# The jump from nonlinear to hybrid systems: where's the impact?

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(SontagFest)
Rutgers University, May23-27, 2011

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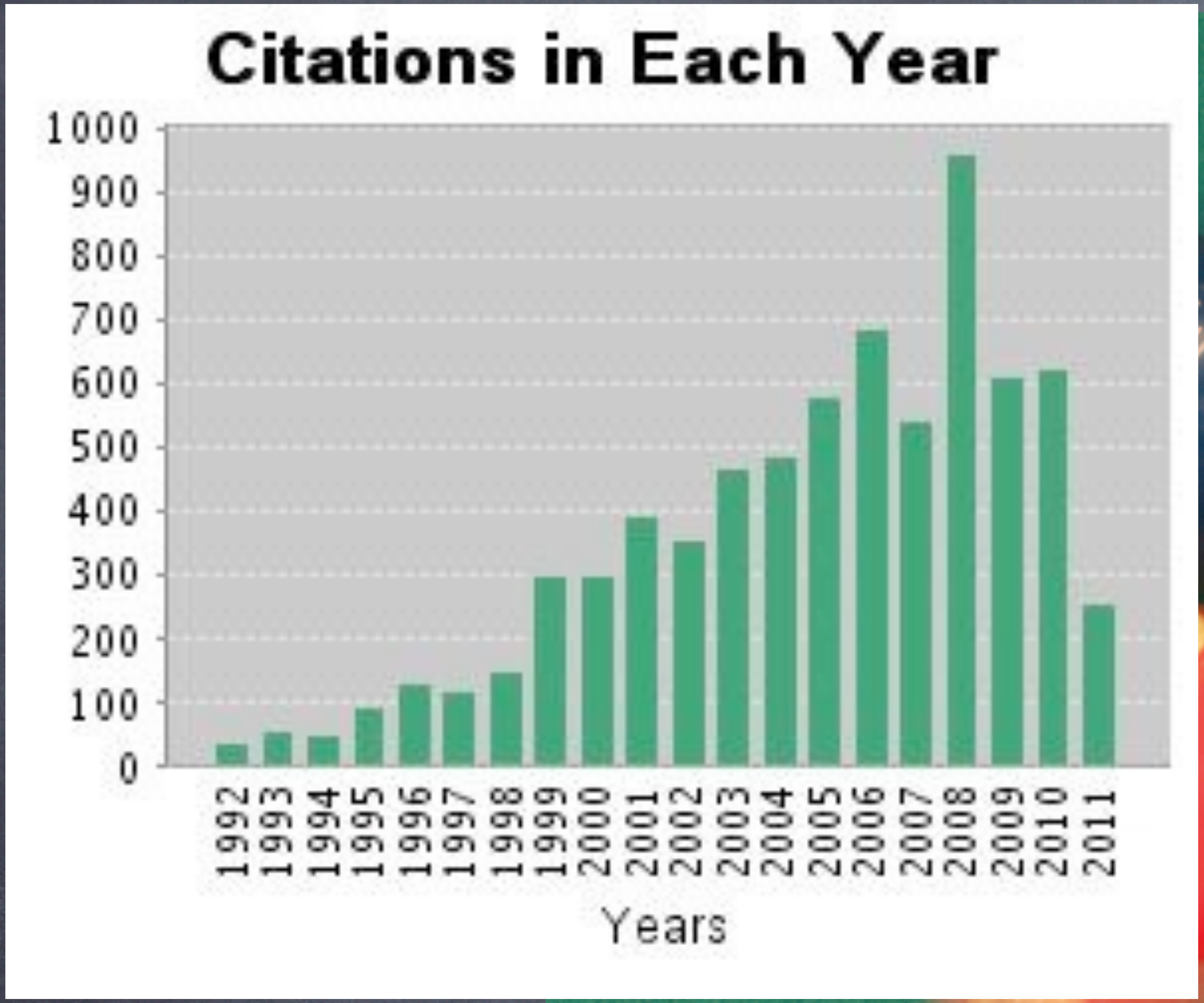
# Eduardo's research: where's the impact?




The impact is extensive

# I have favorites ... My top 5:

ISI Web of Knowledge [v.4.10] – Web of Science

http://apps.isiknowledge.com/summary.do?product=WOS&search_mode=CitationReport&qi

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| | Use the checkboxes to remove individual items from this Citation Report | 2007 | 2008 | 2009 | 2010 | 2011 | Total | Average Citations per Year |
|---|---|---|---|---|---|---|---|---|
| 1. | Title: SMOOTH STABILIZATION IMPLIES COPRIME FACTORIZATION<br>Author(s): SONTAG ED<br>Source: **IEEE TRANSACTIONS ON AUTOMATIC CONTROL** Volume: **34** Issue: **4** Pages: **435-443** Published: **APR 1989** | 28 | 83 | 51 | 26 | 16 | 677 | 29.43 |
| 2. | Title: A smooth converse Lyapunov theorem for robust stability<br>Author(s): Lin YD, Sontag ED, Wang Y<br>Source: **SIAM JOURNAL ON CONTROL AND OPTIMIZATION** Volume: **34** Issue: **1** Pages: **124-160** Published: **JAN 1996** | 13 | 30 | 26 | 12 | 4 | 240 | 15.00 |
| 3. | Title: Comments on integral variants of ISS<br>Author(s): Sontag ED<br>Source: **SYSTEMS & CONTROL LETTERS** Volume: **34** Issue: **1-2** Pages: **93-100** Published: **MAY 25 1998** | 15 | 20 | 14 | 14 | 10 | 199 | 14.21 |
| 4. | Title: FURTHER FACTS ABOUT INPUT TO STATE STABILIZATION<br>Author(s): SONTAG ED<br>Source: **IEEE TRANSACTIONS ON AUTOMATIC CONTROL** Volume: **35** Issue: **4** Pages: **473-476** Published: **APR 1990** | 3 | 12 | 3 | 6 | 1 | 123 | 5.59 |
| 5. | Title: On finite-gain stabilizability of linear systems subject to input saturation<br>Author(s): Liu WS, Chitour Y, Sontag E<br>Source: **SIAM JOURNAL ON CONTROL AND OPTIMIZATION** Volume: **34** Issue: **4** Pages: **1190-1219** Published: **JUL 1996** | 4 | 5 | 2 | 1 | 1 | 68 | 4.25 |

# Eduardo introduced me to nonlinear robustness margins

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 35, NO. 4, APRIL 1990

**Further Facts about Input to State Stabilization**

EDUARDO D. SONTAG

***Abstract*—Previous results about input to state stabilizability are shown to hold even for systems which are not linear in controls, provided that a more general type of feedback be allowed. Applications to certain stabilization problems and coprime factorizations, as well as comparisons to other results on input to state stability, are also briefly discussed.**

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$$\langle \nabla V(x), f(x,d)\rangle < 0 \qquad \forall (x,d) : |d| < \sigma(|x|)$$

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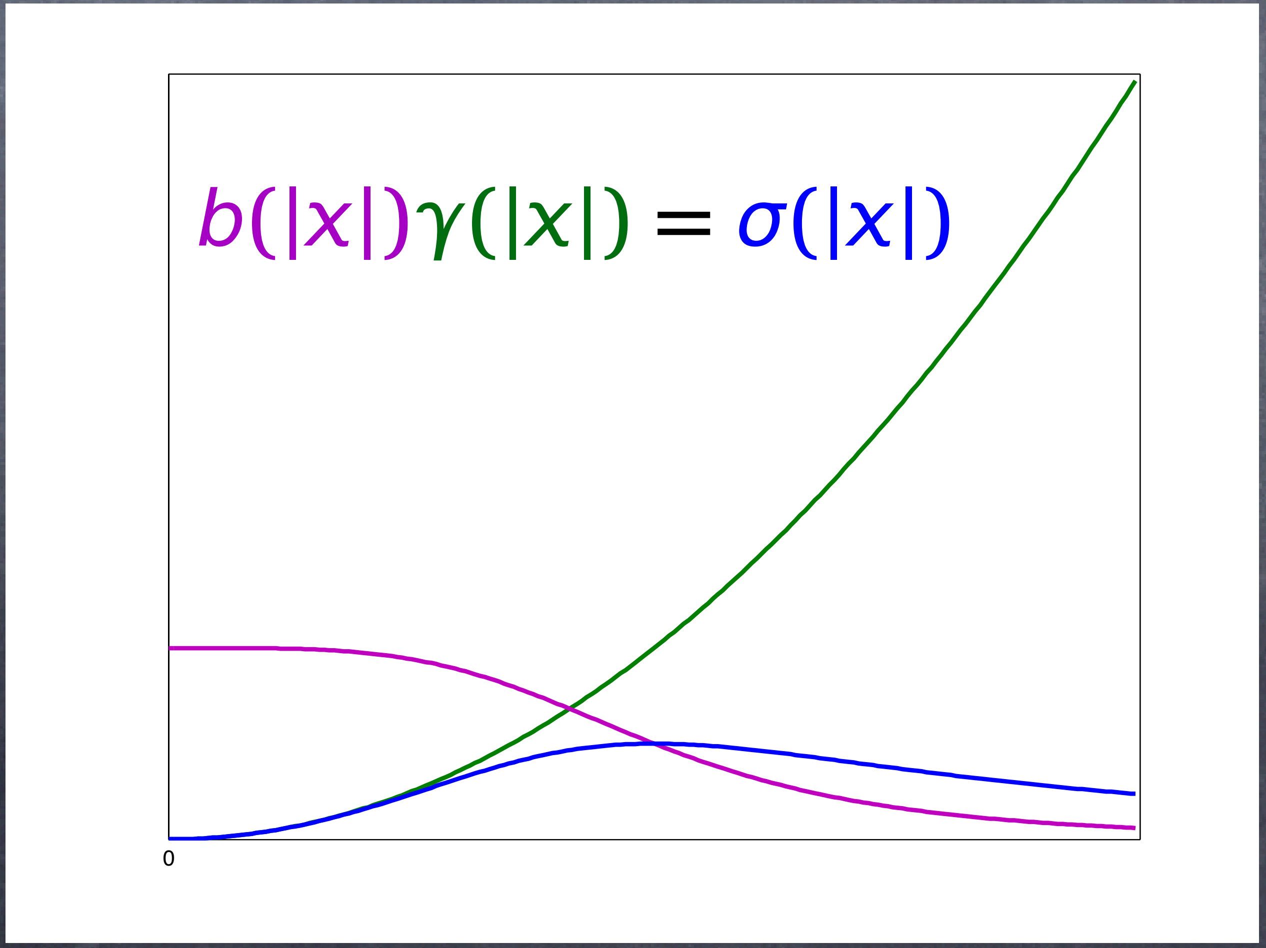


# Eduardo led the renaissance of converse Lyapunov theorems

SIAM J. CONTROL AND OPTIMIZATION
Vol. 34, No. 1, pp. 124–160, January 1996


## A SMOOTH CONVERSE LYAPUNOV THEOREM FOR ROBUST STABILITY*

YUANDAN LIN†, EDUARDO D. SONTAG‡, AND YUAN WANG§

**Abstract.** This paper presents a converse Lyapunov function theorem motivated by robust control analysis and design. Our result is based upon, but generalizes, various aspects of well-known classical theorems. In a unified and natural manner, it (1) allows arbitrary bounded time-varying parameters in the system description, (2) deals with global asymptotic stability, (3) results in smooth (infinitely differentiable) Lyapunov functions, and (4) applies to stability with respect to not necessarily compact invariant sets.

THEOREM 2.9. *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a nonempty, compact, invariant subset for the system (1). Then, (1) is UGAS with respect to $\mathcal{A}$ if and only if there exists a smooth Lyapunov function $V$ with respect to $\mathcal{A}$.*

$$(1) \qquad \dot{x} \in F(x) = \{ v : v = f(x,d)\ ,\ d \in K \}$$

# Eduardo has an uncanny knack for anticipating what we will need later

ELSEVIER

Systems & Control Letters 34 (1998) 93–100

SYSTEMS & CONTROL LETTERS

## Comments on integral variants of ISS [1]

Eduardo D. Sontag *

Although not needed here, it is worth stating the "exponential" version of the above lemma:

**Corollary 10.** *For each* $\gamma \in \mathcal{K}_\infty$ *there exist* $\sigma_1$ *and* $\sigma_2$ *in* $\mathcal{K}_\infty$ *so that*

$$\gamma(rs) \leqslant \sigma_1(r)\sigma_2(s)$$

*for all* $r, s \geqslant 0$.

**Proposition 7.** *Assume that* $\beta \in \mathcal{KL}$. *Then, there exist* $\theta_1, \theta_2 \in \mathcal{K}_\infty$ *so that*

$$\beta(s,t) \leqslant \theta_1(\theta_2(s)e^{-t}) \quad \forall s \geqslant 0, \ t \geqslant 0. \qquad (11)$$

The jump from nonlinear $\dot{x} \in F(x)$

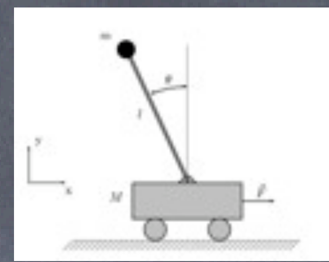
to hybrid systems
$$\begin{aligned} \dot{x} &\in F(x) \quad x \in C \\ x^+ &\in G(x) \quad x \in D \end{aligned}$$

Where's the impact?

And, along the way, we see Eduardo's impact.

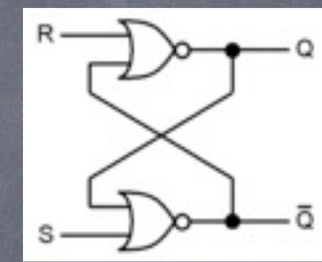
# A compact model takes us from nonlinear to hybrid

- The state



physical variables

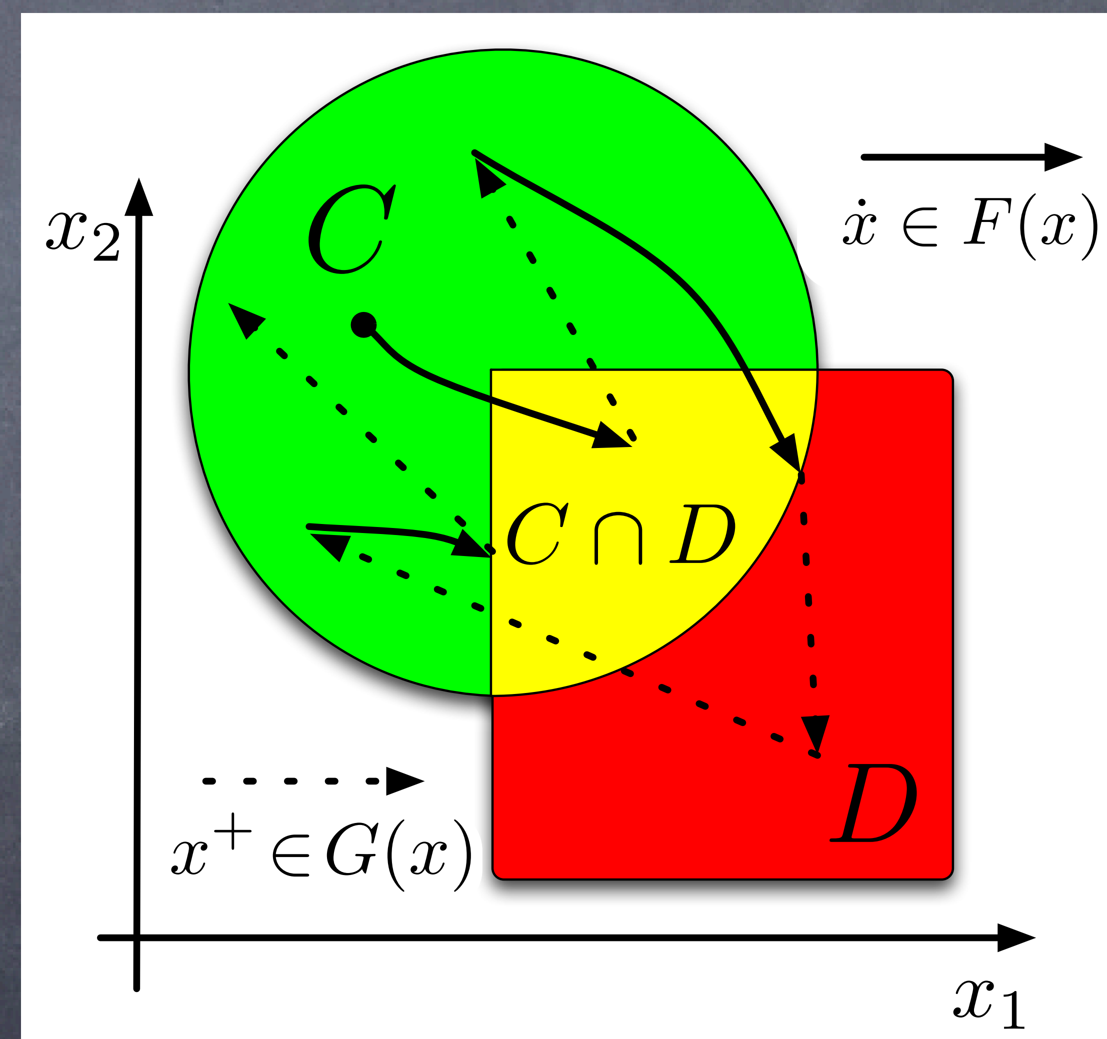
timers




logic states


counters

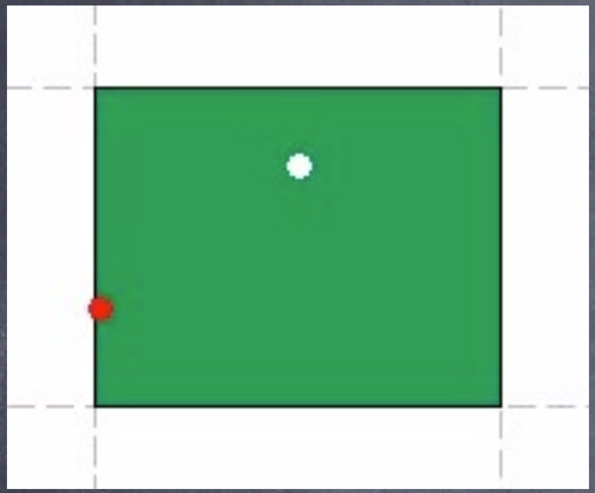
$$x = (\xi, \tau, q, \ell, \ldots) \in \mathbb{R}^n$$



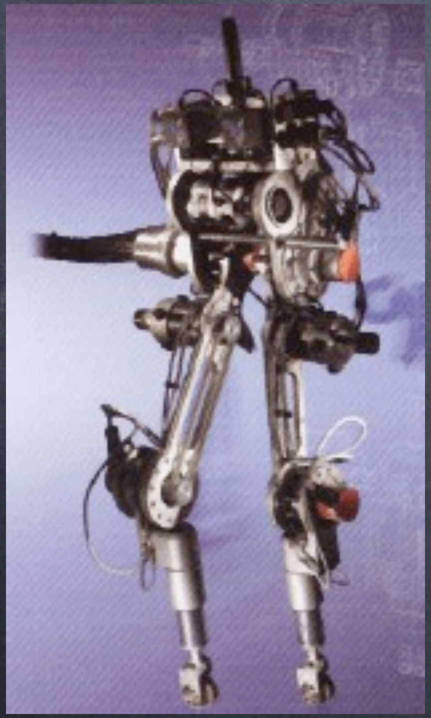

- Continuous change
  - where $x \in C$
  - how $\dot{x} \in F(x)$
- Instantaneous change
  - where $x \in D$
  - how $x^+ \in G(x)$

# Hybrid systems appear in nature & control

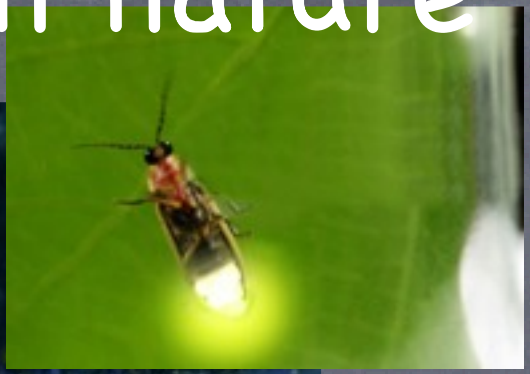
Mechanical systems w/ impacts



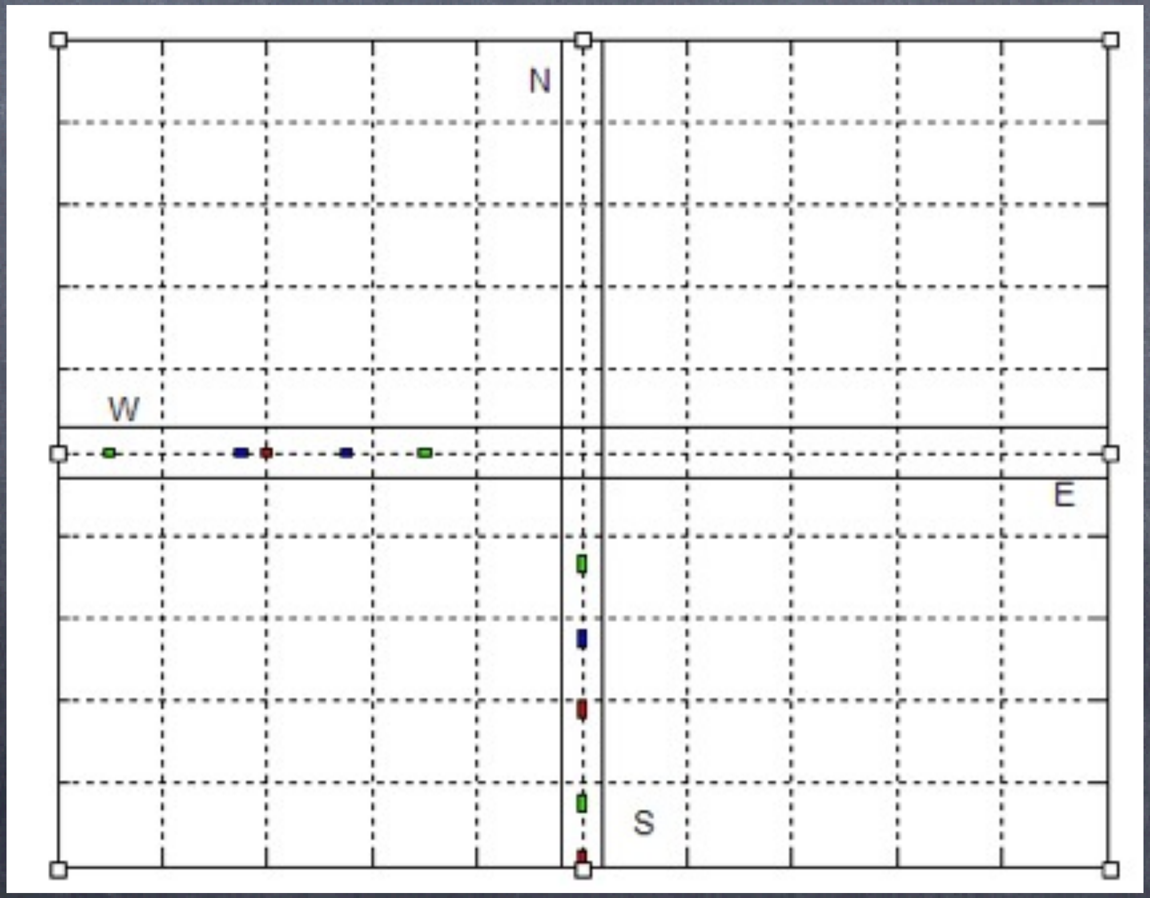
Billiards



Walking robots

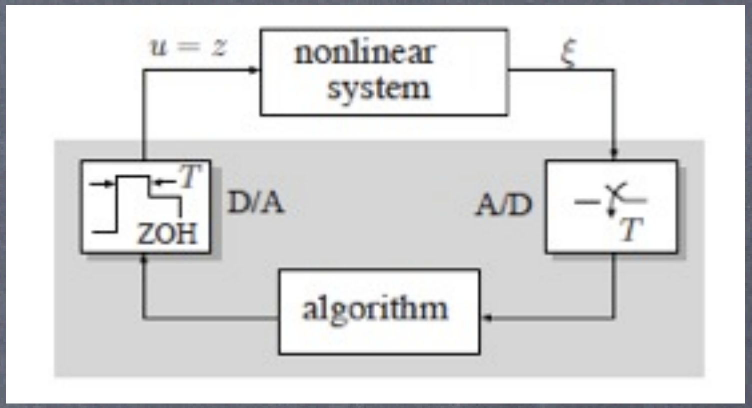


Networks of impulsive biological oscillators

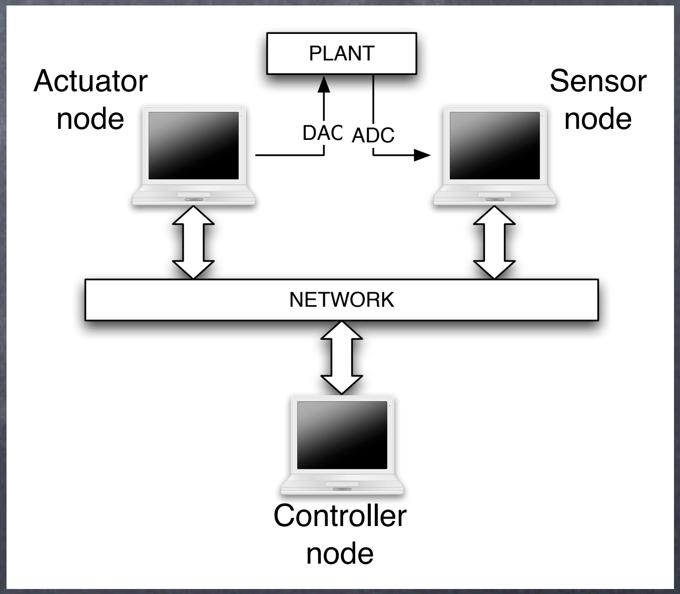



Automated traffic systems

Digital control systems




Sample-and-hold control




Networked control systems

# Under weak regularity conditions, asymptotic stability has nonzero robustness margins

Goebel/T.
Automatica 2006

$C \Rightarrow C_\sigma = \{x : (x + \sigma(x)\mathbf{B}) \cap C \neq \emptyset\}$

$F(x) \Rightarrow F_\sigma(x) = \overline{\text{co}}F((x + \sigma(x)\mathbf{B}) \cap C) + \sigma(x)\mathbf{B}$

$D \Rightarrow D_\sigma = \{x : (x + \sigma(x)\mathbf{B}) \cap D \neq \emptyset\}$

$G(x) \Rightarrow G_\sigma(x) = G((x + \sigma(x)\mathbf{B}) \cap D) + \sigma(x)\mathbf{B}$

## Robustness margins accommodate many corollaries

- Linearization principle
- Reduction principle
- Averaging theory
- Singular perturbations
- Small perturbations
- Converse Lyapunov theorems

(also uses Proposition 7)

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# Lyapunov functions are natural for hybrid systems

If the compact set $\mathcal{A}$ is GAS then there exists a (exponentially decreasing) smooth, global Lyapunov function.

Cai/Goebel/T. IEEE TAC 2008

$\exists \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ :

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \qquad \forall x \in \mathbb{R}^n$$

$$\langle \nabla V(x), f \rangle \leq -V(x) \qquad \forall x \in C \ , \ \ f \in F(x)$$

$$V(g) \leq \exp(-1)V(x) \qquad \forall x \in D \ , \ \ g \in G(x)$$

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# Like for classical systems, the invariance principle can be used as the basis for stability analysis

The compact set $\mathcal{A}$ is GAS if
1) there exists a weak, global Lyapunov function and
2) no solution with a unbounded domain makes the Lyapunov function remain at a positive constant.

Sanfelice/Goebel/T. IEEE TAC 2007

A weak, global Lyapunov function has weakened decrease conditions:

$$\begin{aligned} \langle \nabla V(x), f \rangle &\leq 0 \qquad x \in C \;,\; f \in F(x) \\ V(g) - V(x) &\leq 0 \qquad x \in D \;,\; g \in G(x) \end{aligned}$$

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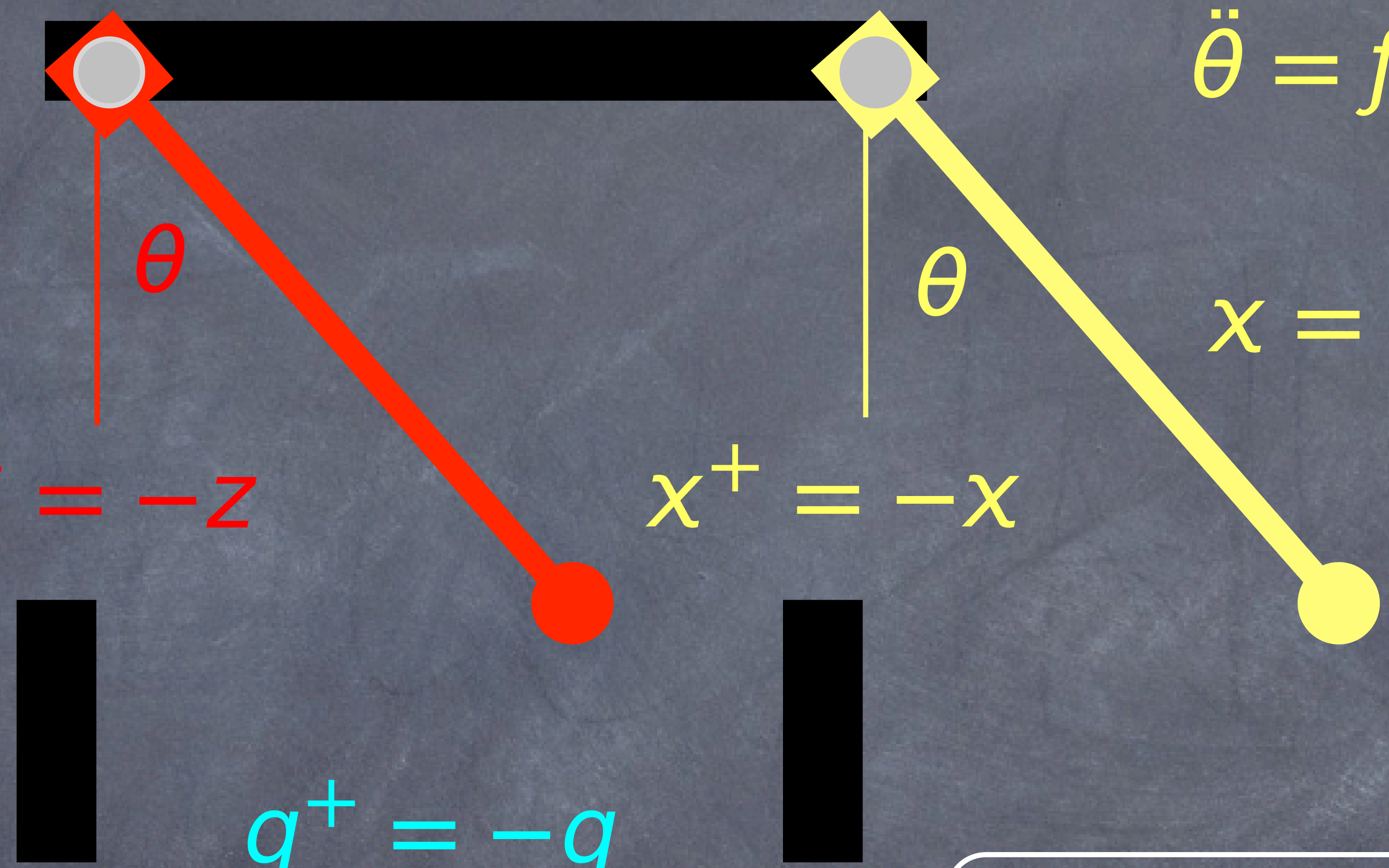
# Converse Lyapunov theorems motivate Lyapunov-based hybrid feedback algorithms

$\ddot{\theta} = f(\theta, \dot{\theta})$

$z = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

$z^+ = -z$




$\ddot{\theta} = f(\theta, \dot{\theta}) + \tau$

$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

$x^+ = -x$

Forni/Zaccarian/T.
IEEE CDC 2011 (s)

Goal: regulate

$x - qz$

$q^+ = -q$

$q \in \{-1, 1\}$

$(x - qz)^+ = x - (-q)(-z) = x - qz$

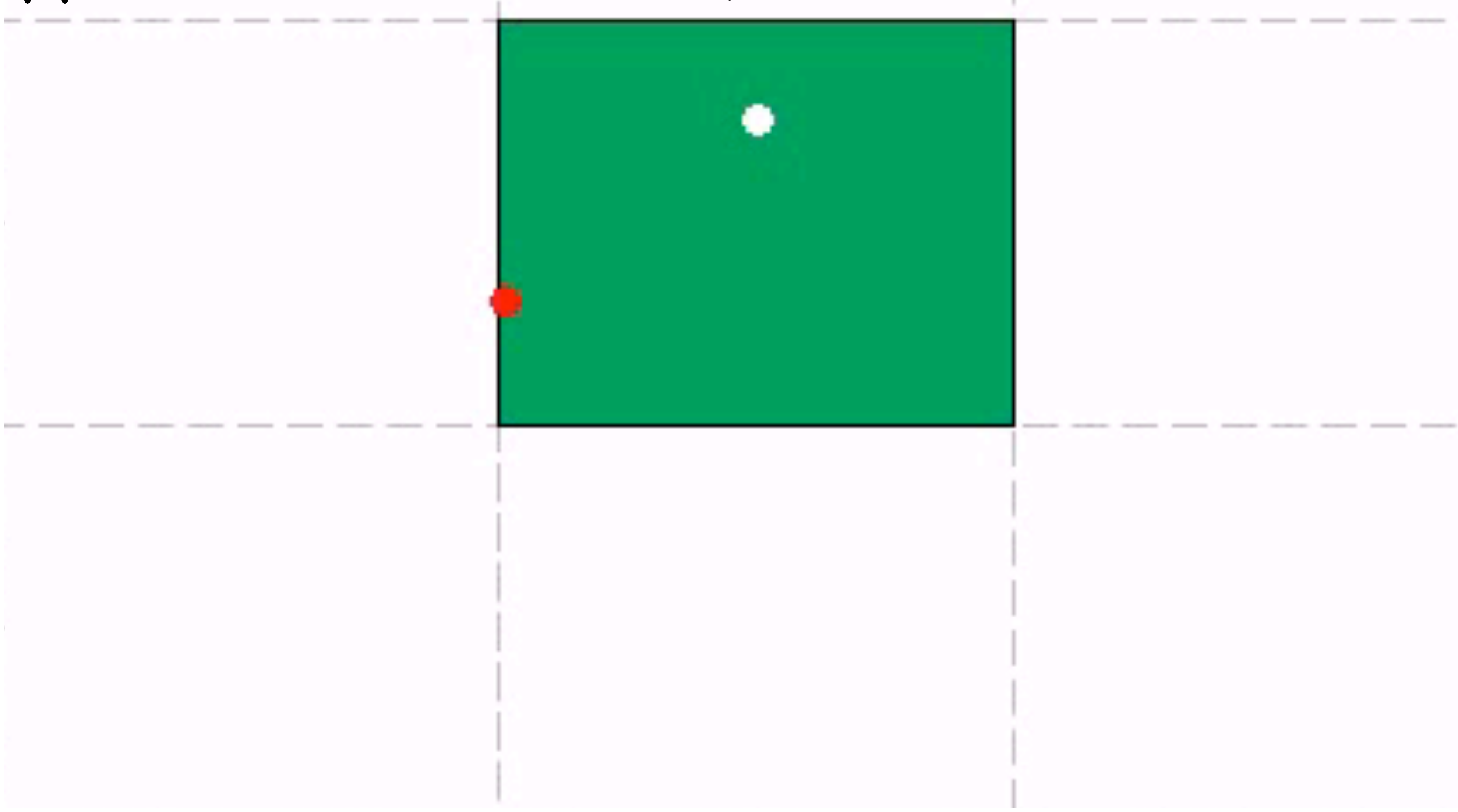
$(x - qz)^+ = -x - (-q)z = -(x - qz)$

$V(q, z, x) = W(x - qz)$

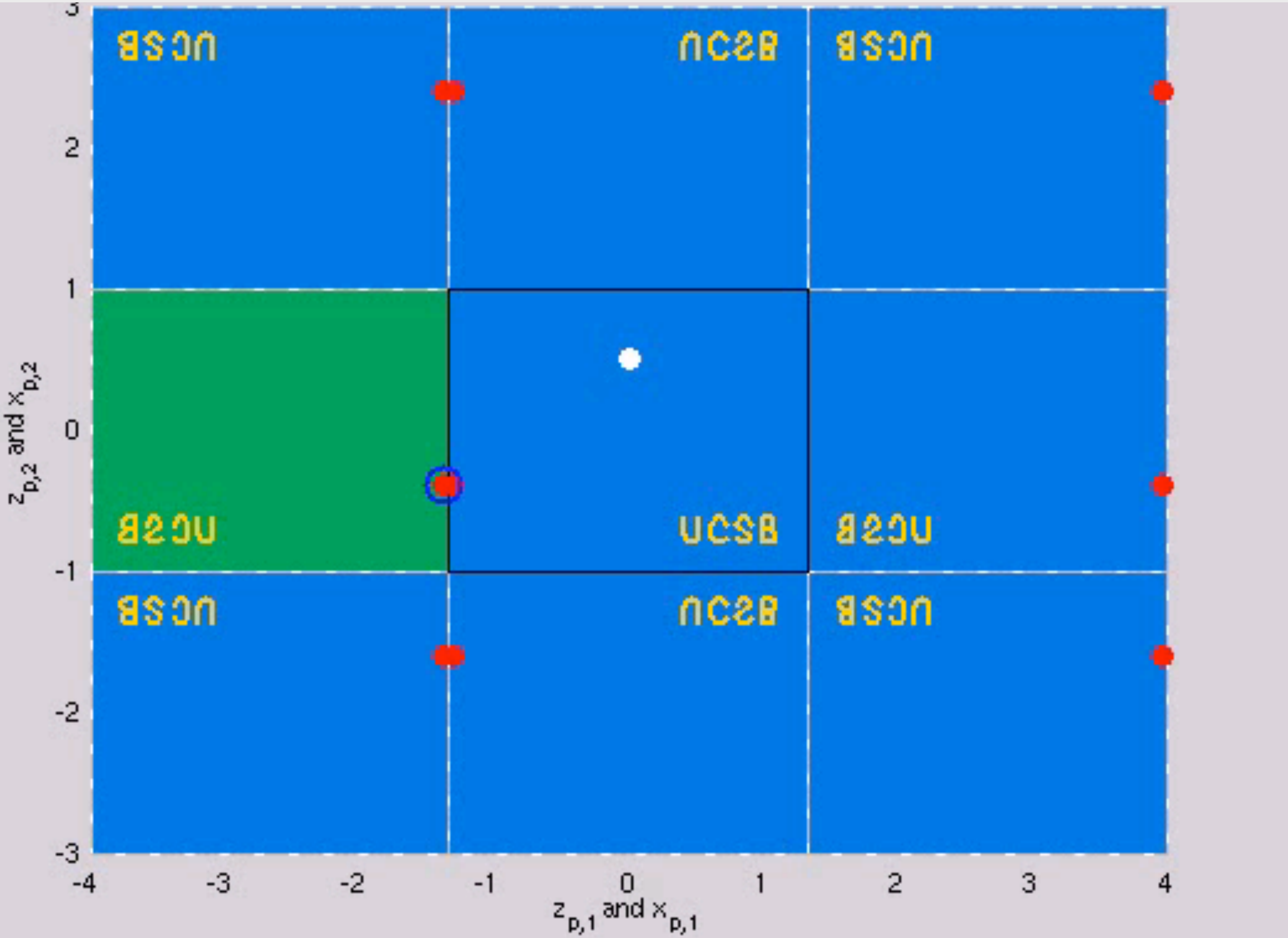
- W a CLF for double integrator
- $W(e) = W(-e)$

# This hybrid feedback approach to tracking extends to billiard systems

First note that the naive, non-hybrid tracking approach fails miserably …

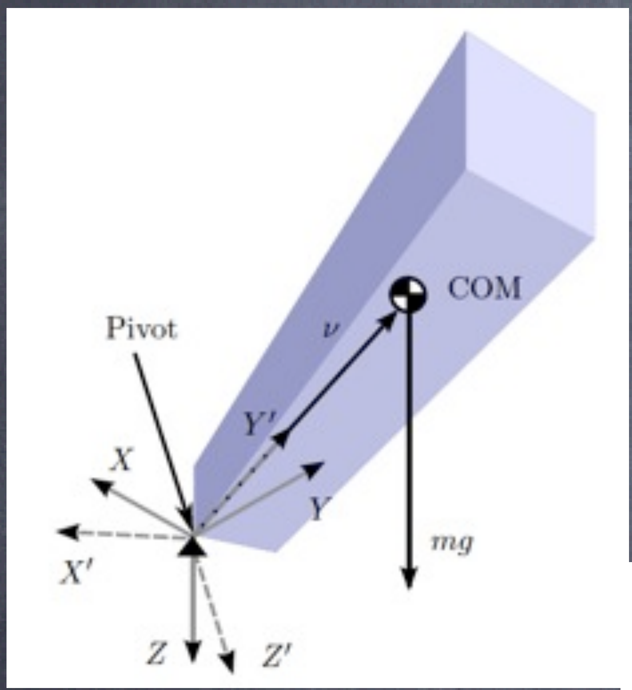


# Lyapunov-based hybrid strategy has no difficulty



# Backstepping of Lyapunov-based hybrid feedbacks is often possible

… including global stabilization for systems evolving on compact manifolds.




3D pendulum

Rigid body dynamics

ESAIM: Control, Optimisation and Calculus of Variations October 1999, Vol. 4, p. 537–557
URL: http://www.emath.fr/cocv/

**CLOCKS AND INSENSITIVITY TO SMALL MEASUREMENT ERRORS***

EDUARDO D. SONTAG[1]

**Abstract.** This paper deals with the problem of stabilizing a system in the presence of small measurement errors. It is known that, for general stabilizable systems, there may be no possible memoryless state feedback which is robust with respect to such errors. In contrast, a precise result is given here, showing that, if a (continuous-time, finite-dimensional) system is stabilizable in any way whatsoever (even by means of a dynamic, time varying, discontinuous, feedback) then it can also be semiglobally and practically stabilized in a way which is insensitive to small measurement errors, by means of a hybrid strategy based on the idea of sampling at a "slow enough" rate.

# Algorithms based on ``synergistic potentials" provide an illustration

Mayhew/Sanfelice/T. ACC 2011

$$\left.\begin{aligned}\dot{z} &= \psi(z)v \\ \dot{v} &= u\end{aligned}\right\} \quad (z, v) \in M \times \mathbf{R}^m \qquad e.g., M = \mathbf{S}^n, SO(3), \ldots$$

Synergy condition on a family of potential functions:

$$\psi(z)^T \nabla V_q(z) = 0,\ (q, z) \notin \mathcal{A} \implies$$

$$\mu_V(q, z) := V_q(z) - \min_{s \in Q} V_s(z) > \delta > 0$$

"synergy gap"

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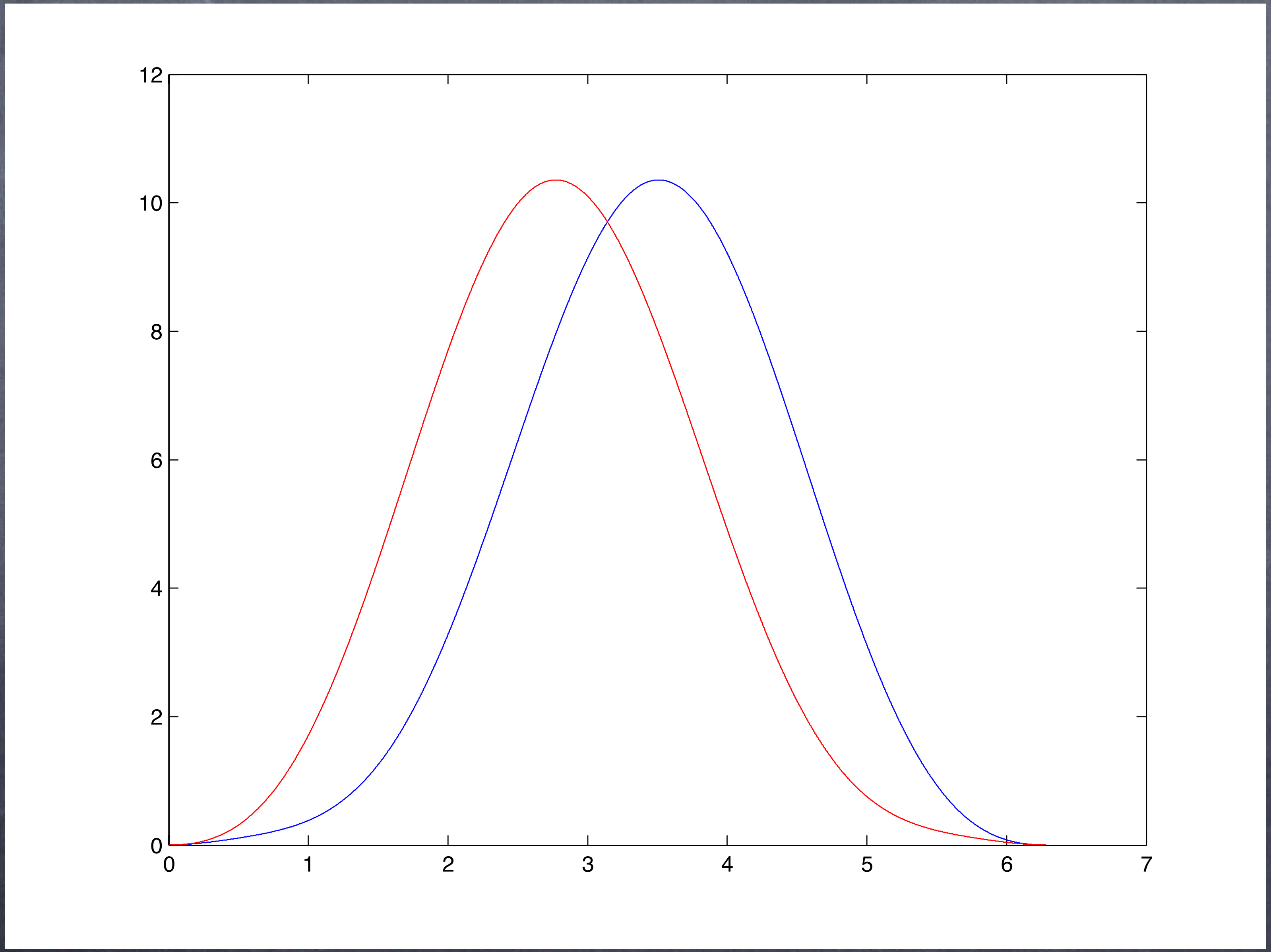
$$M = \mathbf{S}^1$$

$$\psi(z) = \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix}$$




$V_q(\cos(\theta), \sin(\theta))$ vs. $\theta$

$$\left.\begin{aligned}\dot{z} &= \psi(z)v \\ \dot{v} &= u\end{aligned}\right\} \quad (z,v) \in M \times \mathbf{R}^m \qquad e.g., M = \mathbf{S}^n, SO(3), \ldots$$

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Hybrid controller (no backstepping)

$$v = -\psi(z)^T \nabla V_q(z)$$

$$C = \{(q,z) \in Q \times M : \mu_V(q,z) \leq \delta\}$$

$$D = \{(q,z) \in Q \times M : \mu_V(q,z) \geq \delta\}$$

$$G_c(z) = \{s \in Q : \mu_V(s,z) = 0\}$$

Lyapunov function: $W(q,z) = V_q(z)$

$$\left.\begin{aligned}\dot{z} &= \psi(z)v \\ \dot{v} &= u\end{aligned}\right\} \quad (z,v) \in M \times \mathbf{R}^m \qquad e.g., M = \mathbf{S}^n, SO(3), \ldots$$

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Hybrid controller (backstepping)

$$u = -\psi(z)^T \nabla V_q(z) - v$$

$$C = \{(q,z) \in Q \times M : \mu_V(q,z) \leq \delta\} \times \mathbf{R}^m$$

$$D = \{(q,z) \in Q \times M : \mu_V(q,z) \geq \delta\} \times \mathbf{R}^m$$

$$G_c(z) = \{s \in Q : \mu_V(s,z) = 0\}$$

Lyapunov/LaSalle function: $W(q,z) = V_q(z) + 0.5 v^T v$

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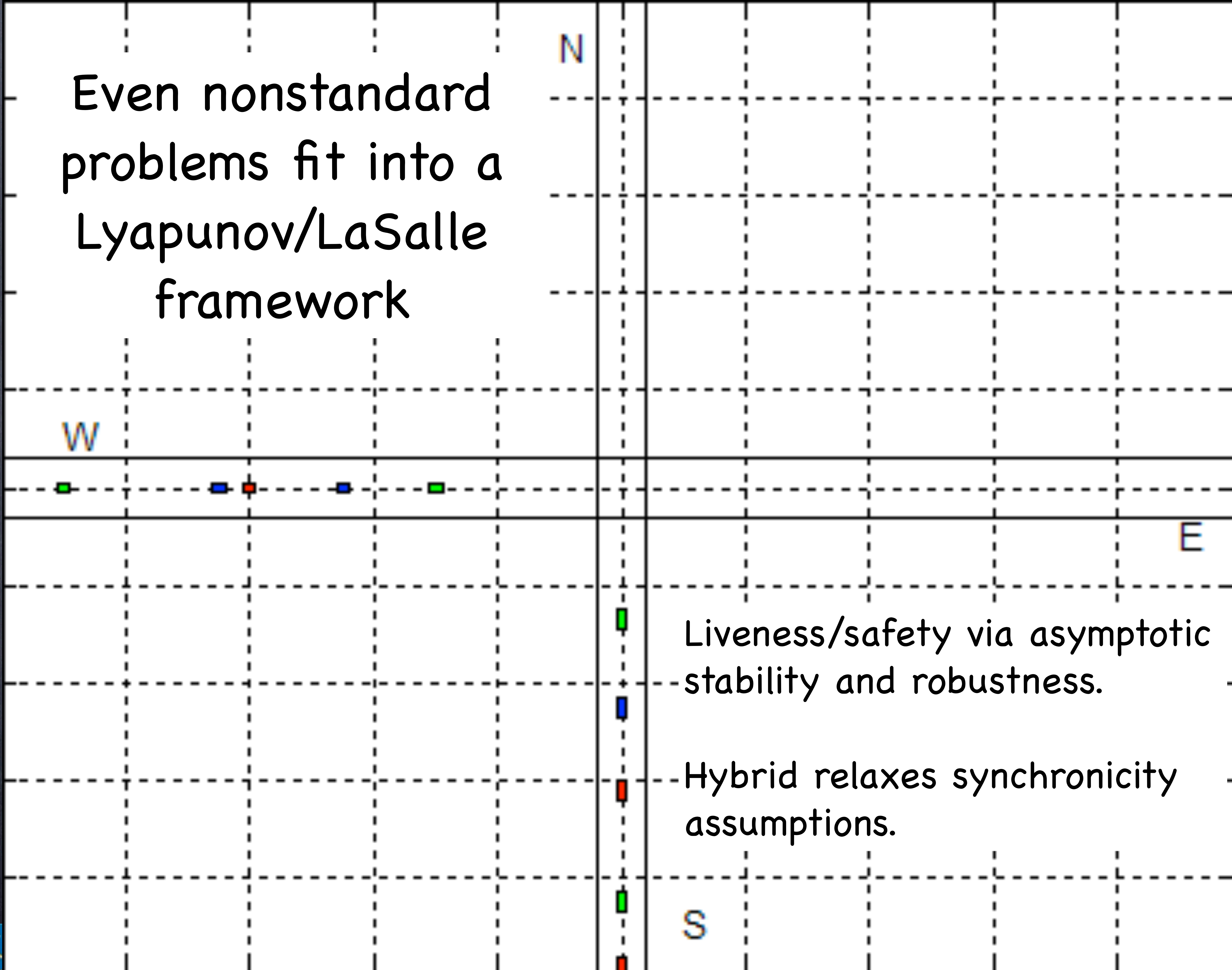
Even nonstandard problems fit into a Lyapunov/LaSalle framework




Liveness/safety via asymptotic stability and robustness.

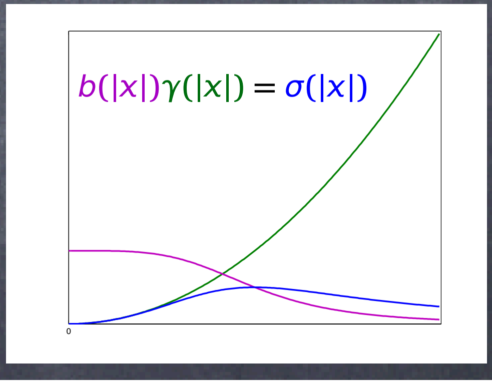
Hybrid relaxes synchronicity assumptions.

# The jump from nonlinear to hybrid systems: where's the impact?

- We can systematically approach problems that we could not touch before, using tool with which we are very familiar.

- Full-scale systems, products, $$, ... ???

What is much clearer is Eduardo's impact on the recent developments in the field, which is extensive.

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