### On coherent dynamical systems

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# Happy birthday, Eduardo!

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This implies the flow  $\Phi$  of *F* is **monotone**:

If x > y and t > 0, then  $\Phi_t x > \Phi_t y$ .

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**Feedback Loop:** Sequence  $i_0, \ldots, i_m = i_0$ ,  $m \ge 1$  such that

$$i_k \neq i_{k-1}$$
 and  $S_k := \frac{\partial F_{i_k}}{\partial x_{i_{k-1}}} \not\equiv 0$ ,  $(k = 1, \dots, m)$ .

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#### Coherence:

A system is **coherent** if every feedback loop is positive.

# The fundamental theorem on coherent systems

### Cascade Decomposition Theorem:

(Angeli-Hirsch-Sontag)

A coherent system ẋ = F(x) in X ⊂ ℝ<sup>n</sup> is transformed, by permuting and changing signs of variables, to

 $\dot{z} = H(z, y), \ \dot{y} = G(y), \quad (z, y) \in X \subset \mathbf{R}^{n-m} \times \mathbf{R}^m$ 

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is invariant under F.

• The fibre system  $\dot{z} = H(z, p), z \in \mathbb{R}^{n-m}$  is coherent.

### Attracting sets

An **attracting set** for a system with flow  $\Phi$  is nonempty compact invariant set K that attracts all points in an open set  $U \supset K$ :

$$\lim_{t\to\infty} \operatorname{dist}(\Phi_t x, K) = 0, \qquad (x \in U).$$

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#### Attractors in applications

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#### Attractors in applications

The ODEs that model interacting species, chemical reactions, or dissipative mechanical systems, usually have global attractors, but volume preserving systems have no attractors.

#### Theorem

-Angeli-Hirsch-Sontag

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For coherent systems one proceeds by induction on dimension, exploiting the Cascade Decomposition Theorem.

### Periodic points

- $p \in X$  is **periodic** with **period**  $\lambda > 0$  if and  $\Phi_{\lambda}p = p$ .
- $\mathcal{P}_{\lambda}$  = the set of points of **minimal period**  $\lambda > 0$ .
- $\mathcal{P}=$  the set of all periodic points, including fixed points.
- A system is **globally periodic** if all points have a common period.

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A system is globally periodic if all points have a common period.

#### Theorem (Resonance in monotone systems)

Assume  $\Phi$  is monotone and  $p \in \mathcal{P}_{\lambda}$ . Then there is a neighborhood U of p such that:

If  $q \in \mathcal{P}_{\mu} \cap U$  and  $q \ge p$  or  $q \le p$ , then  $\frac{\mu}{\lambda}$  is **rational**.

#### Theorem

Assume  $F : X \to \mathbf{R}^n$  is coherent.

If periodic points are dense in an open set  $W \subset X$ , there is a dense open subset  $V \subset W$  such that F is globally periodic in each component of V.

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The same conclusions hold for monotone **maps**  $f : X \to X$ .

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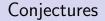
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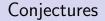
The same conclusions hold for monotone maps  $f : X \rightarrow X$ . **Proofs:** 

- (1) For monotone maps: Lattice properties of  $\mathbf{R}^n$
- (2) For cooperative systems: Resonance.
- (3) For coherent systems: Cascade Decomposition and induction on dimension.



In coherent systems, periodic orbits are nowhere dense in attractors.

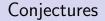
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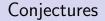
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