	address
A	-	0 0 0 0 0
B	-	1 * 0 0 *
C	-	0 1 0 0 *
D	-	1 * 1 * 0
E	-	0 1 0 1 *
F	-	* * 1 1 1

	A	B	C	D	E	F
A	0	1	1	2	2	3
B	1	0	1	1	2	2
C	1	1	0	2	1	2
D	2	1	2	0	2	1
E	2	2	1	2	0	1
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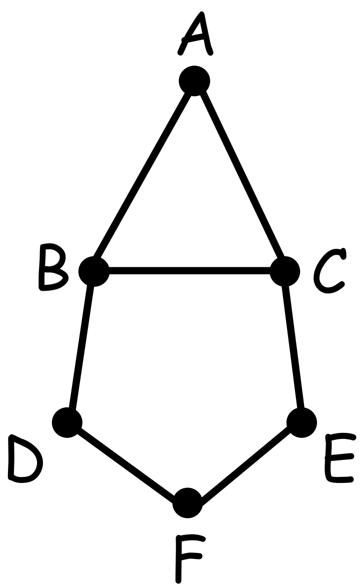

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D	-	1 * 1 * 0
E	-	0 1 0 1 *
F	-	* * 1 1 1

	A	B	C	D	E	F
A	0	1	1	2	2	3
B	1	0	1	1	2	2
C	1	1	0	2	1	2
D	2	1	2	0	2	1
E	2	2	1	2	0	1
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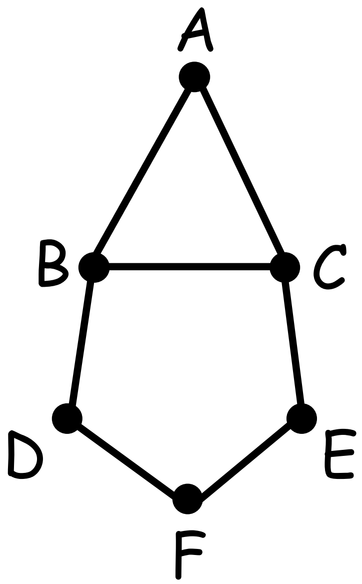
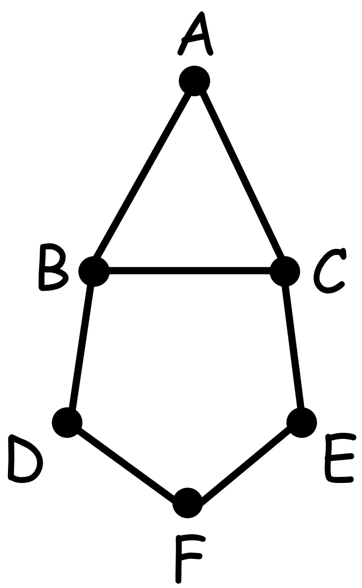


vertex	-	address
A	-	0 0 0 0 0
B	-	1 * 0 0 *
C	-	0 1 0 0 *
D	-	1 * 1 * 0
E	-	0 1 0 1 *
F	-	* * 1 1 1

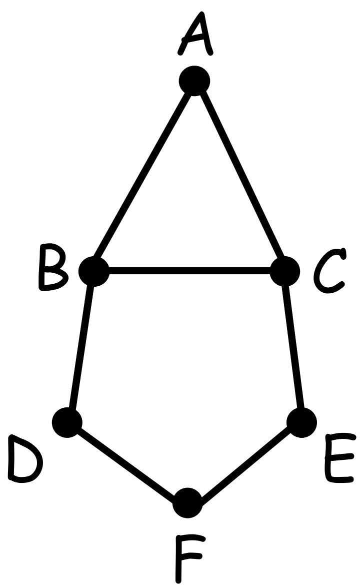
	A	B	C	D	E	F
A		1		1		
B						
C		1		1		
D						
E		1		1		
F						



vertex	-	address
A	-	0 0 0 0 0
B	-	1 * 0 0 *
C	-	0 1 0 0 *
D	-	1 * 1 * 0
E	-	0 1 0 1 *
F	-	* * 1 1 1

	A	B	C	D	E	F
A		1		1		
B						
C		1		1		
D						
E		1		1		
F						

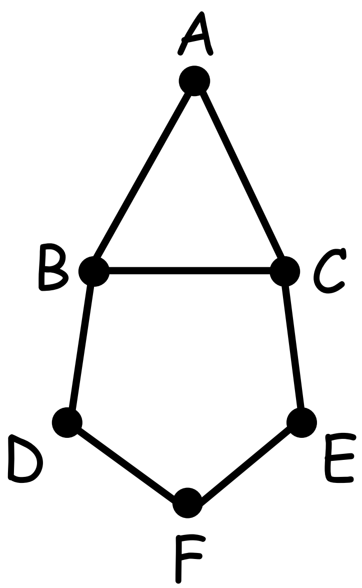
A C E x B D



vertex	-	address
A	-	00000
B	-	1*00*
C	-	0100*
D	-	1*1*0
E	-	0101*
F	-	**111

	A	B	C	D	E	F
A						
B	1		1		1	
C						
D	1		1		1	
E						
F						

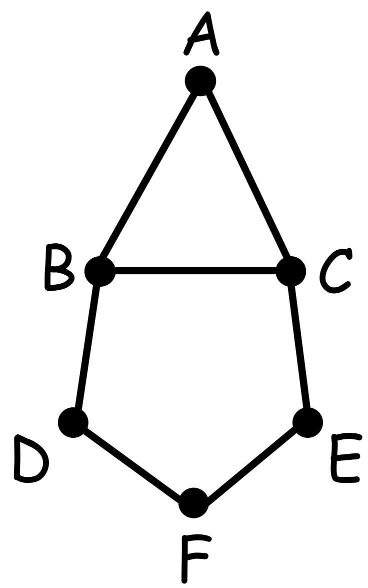
ACE x BD = BD x ACE



vertex	-	address
A	-	0 0 0 0 0
B	-	1 * 0 0 *
C	-	0 1 0 0 *
D	-	1 * 1 * 0
E	-	0 1 0 1 *
F	-	* * 1 1 1

	A	B	C	D	E	F
A			1		1	
B						
C						
D						
E						
F						

A × C E



vertex	-	address
A	-	0 0 0 0 0
B	-	1 * 0 0 *
C	-	0 1 0 0 *
D	-	1 * 1 * 0
E	-	0 1 0 1 *
F	-	* * 1 1 1

	A	B	C	D	E	F
A	0	1	1	2	2	3
B	1	0	1	1	2	2
C	1	1	0	2	1	2
D	2	1	2	0	2	1
E	2	2	1	2	0	1
F	3	2	2	1	1	0

Distance matrix $D(G) = (d_{ij})$

column	**contribution**
1	ACE x BD
2	A x CE
3	ABCE x DF
4	ABC x EF
5	AD x F

$$Q(G) = \tfrac{1}{2} \sum_{1 \le i,j \le n} d_{ij} x_i x_j = (x_1 + x_3 + x_5)(x_2 + x_4)$$

$$+ x_1 (x_3 + x_5)$$

$$+ (x_1 + x_2 + x_3 + x_5)(x_4 + x_6)$$

$$+ (x_1 + x_2 + x_3)(x_5 + x_6)$$

$$+ (x_1 + x_4) x_6$$

A valid extended addressing of G using N-tuples corresponds exactly to a decomposition of $Q(G) = \frac{1}{2} \sum_{1 \le i, j \le n} d_{ij} x_i x_j$ into a sum of N terms of form

$(x_{i_1} + x_{i_2} + \ldots + x_{i_r})(x_{j_1} + x_{j_2} + \ldots + x_{j_s})$.

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However, since $AB = \frac{1}{4}[(A+B)^2 - (A-B)^2]$

then

$$Q(G) = \sum_{N \text{ terms}} (x_{i_1} + x_{i_2} + \dots + x_{i_r})(x_{j_1} + x_{j_2} + \dots + x_{j_s})$$

$$= \sum_{N} \frac{1}{4}[(x_{i_1} + \dots + x_{i_r} + x_{j_1} + \dots + x_{j_s})^2 - (x_{i_1} + \dots + x_{i_r} - x_{j_1} - \dots - x_{j_s})^2]$$

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then

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$$= \sum_{N} \tfrac{1}{4}[(x_{i_1} + \ldots + x_{i_r} + x_{j_1} + \ldots + x_{j_s})^2 - (x_{i_1} + \ldots + x_{i_r} - x_{j_1} - \ldots - x_{j_s})^2]$$

Thus, Q(G) is congruent to a quadratic form which has N **positive** squares and N **negative** squares.

Hence, by Sylvester's law of inertia,

$$N \geq n_+(G) = \text{ number of positive eigenvalues of } D(G);$$

and

$$N \geq n_-(G) = \text{ number of negative eigenvalues of } D(G);$$

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Theorem: (Graham, Pollak, Witsenhausen)

$$N(G) \geq \max\{n_+(G), n_-(G)\}$$

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H.S. Witsenhausen

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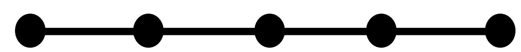
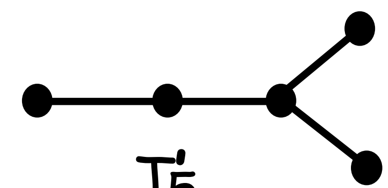
Question: How close to the truth is this bound?

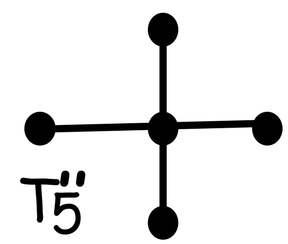
T_n - a tree with n vertices



T_5



T_5'



T_5''

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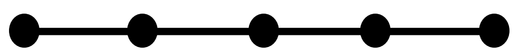
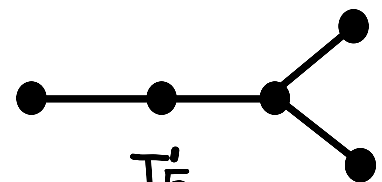
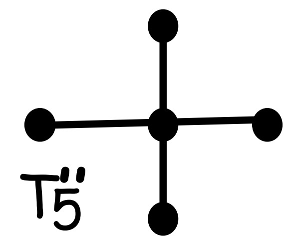


T_5



T_5'



T_5''

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

$D(T_5)$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 3 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 & 2 \\ 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

$D(T_5')$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

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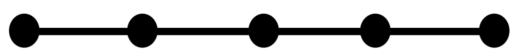
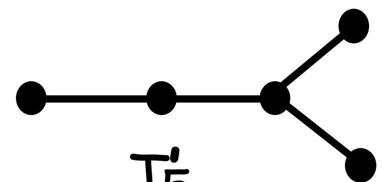


T_5

$$D(T_5) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

$n_+ = 1$

$n_- = 4$



T_5'

$$D(T_5') = \begin{bmatrix} 0 & 1 & 2 & 3 & 3 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 2 & 1 & 0 & 2 \\ 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

$n_+ = 1$

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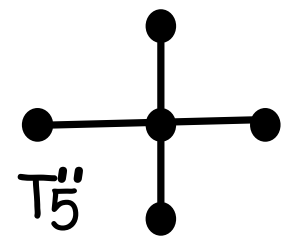


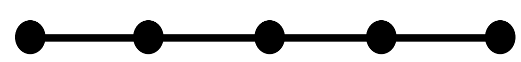
T_5''

$$D(T_5'') = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

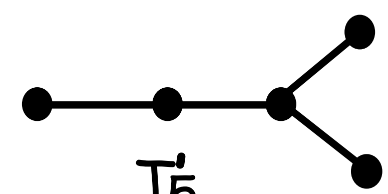
$n_+ = 1$

$n_- = 4$

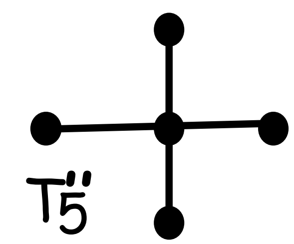
T_n - a tree with n vertices



T_5



T_5'



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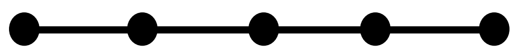
$D(T_5'')$

$n_+ = 1$

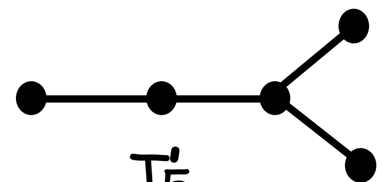
$n_- = 4$

$\det D(T_5) = 32$

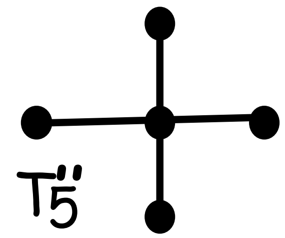
T_n - a tree with n vertices



T_5



T_5'



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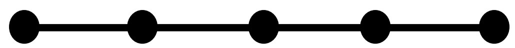
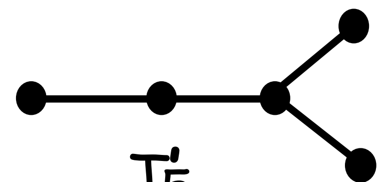
$\det D(T_5') = 32$!

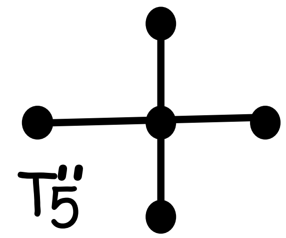
T_n - a tree with n vertices



T_5



T_5'



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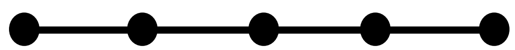
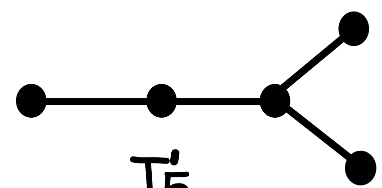
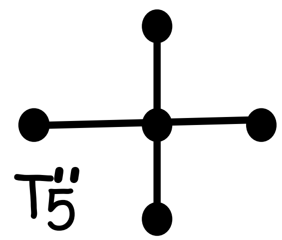
T_n - a tree with n vertices



T_5



T_5'



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$D(T_5'')$

$n_+ = 1$

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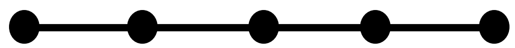
$\det D(T_5'') = 32$!!

A coincidence ?

T_n - a tree with n vertices



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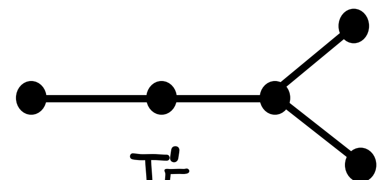
$\det D(T_5) = 32$



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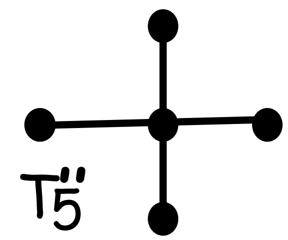

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$n_+ = 1$

$n_- = 4$

$\det D(T_5'') = 32$!!

A coincidence ? (or an example of the law of small numbers?)

If T_n is a tree with n vertices then

$$\det D(T_n) = (-1)^{n-1}(n-1)\,2^{n-2}$$

independent of the structure of the tree.

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independent of the structure of the tree.

This implies $n_+(T_n) = 1, \quad n_-(T_n) = n-1$

and so,

$$N(T_n) = n-1$$

for any tree T_n tree with n vertices.

Some questions

Is it true that $N(G) = \max\{n_+(G), n_-(G)\}$?

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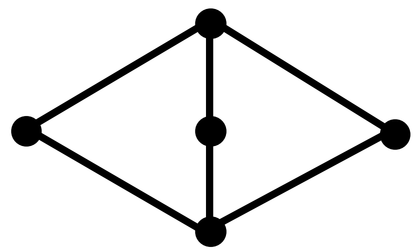
No! Take $G = K_{2,3}$.



$n_+(G) = 2$, $n_-(G) = 3$, $N(G) = 4$

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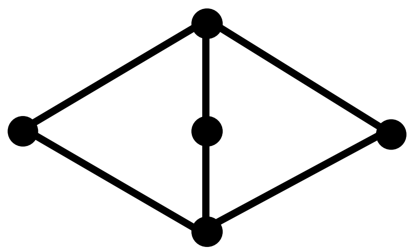

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What is the value of $N(K_{s,t})$ in general?

(It is between s+t-2 and s+t-1).

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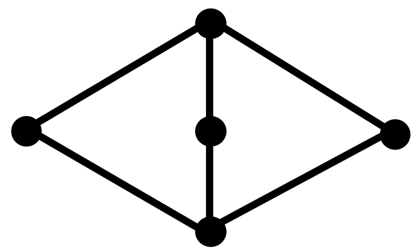

$n_+(G) = 2$, $n_-(G) = 3$, $N(G) = 4$

What is the value of $N(K_{s,t})$ in general?

(It is between s+t-2 and s+t-1).

Why is $n_+(G)$ so small in general?

What does det D(G) mean?

For example, $\det D(T_n) = (-1)^{n-1}(n-1)\,2^{n-2}$ for any tree T_n.

$n - 1$ is the number of edges in T_n.

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For example, $\det D(T_n) = (-1)^{n-1}(n-1)\,2^{n-2}$ for any tree T_n.

$n - 1$ is the number of edges in T_n.

In general, one could look at the **characteristic polynomial** of D(G), i.e., det (D(G) – xI) (where I denotes the n by n identity matrix).

The constant term is just det D(G).

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For example, $\det D(T_n) = (-1)^{n-1}(n-1)\,2^{n-2}$ for any tree T_n.

n - 1 is the number of edges in T_n.

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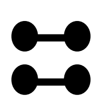
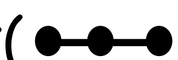
For G = T_n, we understand them (Graham/Lovász).

For example, the coefficient of x is

4 #() + 2 #() + 4 #() - 4

Which graphs have valid addressings which use only 0's and 1's (i.e., no *'s)?

That is, which graphs can be isometrically embedded in an N-cube?

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Theorem (Djoković)

G can be isometrically embedded into an N-cube if and only if for every edge {u,v} of *G*, the set of vertices S(u) which are closer to u than to v is closed under taking shortest paths, i.e., all shortest paths between any two vertices in S(u) stay within S(u).

What about addressing directed graphs?

Again, we use N-tuples of 0's, 1's and *'s. Now, however, we modify the "Hamming distance" between two N-tuples so that $d_{H^*}(a,b) = 1$ if and only if a=0 and b=1 (so that $d_{H^*}(1,0) = 0$).

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<u>Theorem</u> (Chung, Graham, Winkler)

Any strongly connected directed graph has a valid addressing using 0's, 1's and *'s.

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Theorem If G has n vertices then $N^*(G) \leq \frac{3}{4}n^2 + o(n^2)$.

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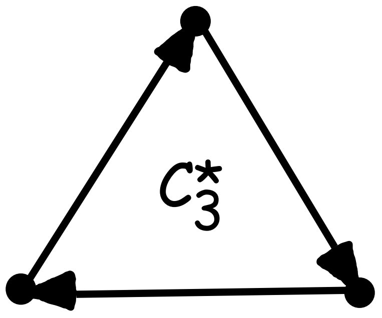
What is the right constant here??

The simplest strongly connected directed graph C_n^*

(a directed cycle on n vertices)



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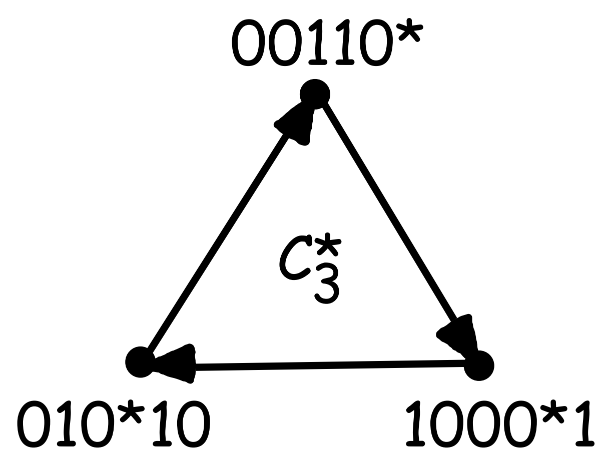

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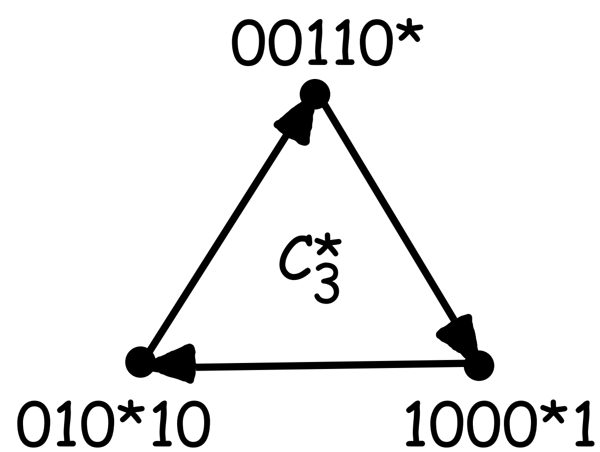

$N^*(C_3^*) \leq 6$

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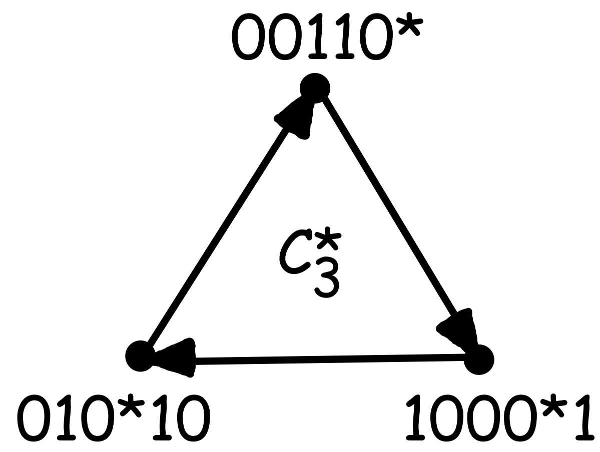

$N^*(C_3^*) \leq 6$

There exists positive constants c and c′ such that

$$cn^{\frac{3}{2}} < N^*(C_n^*) < c' n^{\frac{5}{3}} (\log n)^{\frac{1}{3}}$$

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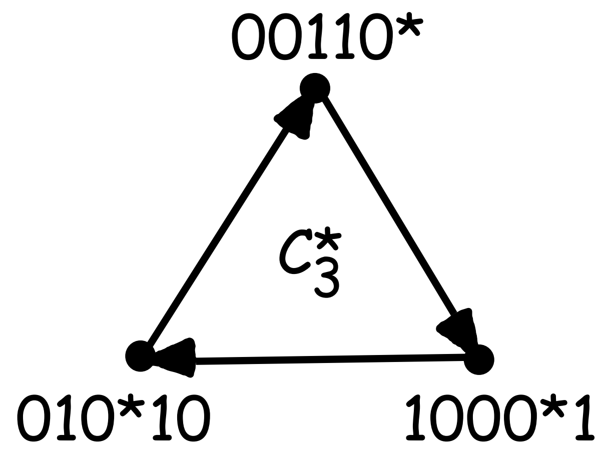

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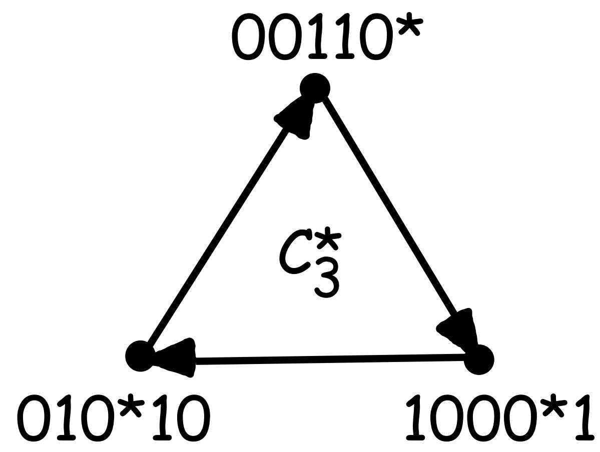

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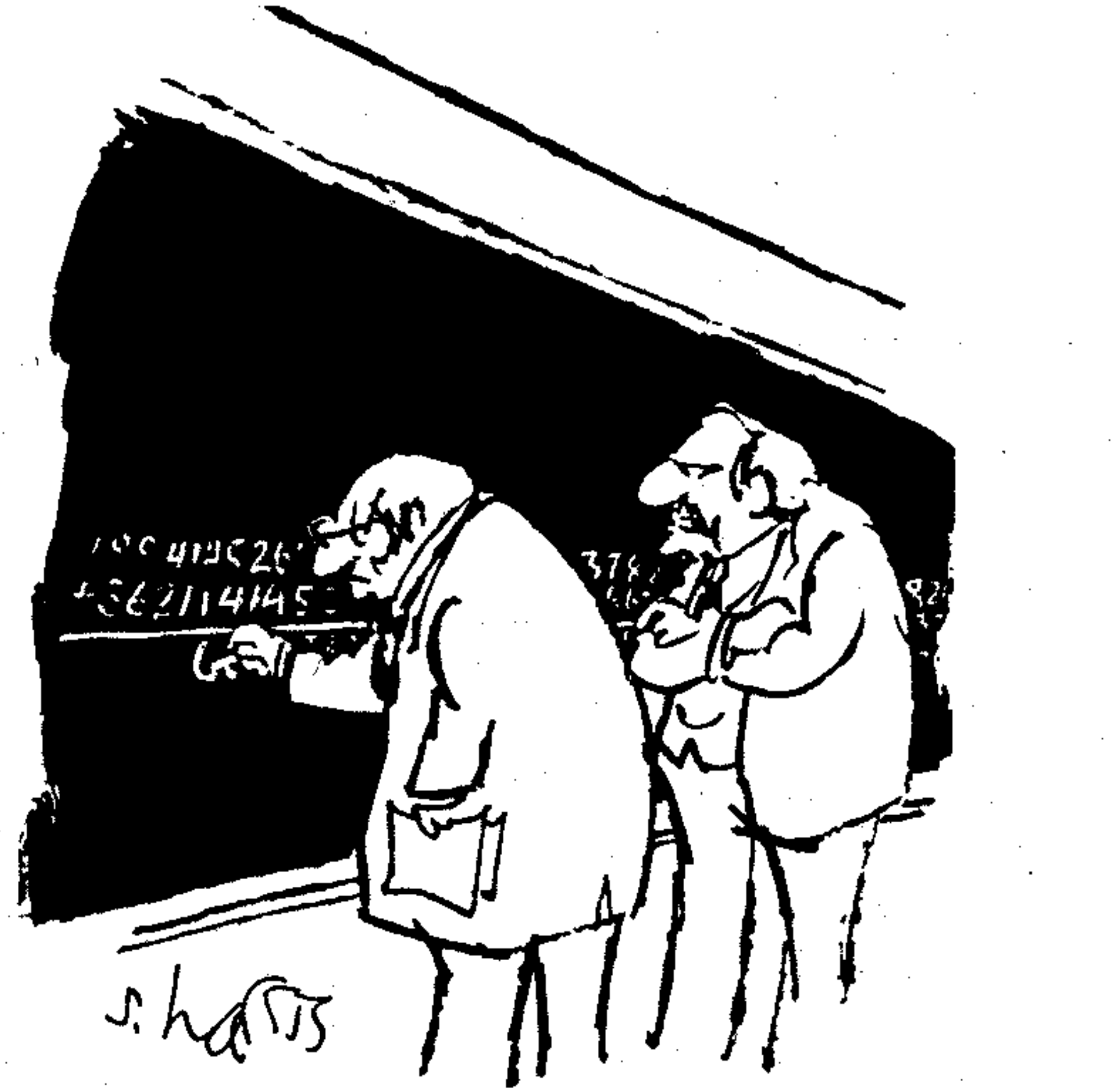
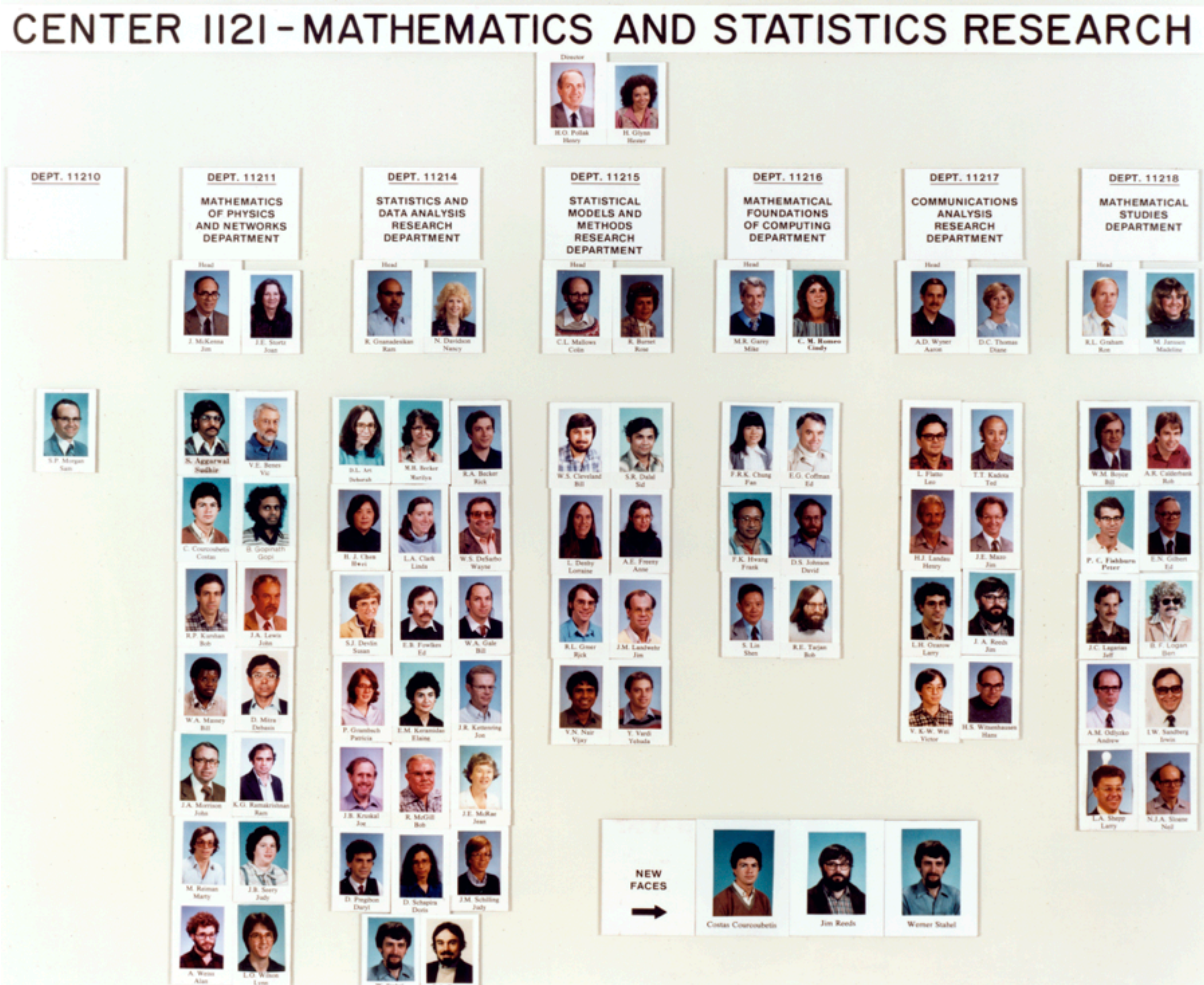
Clearly, there is **lots** more to be done!



"Adding two numbers which probably have never been added before is ***not*** considered a mathematical breakthrough."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Costas Courcoubetis
Jim Reeds
Werner Stahel

Really LARGE numbers

Really LARGE numbers

41

Super-base-2 expansion

$$41 = 2^{5} + 2^{3} + 1$$

$$(32 + 8 + 1)$$

Super-base-2 expansion

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First step: Replace each 2 by 3, subtract 1, and write the result in a super-base-3 expansion;

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$$3^{3^3+1} + 3^{3+1} \rightarrow 4^{4^4+1} + 4^{4+1} - 1$$

$$= 4^{4^4+1} + 3 \cdot 4^4 + 3 \cdot 4^3 + 3 \cdot 4^2 + 3 \cdot 4 + 3$$

$$4^{4^{4} + 1} + 3 \cdot 4^{4} + 3 \cdot 4^{3} + 3 \cdot 4^{2} + 3 \cdot 4 + 3$$

= 5363123171977038839829609999282338450991746328236957/
35108942457748870561202941879072074971926676137107601/
27432745944203415015531247786279785734596024337407

$$4^{4^{4+1}} + 3 \cdot 4^4 + 3 \cdot 4^3 + 3 \cdot 4^2 + 3 \cdot 4 + 3$$

= 53631231719770388398296099992823384509917463282369 57/
3510894245774887056120294187907207497192667613710 7601/
27432745944203415015531247786279785734596024337407

<u>The general step</u>: Replace the current super-base b by b+1, subtract 1, and then express the new number in a super-base-(b+1) expansion.

For example, the next step for us would be

$$4^{4^{4+1}} + 3 \cdot 4^4 + 3 \cdot 4^3 + 3 \cdot 4^2 + 3 \cdot 4 + 3$$

$$\rightarrow 5^{5^{5+1}} + 3 \cdot 5^5 + 3 \cdot 5^3 + 3 \cdot 5^2 + 3 \cdot 5 + 2$$

$$5^{5^{5}+1} + 3 \cdot 5^{5} + 3 \cdot 5^{3} + 3 \cdot 5^{2} + 3 \cdot 5 + 2 \ =$$

**95550629897273876017820227985198229959904052449504716856975639462326026512130
79015060296932598699251327932200778972311176796063943369034861442050734579933
01043980948378597850919640830169023805612987766813050500741325561706573884126
20574654722358848264137814259836875719767877123954660960332094150589358456127
62105350253545323371914354257249751282930972307715917556899245668458899640637
16920215774618427763391798187051052665773015676862662874318454579889345164133
22959149190761514346828643684571132406564587188106816286516082264148974343128
81226811090088366124702838214096800393603569185361776527231780769732005926742
46896359757297252754116374610802924456455472594979974343099771573833469006518
58808179629723987308211002544253973490224356660256658036956711527009943628501
91649006230250985067336985879545136947469619086578934984229498973905340214112
18046891973167632711407852151416221192757541158245483642856085854061616395240
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45037638388966761531003091430062276271215305034474027232923524103254913321596
80480194368129255373537170318143488288351349629324976778988159086951275445665
61164737196517197808066416703641583174912907261343100215389954234405190209368
41624004519367981064598168012915603908368368712666614396484536027452978107034
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35961507277914583733975987152740132023234270013669969303992972329807508762934
82905723784255020784343865451856241267671919642698799374729248525019112506244
64200091329502812564309381496902220367007117353102789265266251745909479485359
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181118344148425104419748071415242369556995834811324974281842617356436647398340
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49834377526252197109116095678611527033357686687124271822831891022850827296609
07702677419680712533224929270165373323427094507406717385732515751897708788931
1405888292938470840454102546**7

(Unbelievable) Fact: If we keep on repeating this process, we will eventually reach 0 !

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Goodstein's Theorem:

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$$G(4) = 3\cdot 2^{z}(2^{3\cdot 2^{z}} - 1) + 4 > 10^{10^{,0}}$$

No one has ever computed $G(5)$ exactly.

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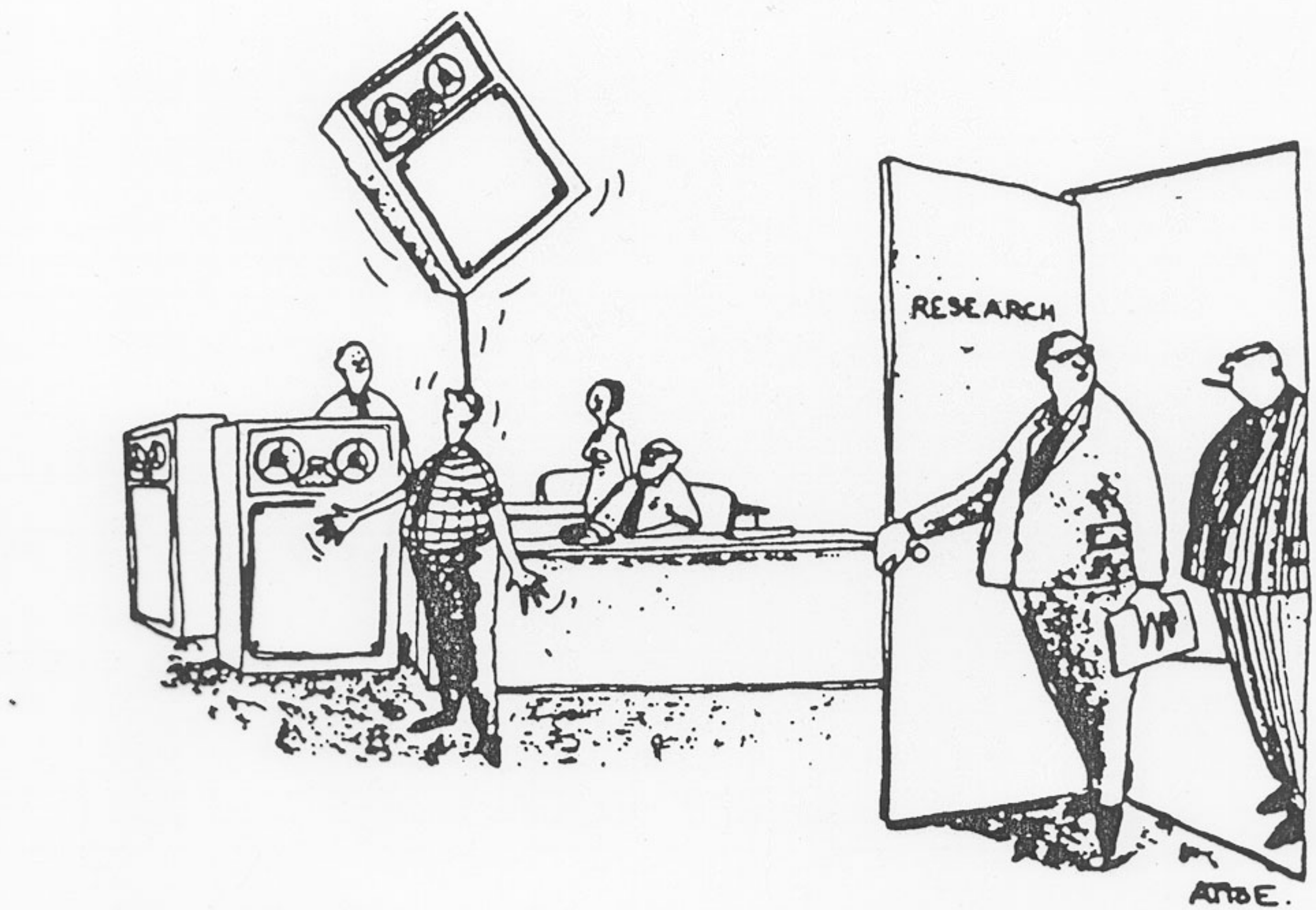
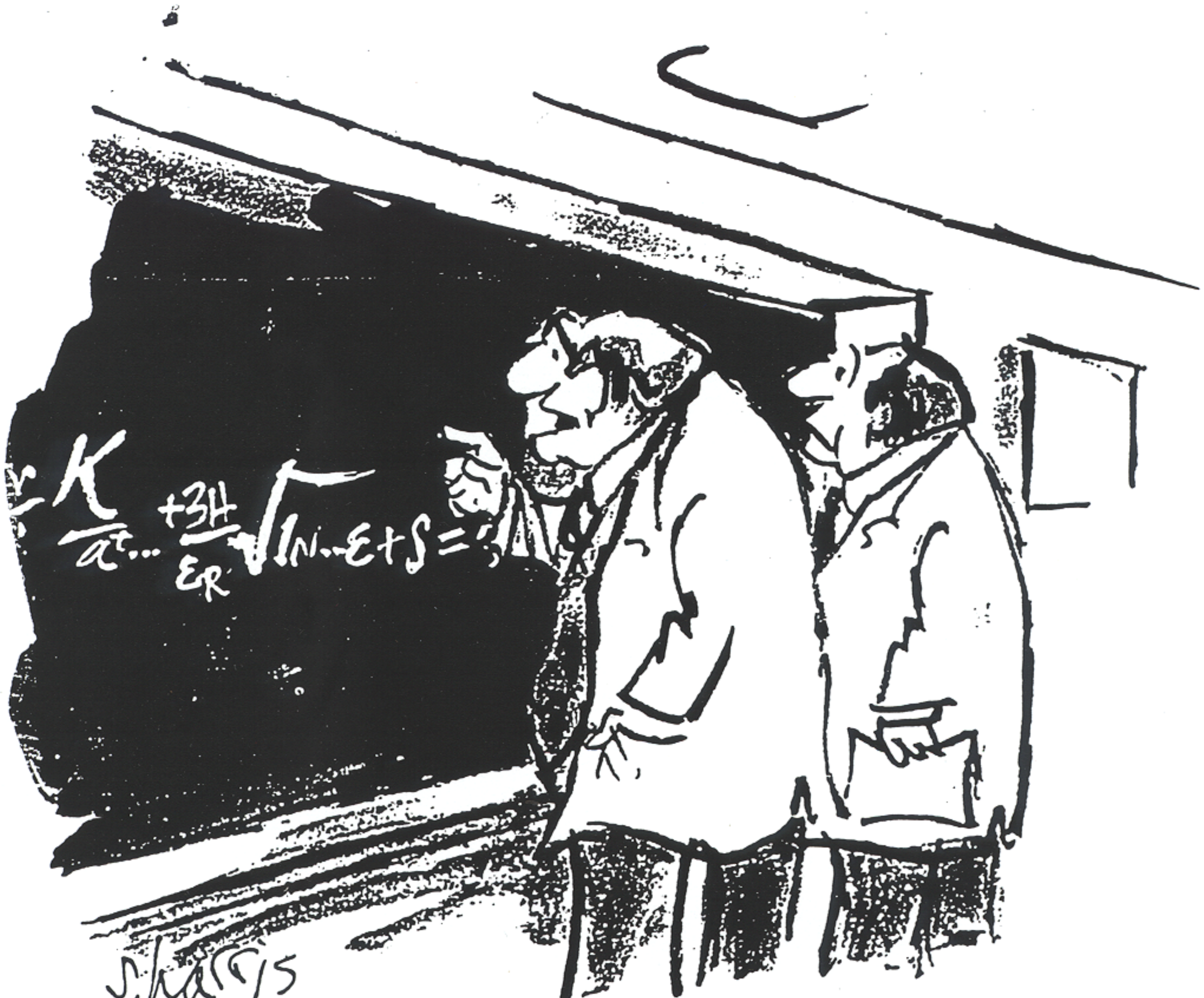
No one has ever computed G(6) exactly. $25

THE CARDINALS

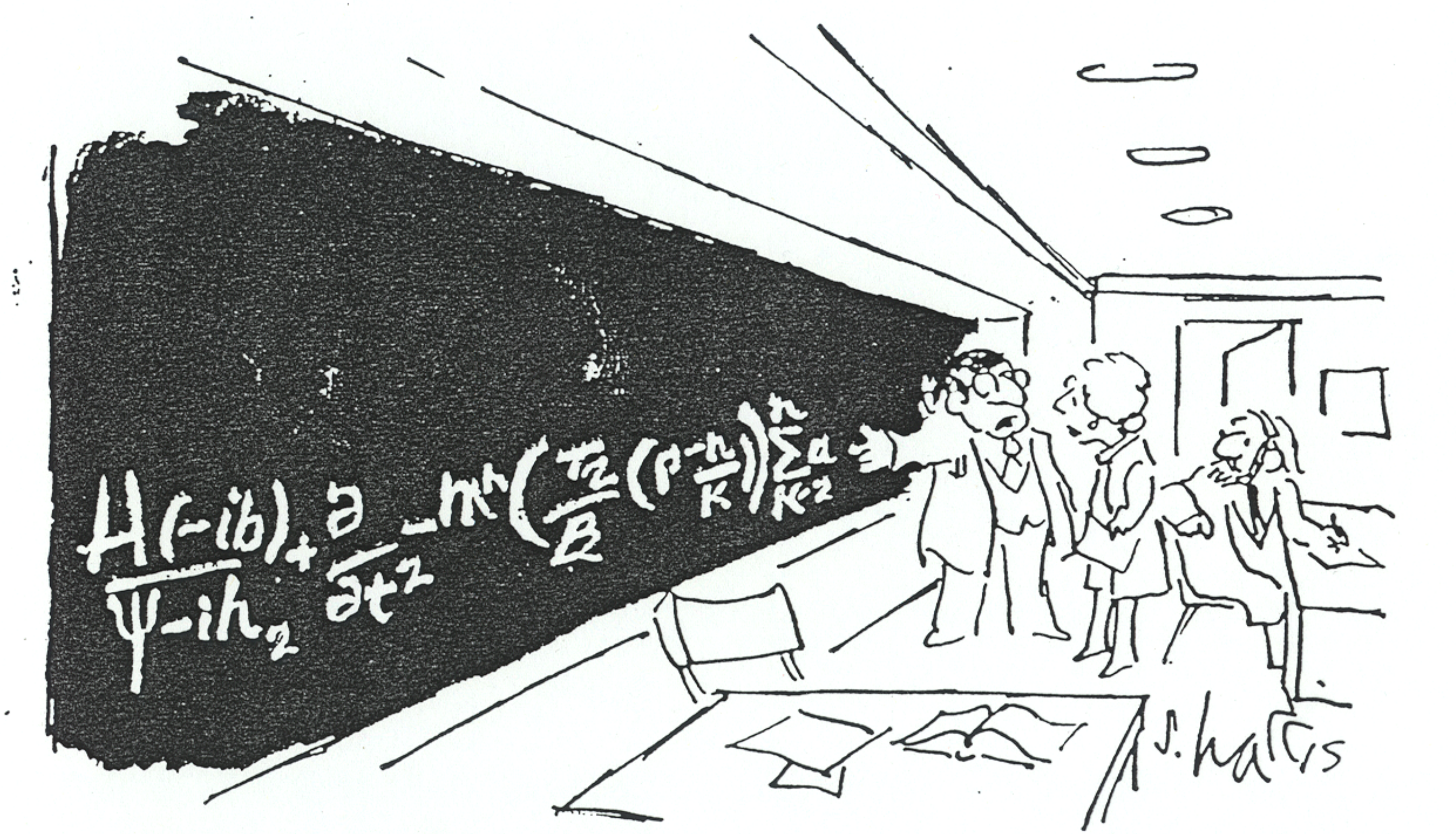
THE ORDINALS



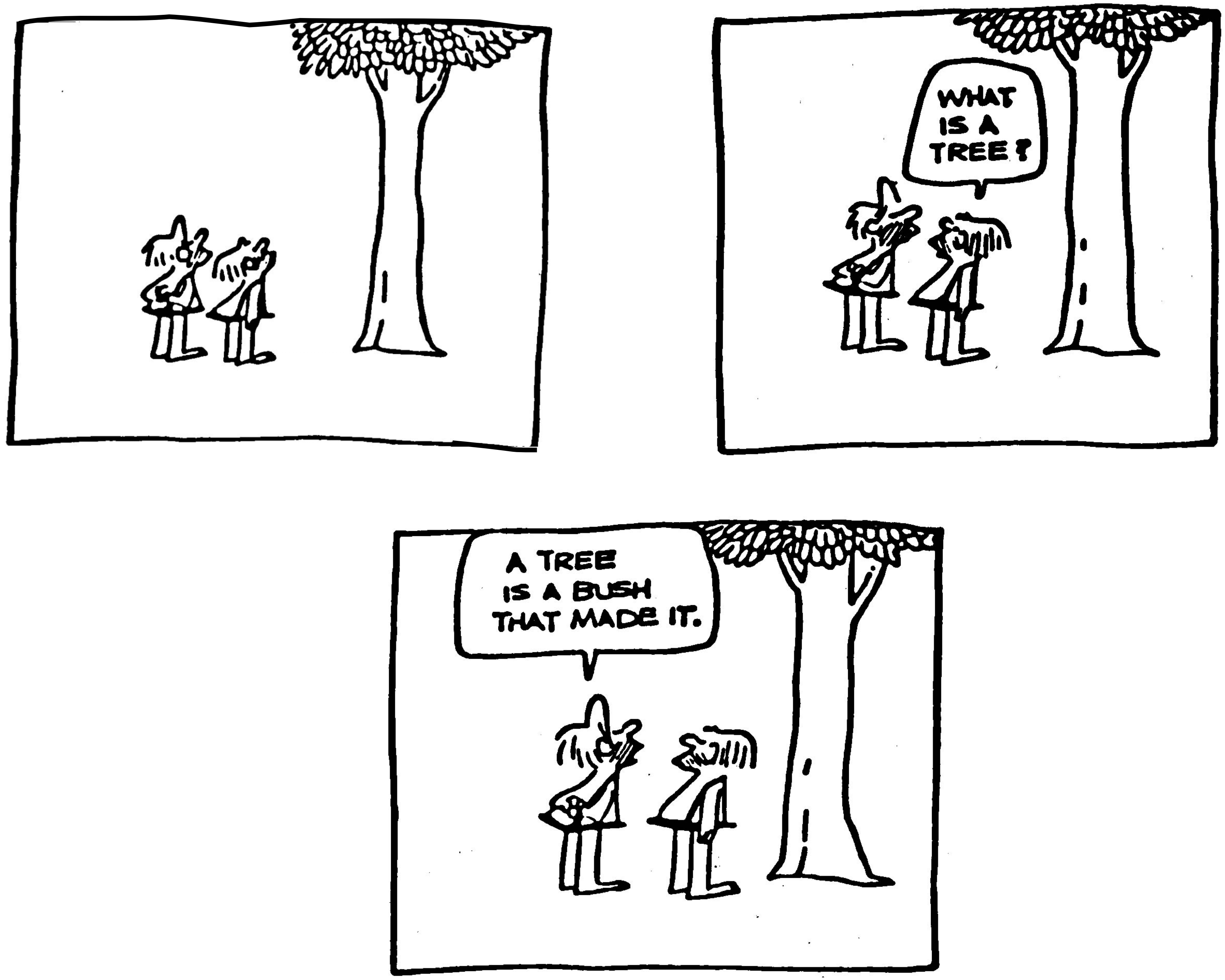
I hear they are doing some amazing things with computers these days.



"Very creative. Very imaginative. Logic……*that's* what's missing."



"But this *is* the simplified version for the general public."


WHAT IS A TREE?
A TREE IS A BUSH THAT MADE IT.

Example: $G = K_n$, the complete graph on n vertices.

$$D(K_n) = \begin{pmatrix} 0,1,1,\ldots\ldots,1 \\ 1,0,1,\ldots\ldots,1 \\ 1,1,0,\ldots\ldots,1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1,1,1,\ldots\ldots,0 \end{pmatrix}$$

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If r is an nth root of unity then

$$\begin{pmatrix} 0,1,1,......,1 \\ 1,0,1,......,1 \\ 1,1,0,......,1 \\ .... \\ 1,1,1,......,0 \end{pmatrix} \begin{pmatrix} 1 \\ r \\ r^2 \\ . \\ . \\ r^{n-1} \end{pmatrix} = (r + r^2 + r^3 + ... + r^{n-1}) \begin{pmatrix} 1 \\ r \\ r^2 \\ . \\ . \\ r^{n-1} \end{pmatrix}$$

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But $(r + r^2 + r^3 + \ldots + r^{n-1}) = \begin{cases} -1 & \text{if } r^n = 1, r \neq 1 \\ n-1 & \text{if } r = 1 \end{cases}$

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$$D(K_n) = \begin{pmatrix} 0,1,1,\ldots\ldots,1 \\ 1,0,1,\ldots\ldots,1 \\ 1,1,0,\ldots\ldots,1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1,1,1,\ldots\ldots,0 \end{pmatrix}$$

If r is an nth root of unity then

$$\begin{pmatrix} 0,1,1,\ldots\ldots,1 \\ 1,0,1,\ldots\ldots,1 \\ 1,1,0,\ldots\ldots,1 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1,1,1,\ldots\ldots,0 \end{pmatrix} \begin{pmatrix} 1 \\ r \\ r^2 \\ . \\ . \\ r^{n-1} \end{pmatrix} = (r + r^2 + r^3 + \ldots + r^{n-1}) \begin{pmatrix} 1 \\ r \\ r^2 \\ . \\ . \\ r^{n-1} \end{pmatrix}$$

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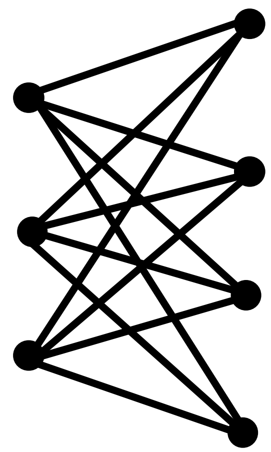
Consequently, $N(K\ ) = n-1$



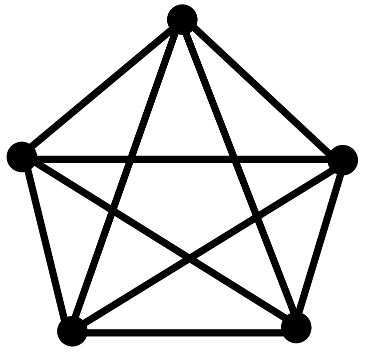
$K_{3,4}$ - complete bipartite graph

$N(K_n) = n-1$

Equivalent statement: K_n cannot be decomposed into fewer than n-1 complete bipartite edge-disjoint subgraphs

(since each term $(x_{i_1} + x_{i_2} + \ldots + x_{i_r})(x_{j_1} + x_{j_2} + \ldots + x_{j_s})$ in the sum corresponds to a complete bipartite subgraph $K_{r,s}$).

For example, K_5



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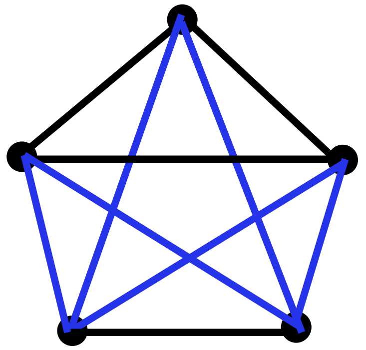

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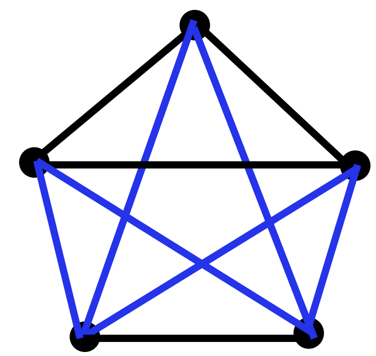

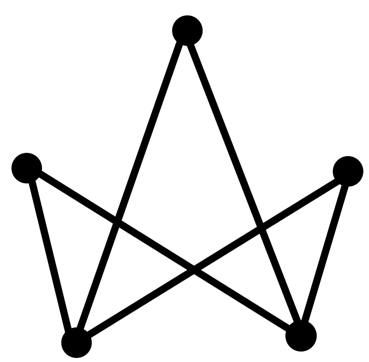

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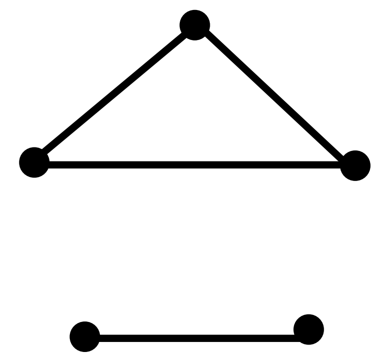

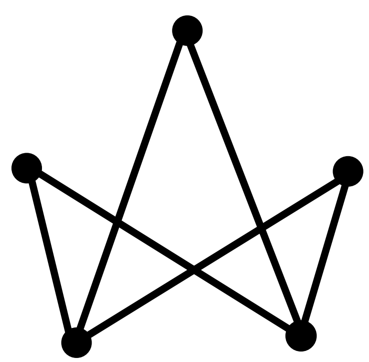

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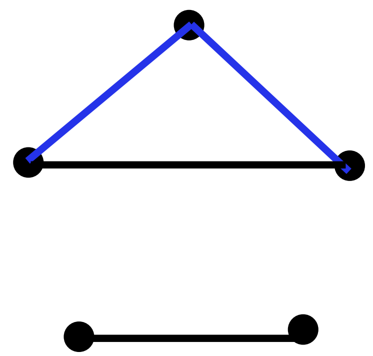

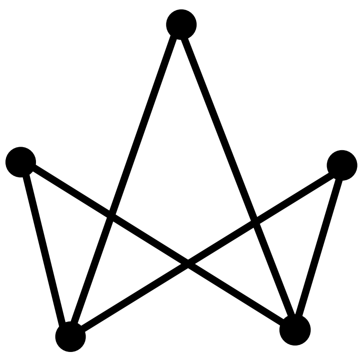

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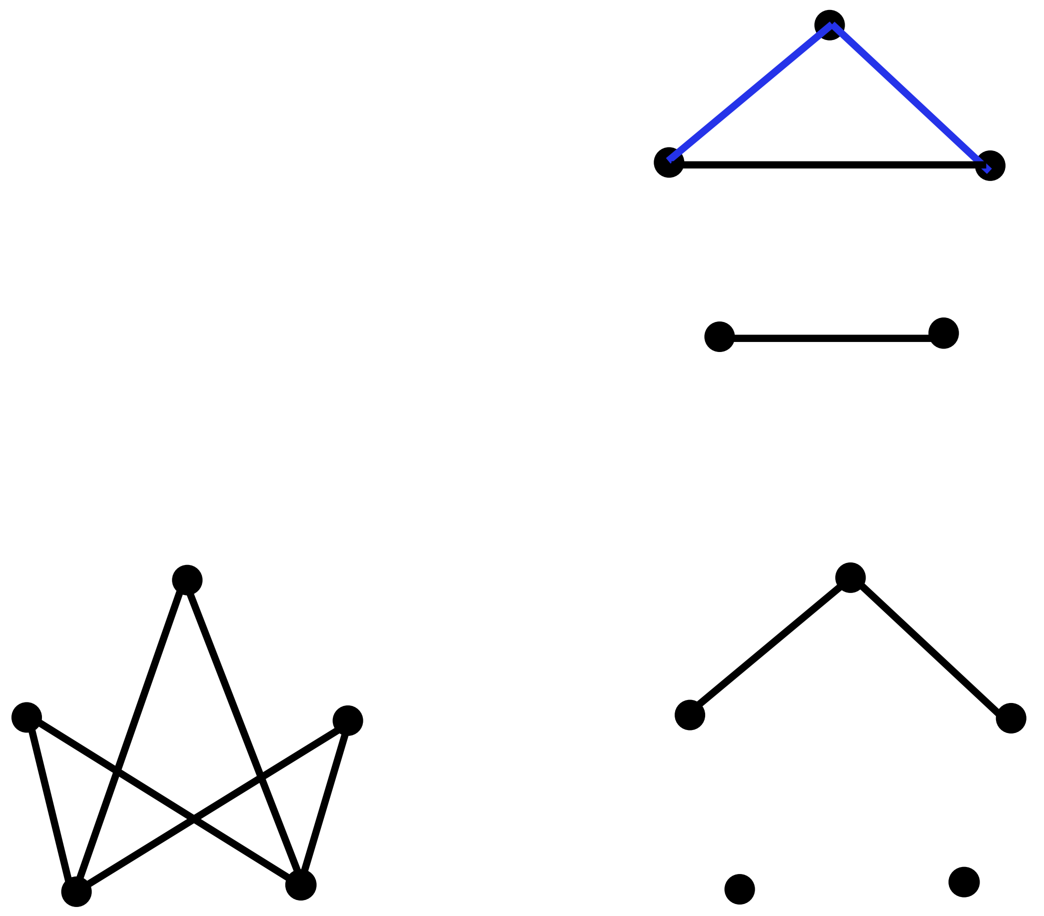

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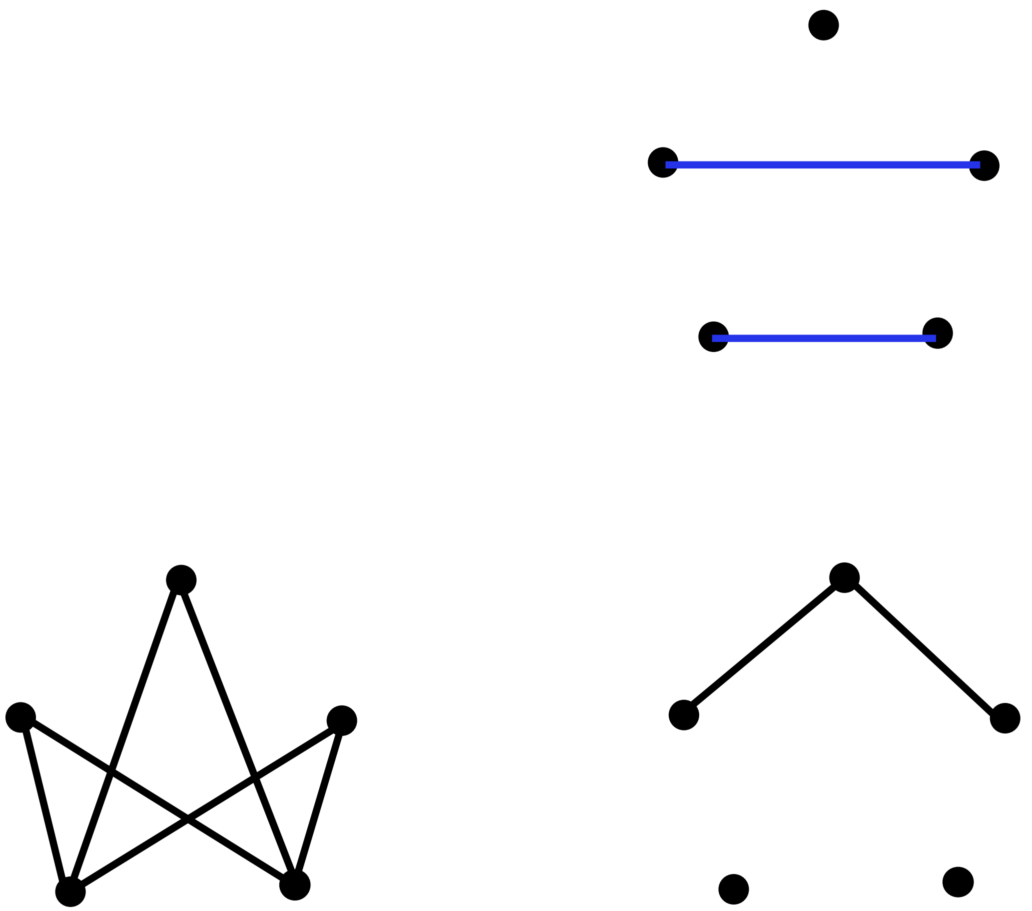

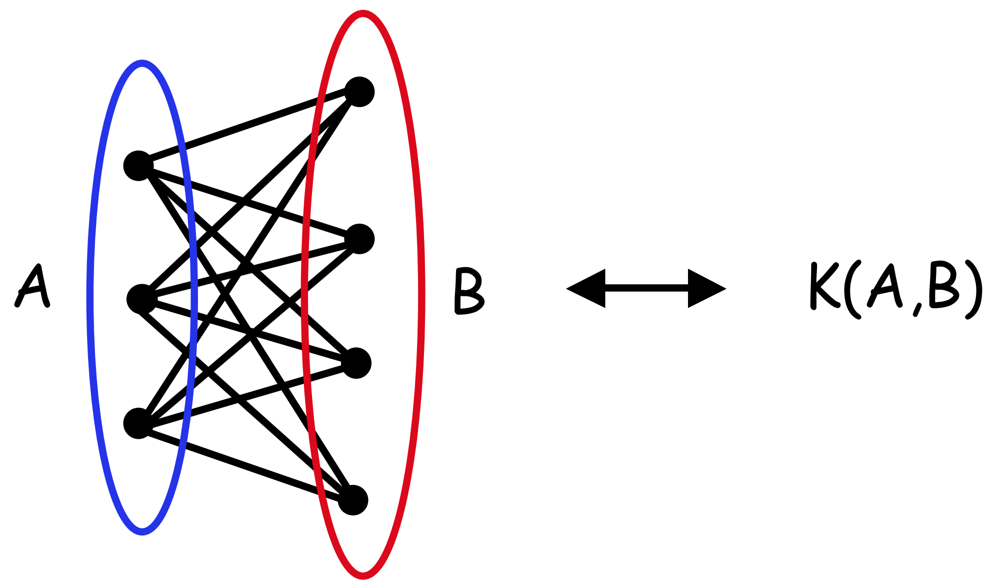

complete bipartite graph on vertex sets A and B

K_n cannot be decomposed into fewer than
n-1 complete bipartite edge-disjoint subgraphs

In other words,

$$K_n = \sum_{k=1}^{t} K(A_k, B_k) \Rightarrow t \geq n-1$$

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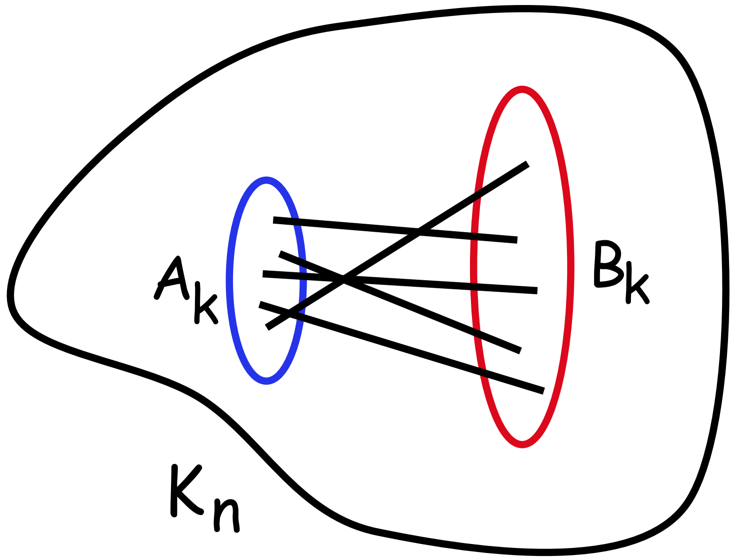

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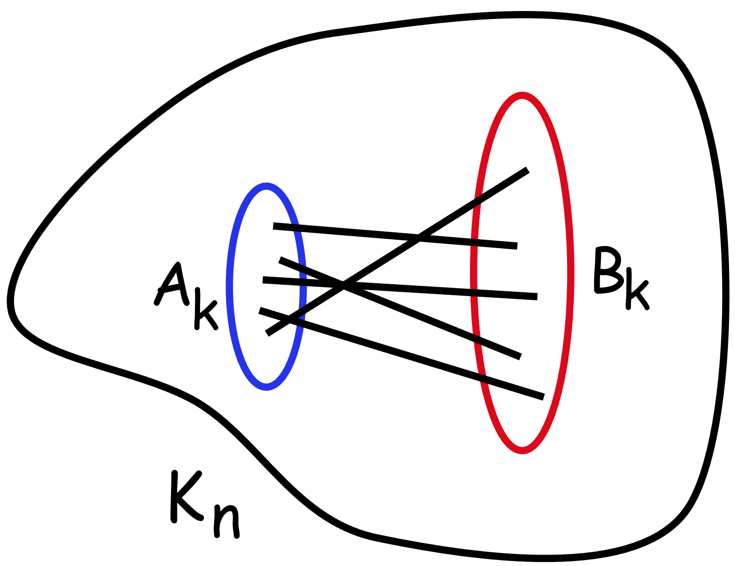
<u>**Non-eigenvalue proof**</u> (H. Tverberg)



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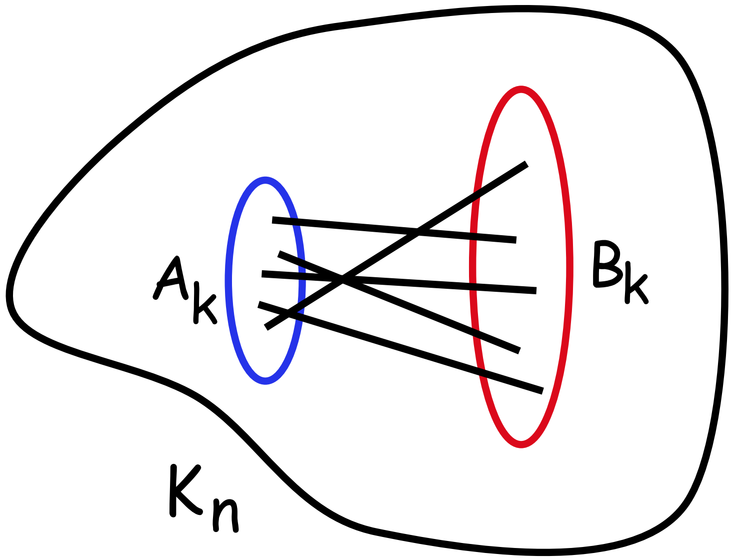

Hypothesis implies

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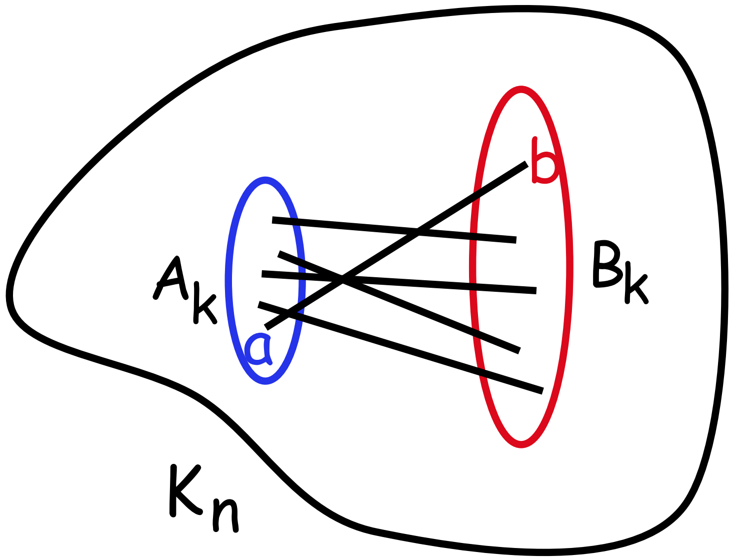

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Consider the system of t+1 homogeneous linear equations in the n variables x_i :

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$$= \sum_{i=1}^{n} x_i^2$$

Therefore, $x_i = 0$ for all i.

Thus, the number of equations must be at least as large as the number of variables, i.e., $t + 1 \ge n$, as claimed. ∎