On coherent dynamical systems

Morris W. Hirsch

University of California at Berkeley
University of Wisconsin at Madison

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A few of E.D. Sontag's articles on monotone and coherent systems:

Graph-theoretic characterizations of monotonicity of chemical networks in reaction coordinates,
J. Math. Biology **61** 2010 (with D. Angeli & P. De Leenheer)

Attractors in coherent systems of differential equations,
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Singularly perturbed monotone systems and an application to double phosphorylation cycles,
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Background

A vector field $F : X \to \mathbf{R}^n$ in a convex open set $X \subset \mathbf{R}^n$ is **cooperative** if $\frac{\partial F_i}{\partial x_j} \geq 0, \ (i \neq j)$.

This implies the flow Φ of F is **monotone**:

If $x > y$ and $t > 0$, then $\Phi_t x > \Phi_t y$.

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Feedback Loop: Sequence $i_0, \dots, i_m = i_0, \ m \geq 1$ such that

$$i_k \neq i_{k-1} \quad \text{and} \quad S_k := \frac{\partial F_{i_k}}{\partial x_{i_{k-1}}} \not\equiv 0, \quad (k = 1, \dots, m).$$

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Coherence:

A system is **coherent** if every feedback loop is positive.

The fundamental theorem on coherent systems

Cascade Decomposition Theorem: (Angeli-Hirsch-Sontag)

- A coherent system $\dot{x} = F(x)$ in $X \subset \mathbf{R}^n$ is transformed, by permuting and changing signs of variables, to

$$\boxed{\dot{z} = H(z, y),\ \ \dot{y} = G(y), \quad (z, y) \in X \subset \mathbf{R}^{n-m} \times \mathbf{R}^m}$$

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- The **fibre system** $\dot{z} = H(z, p),\ z \in \mathbf{R}^{n-m}$ is **coherent**.

Attracting sets

An **attracting set** for a system with flow Φ is nonempty compact invariant set K that attracts all points in an open set $U \supset K$:

$$\boxed{\lim_{t\to\infty} \operatorname{dist}(\Phi_t x, K) = 0, \qquad (x \in U).}$$

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The ODEs that model interacting species, chemical reactions, or dissipative mechanical systems, usually have global attractors, but volume preserving systems have no attractors.

Coherent systems are nonchaotic

Theorem —Angeli-Hirsch-Sontag

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For coherent systems one proceeds by induction on dimension, exploiting the Cascade Decomposition Theorem.

Periodic points

$p \in X$ is **periodic** with **period** $\lambda > 0$ if and $\Phi_\lambda p = p$.

$\mathcal{P}_\lambda =$ the set of points of **minimal period** $\lambda > 0$.

$\mathcal{P} =$ the set of all periodic points, including fixed points.

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Theorem (Resonance in monotone systems)

Assume Φ is monotone and $p \in \mathcal{P}_\lambda$. Then there is a neighborhood U of p such that:

If $q \in \mathcal{P}_\mu \cap U$ and $q \geq p$ or $q \leq p$, then $\frac{\mu}{\lambda}$ is **rational**.

Nondensity of periodic points

Theorem

Assume $F : X \to \mathbf{R}^n$ is coherent.

If periodic points are dense in an open set $W \subset X$, there is a dense open subset $V \subset W$ such that F is globally periodic in each component of V.

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Proofs:

(1) For monotone maps: Lattice properties of $\mathbf{R}^n$

(2) For cooperative systems: Resonance.

(3) For coherent systems: Cascade Decomposition and induction on dimension.

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