Monotone Dynamical Systems: A Quick Tour

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Monotone Dynamical System

1. State space: metric space (X, d) with a closed* partial order relation $\leq$. *$(x_n \leq y_n \wedge x_n \to x \wedge y_n \to y \Rightarrow x \leq y)$
2. Dynamics: discrete-time ($T = \mathbb{Z}_+$) or continuous-time ($T = \mathbb{R}_+$) semiflow $\Phi : T \times X \to X$. Notation $\Phi_t(x) = \Phi(t, x)$:
   - Φ continuous.
   - $\Phi_0 = id_X$
   - $\Phi_t \circ \Phi_s = \Phi_{t+s}, \quad t, s \in T$
3. Order-Preserving: $x \leq y \Rightarrow \Phi_t(x) \leq \Phi_t(y),\ t \in T, x, y \in X$.

Trivial Examples:

- $X = \mathbb{R}$, usual order $\leq$, $x' = f(x)$, $\Phi_t(x_0) = x(t, x_0)$.
- $X = BC(\mathbb{R}, \mathbb{R})$, usual order $\leq$, $u_t = u_{xx} + f(x, u)$, $\Phi_t(u_0) = u(t, \cdot)$.
- $X = \mathbb{R}$, $f \nearrow$, $x(n+1) = f(x(n)),\ n \geq 0,\ \Phi_n(x(0)) = f^{(n)}(x(0))$.

standing assumptions:

- $T = \mathbb{R}_+$.
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Ordered Banach Space Induces $\leq$

$X \subset Y$, Y an ordered Banach space with closed positive cone Y_+:

$$(\mathbb{R}_+)Y_+ \subset Y_+, \; Y_+ + Y_+ \subset Y_+, \; (Y_+) \cap (-Y_+) = \{0\}$$

Partial order: $y \leq x \Leftrightarrow x - y \in Y_+$

Y is strongly ordered if Int $Y_+ \neq \emptyset$. Then $y \ll x \Leftrightarrow x - y \in$ Int Y_+.

Examples:

- $Y = \mathbb{R}^n, \; Y_+ = \mathbb{R}^k_+ \times (-\mathbb{R}^{n-k}_+), \; 0 \leq k \leq n :$

$$x \leq y \Leftrightarrow (x_i \leq y_i, \; i \leq k) \wedge (x_j \geq y_j, \; j > k)$$

- $Y = L^p(\Omega, \mathbb{R}^n), \; C^r(\Omega, \mathbb{R}^n), \; f \leq g \Leftrightarrow f(s) \leq g(s), \; s \in \Omega$

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Equilibria, Sub & Super

Equilibria: $E = \{e \in X : \forall t \geq 0, \Phi_t(e) = e\}$

Sub-equilibria: $E_- = \{x \in X : \forall t \geq 0, \Phi_t(x) \geq x\}$

$$x \in E_- \Rightarrow x \leq \Phi_s(x) \leq \Phi_{t+s}(x),\ t, s \geq 0$$

$\therefore \Phi_t(x) \nearrow e \in E,\ t \nearrow \infty.$

Super-equilibria: $\{x \in X : \forall t \geq 0, \Phi_t(x) \leq x\}$

in applications, these can be identified by the semiflow generator

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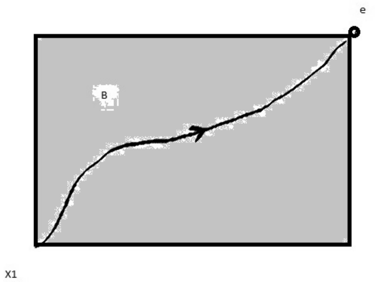
Sub & Super Equilibria Bracket Basin

x_1 is a sub-equilibrium with $\Phi_t(x_1) \nearrow e \in E$. Monotonicity implies

$$B = \{x \in X : x_1 \leq x \leq e\} \subset \text{Basin of attraction of } e$$

because it is "sandwiched": $\Phi_t(x_1) \leq \Phi_t(x) \leq \Phi_t(e) = e$



Strong Monotonicity & Limit Set Dichotomy

Φ strongly monotone (Hirsch) if Y is strongly ordered and
$x < y \Rightarrow \Phi_t(x) \ll \Phi_t(y),\ t > 0$.

Φ is strongly order preserving (Matano) (SOP) if it is monotone and
$x < y \Rightarrow \exists$ nbhds $U, V,\ x \in U, y \in V,\ \exists\ t_0 \geq 0$ such that

$$\Phi_{t_0}(U) \leq \Phi_{t_0}(V)$$

Theorem[LSD, Hirsch(1982)]: Let Φ be SOP. If $x < y$ then either

(a) $\omega(x) < \omega(y)$, or

(b) $\omega(x) = \omega(y) \subset E$

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Generic Convergence

Theorem*: Assume $X \subset Y$, Y an ordered Banach space, and X is either convex or the closure of an open set. Let

$$C = \{x \in X : \Phi_t(x) \to e,\ e \in \text{ Equilibria}\}$$

If Φ is SOP on X and some mild smoothness and compactness assumptions hold (†), then Int C is dense in X.

*Inspired by: M. Hirsch. Systems of differential equations which are competitive or cooperative II: convergence almost everywhere, SIAM J. Math. Anal., 16, 1985.

(†)$\exists \tau > 0$:

- $x_1 < x_2 \Rightarrow \Phi_\tau x_1 \ll \Phi_\tau x_2$
- Φ_τ is locally C^1 at each $e \in E$, $\Phi'_\tau(e)$ is Krein-Rutman operator.

ODEs-A Canonical Form

$x' = F(x)$ is a monotone system w.r.t. **orthant cone** $\mathbb{R}^k_+ \times (-\mathbb{R}^{n-k}_+)$ in domain X if, on permuting variables $x = (x_1, x_2),\ x_1 \in \mathbb{R}^k,\ x_2 \in \mathbb{R}^{n-k}$

$$\begin{aligned} x_1' &= f_1(x_1, x_2) \\ x_2' &= f_2(x_1, x_2) \end{aligned}$$

- diagonal blocks $\frac{\partial f_i}{\partial x_i}(x)$ have nonnegative off-diagonal entries.
- off-diagonal blocks $\frac{\partial f_i}{\partial x_j}(x) \leq 0,\ i \neq j$ have nonpositive entries.

$$\text{Jacobian} = \begin{bmatrix} * & + & - & - \\ + & * & - & - \\ - & - & * & + \\ - & - & + & * \end{bmatrix},\ + \geq 0,\ - \leq 0$$

Components cluster into two subgroups. positive within-group interactions, negative between-group interactions.

Strong monotonicity holds if the Jacobian is irreducible at a.e. $x \in X$!

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Repressilator with 2 genes

x_i = [protein] product of gene i
y_i = [mRNA] of gene i.
x_{i-1} represses transcription of y_i:

$$\begin{aligned} x_i' &= \beta_i(y_i - x_i) \\ y_i' &= \alpha_i f_i(x_{i-1}) - y_i, \ i = 1, 2, \quad \text{mod } 2 \end{aligned}$$

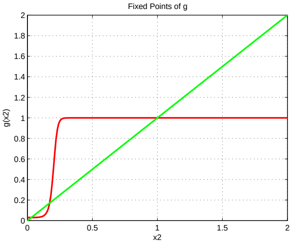
where $\alpha_i, \beta_i > 0$ and $f_i > 0$ satisfies $f_i' < 0$.

$$\text{Jacobian} = \begin{bmatrix} - & + & 0 & 0 \\ 0 & - & - & 0 \\ 0 & 0 & - & + \\ - & 0 & 0 & - \end{bmatrix}$$

Gardner et al, “Construction of a genetic toggle switch in E. coli", Nature(403),2000.

Dynamics of Repressilator

Equilibria $u = (x_1, y_1, x_2, y_2)$ are in 1-to-1 correspondence with fixed points of increasing map $g \equiv \alpha_2 f_2 \circ \alpha_1 f_1$



Theorem: If g has no degenerate fixed points, $\exists$ odd number of equilibria $u^1, u^2, \cdots, u^{2m+1}$ indexed by increasing values of x_2. u_{2i+1} are stable, u_{2i} are unstable. If $B(u_i)$ denotes the basin of attraction of u_i, then

$$\cup_{\text{odd } i} B(u_i)$$

is open and dense in $\mathbb{R}^4_+$. u_1 is globally attracting if $m = 0$.

Repressilator with transcription and translation delays

$$\begin{aligned} x_i'(t) &= \beta_i[y_i(t-\mu_i) - x_i(t)] \\ y_i'(t) &= \alpha_i f_i(x_{i-1}(t-\tau_{i-1})) - y_i(t),\ i = 1,2 \end{aligned}$$

Generates a SOP semiflow on:

$$\begin{aligned} X &= C([-\tau_1, 0], \mathbb{R}_+) \times C([-\mu_1, 0], \mathbb{R}_+) \\ &\quad \times C([-\tau_2, 0], \mathbb{R}_+) \times C([-\mu_2, 0], \mathbb{R}_+) \end{aligned}$$

Previous Theorem holds without change for delayed repressilator.
Extends to arbitrary even number of genes.

Test for Orthant-Cone Monotone ODE $x' = f(x)$

- $\forall i \neq j$, $\frac{\partial f_i}{\partial x_j}(x)$ does not change sign in X.
- Feedback Symmetry: $\frac{\partial f_i}{\partial x_j}(x)\frac{\partial f_j}{\partial x_i}(y) \geq 0,\ i \neq j$. golden rule
- Construct signed, influence graph:
  - un-directed edge joins i to $j \neq i$ if $\exists x \in X$, $\frac{\partial f_i}{\partial x_j}(x) \neq 0$.
  - append $+$ sign to edge if derivative is positive, $-$ sign if negative.
- balanced graph (‡): every loop (cycle) has even number of "$-$" signs.

‡ This is Harary's Theorem: "a balanced network is clusterable". See "Networks: An Intro.", M. Newman
An algorithm is given for clustering, i.e, permuting indices into subsets $I = \{1, 2, \cdots, k\}$ and I^c.

Systems of Parabolic PDEs

Given elliptic operators L_i, the parabolic system

$$\begin{aligned} \partial_t u_1 &= L_1 u_1 + f_1(x, u_1, u_2) \\ \partial_t u_2 &= L_2 u_2 + f_2(x, u_1, u_2), \ x \in \Omega, \ t > 0 \end{aligned}$$

where $f = (f_1(x, \cdot, \cdot), f_2(x, \cdot, \cdot))$ in canonical form, and boundary conditions

$$0 = \alpha_i \frac{\partial u_i}{\partial n} + \beta_i u_i, \ x \in \partial\Omega$$

where $\alpha_i, \beta_i \geq 0$, generates a monotone semiflow on spaces

$$C_0^r(\overline{\Omega}) := \{v \in C^r(\overline{\Omega}) : v|\partial\Omega = 0\}$$

$r = 0, 1$ for Dirichlet B.C., or

$$C_{\alpha,\beta}^r(\overline{\Omega}) := \left\{ v \in C^r(\overline{\Omega}) : \beta v + \alpha \frac{\partial v}{\partial n} = 0, \ x \in \partial\Omega \right\}$$

for Robin or Neumann B.C.

Convergence to uniform equilibria

$$
\begin{aligned}
u_t &= D\nabla^2 u + f(u) \\
\frac{\partial u}{\partial n} &= 0,\ x \in \partial\Omega \\
u(x,0) &= u_0(x),\ x \in \Omega
\end{aligned}
$$

$\Omega \subset \mathbb{R}^n$ smooth, bounded, convex.
$D = \mathrm{diag}(d_i), d_i > 0$.
$f : \mathbb{R}^m \to \mathbb{R}^m$ is C^2, cooperative, and irreducible.

Theorem[Enciso, Hirsch, S. (2008)]: Let solutions of $u' = f(u)$ be bounded on $t \geq 0$. The set of $u_0 \in C(\overline{\Omega}, \mathbb{R}^m)$ such that $u(x,t)$ converges to a spatially-uniform equilibrium is prevalent in $C(\overline{\Omega}, \mathbb{R}^m)$.

W is prevalent if its complement is shy. Borel set $W \subset X = C(\overline{\Omega}, \mathbb{R}^m)$ is shy if $\exists$ a nonzero compactly supported Borel measure μ on X, such that $\mu(W + x) = 0, \forall x \in X$. Hunt,Sauer,Yorke,1993

Unmentioned & New Directions

- Monotone Maps: No LSD, generic convergence to periodic points.
- Non-autonomous theory-Skew-Product Semiflows: J. Mierczynski, W. Shen, X. Zhao
- Monotone Random Systems: See I. Chueshov, Springer Lect. Notes in Math.
- Control Theory: Sontag, Angeli, De Leenheer, Enciso, Wang

My Favorite References

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