Towards a Unified View of Communication and Control

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Slides courtesy of Sekhar Tatikonda

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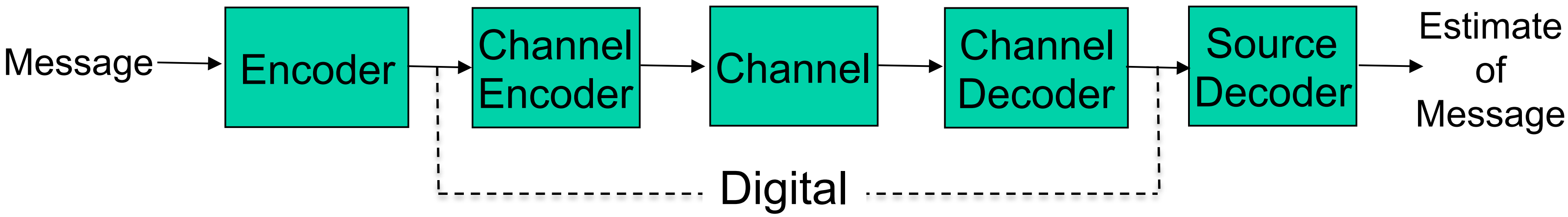
Shannon Picture



Source Encoder: Compression
Channel Encoder: Expansive Map

- Source Coding Theorem
- Noisy Channel Coding Theorem
- Source Channel Separation
- Can recover message with probability of error going to zero (block length $\longrightarrow \infty$), provided

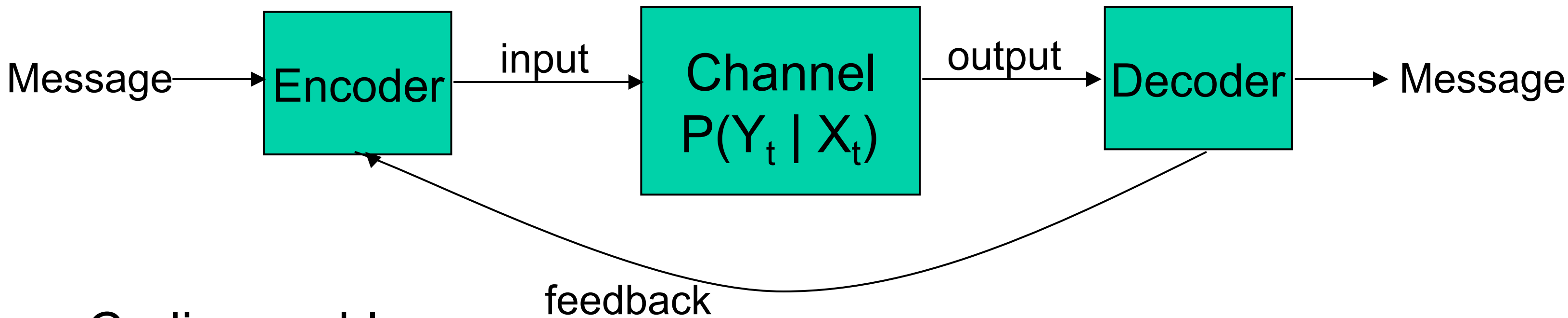
$$R < C, \quad C = \text{capacity}$$

- $R > C$, cannot recover message with probability of error $\longrightarrow 0$
- Separation architecture

Feedback Communication Problems

- Feedback from output of the channel to input of the channel encoder
- Causality
- Can we use the language and machinery of ergodic partially observed stochastic control to understand fundamental limitations of feedback communication problems?

Review of Basic Set-Up



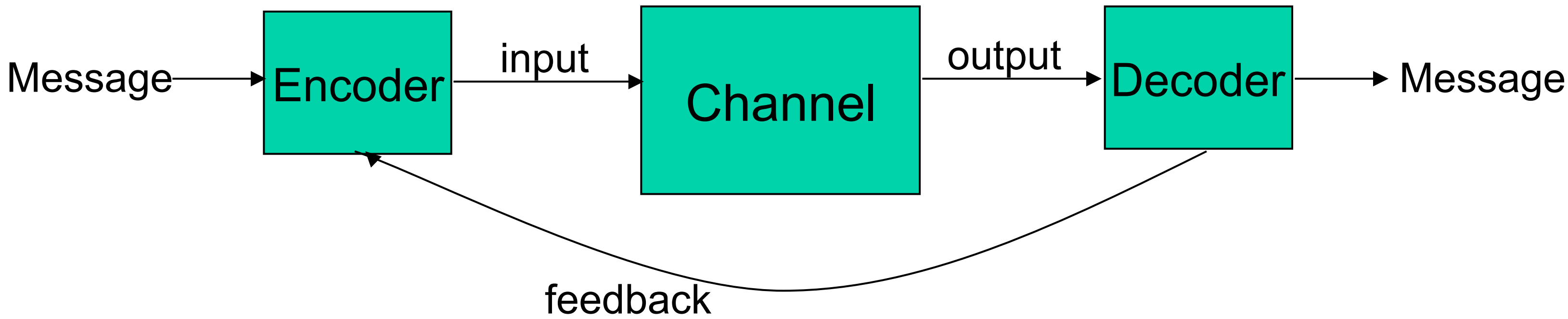
- Coding problem:
  - memoryless channel: $P(Y_t \mid X_t)$
  - messages: { m }
  - codewords: encoder maps $m \rightarrow X^T = (X_1,\ldots,X_T)$
  - decoder: $Y^T \rightarrow \hat{m}$

- Objective: maximize number of messages subject to small error probability: $P(m \neq \hat{m})$

- Shannon's theorem: $C = \max_{P(X)} I(X; Y)$
  - single letter characterization

- If no memory (or state) then feedback does not increase capacity

Time-Ordering of Events and Code-Functions



- With feedback one can adapt the channel input symbol. Use code-functions: $X_t = F_t(Y^{t-1})$

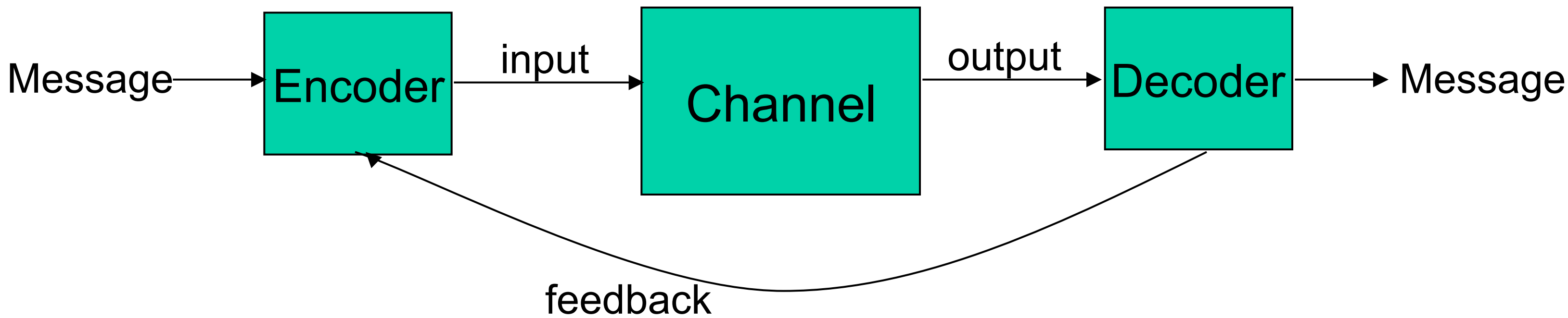
- Time order of event.

  non-causal: $M, X_1, X_2, \ldots, X_T, Y_1, Y_2, \ldots, Y_T, M$

  causal: $M, X_1, Y_1, X_2, Y_2, \ldots, X_T, Y_T, M$

- Every message is assigned a sequence of code-functions $F_1, F_2, \ldots, F_T$ (as opposed to a codeword.)

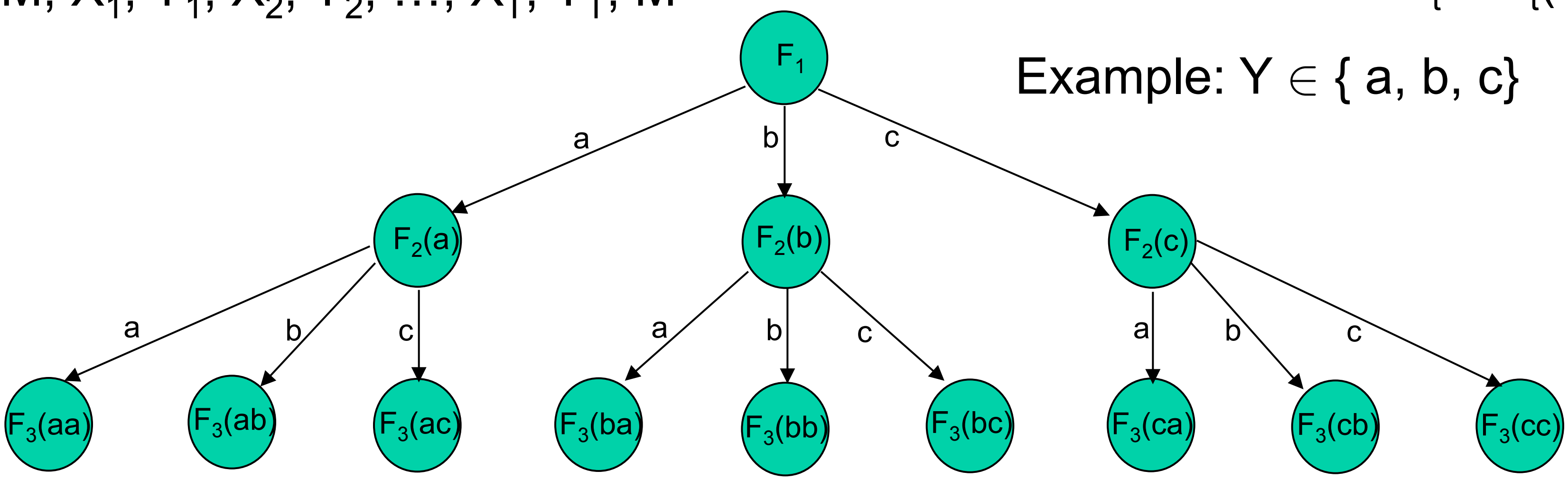
Code-Functions



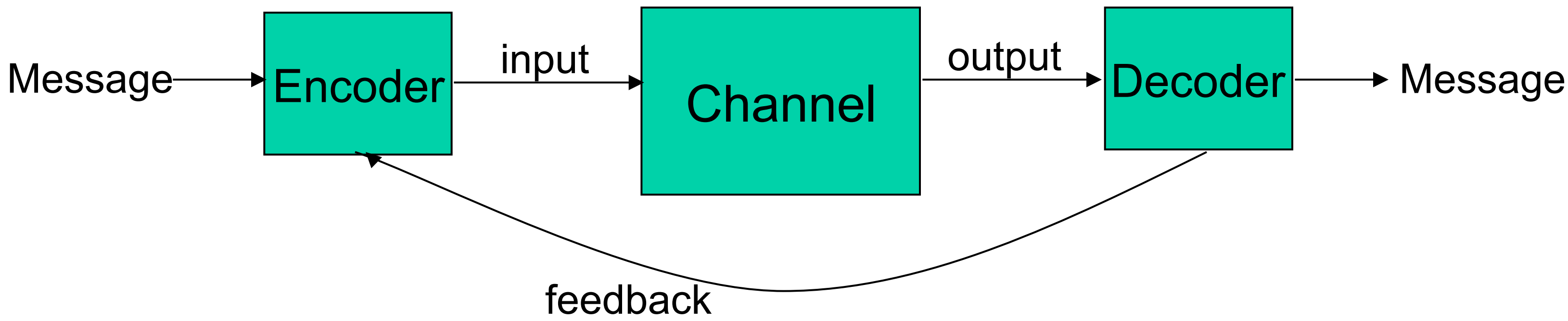
Time order:
$M, X_1, Y_1, X_2, Y_2, \ldots, X_T, Y_T, M$

Code-functions: $X_t = F_t(Y^{t-1})$

Example: $Y \in \{ a, b, c\}$



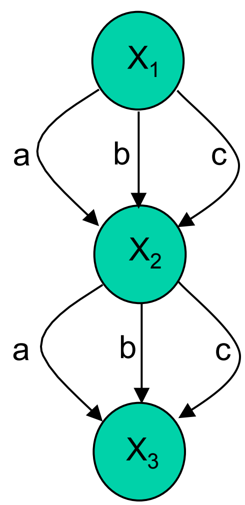
Codewords versus Code-Functions

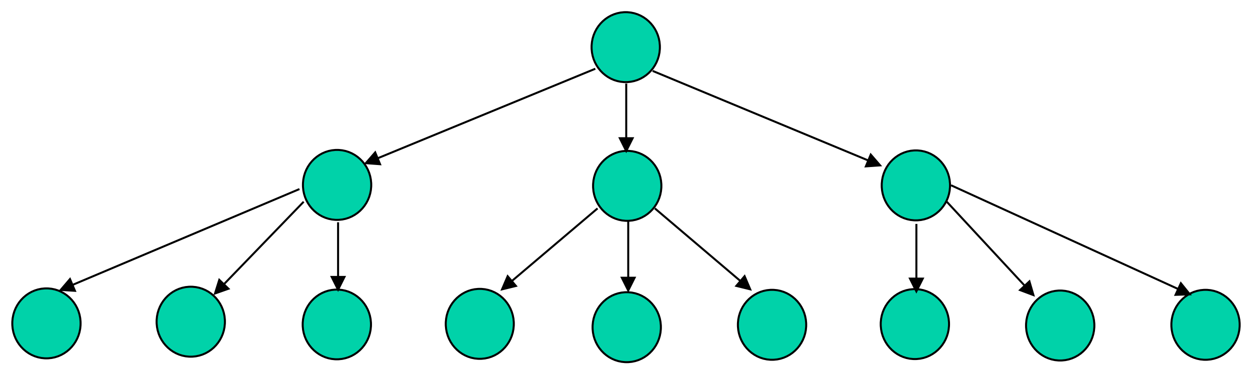


A codeword is a code-function that does not care about the feedback.

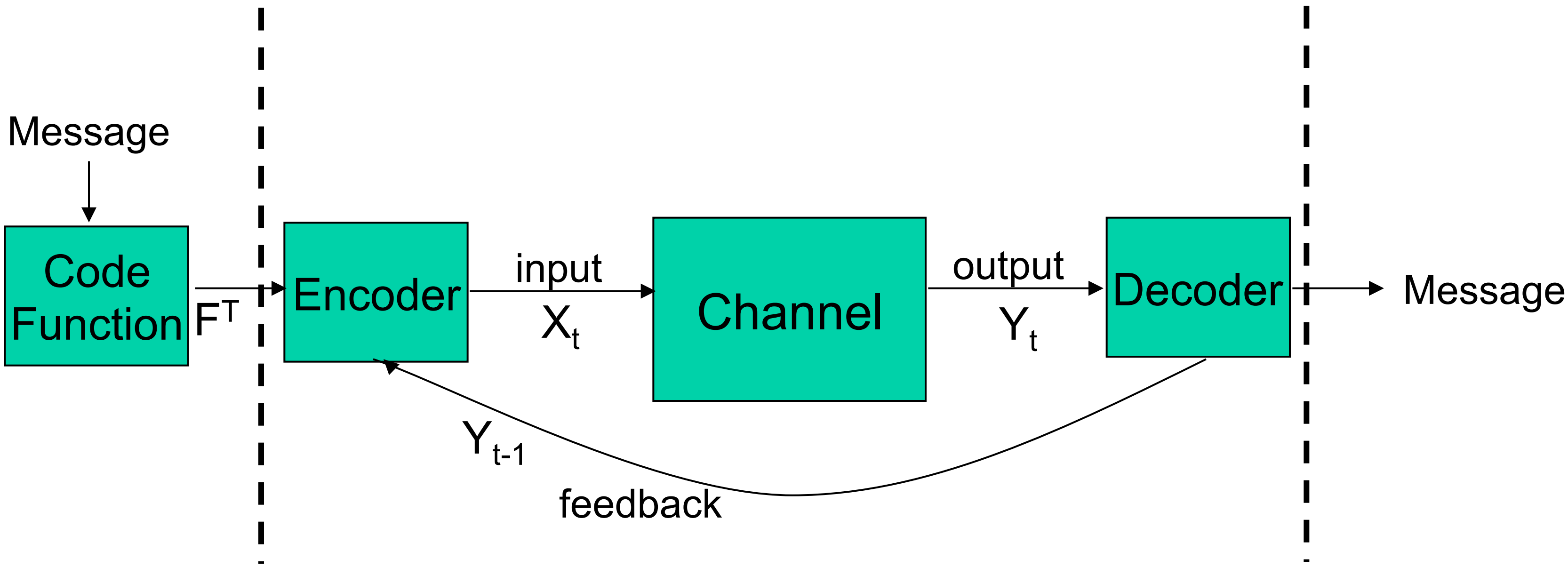
Code-functions: $X_t = F_t(Y^{t-1})$

Example: $Y \in \{ a, b, c\}$



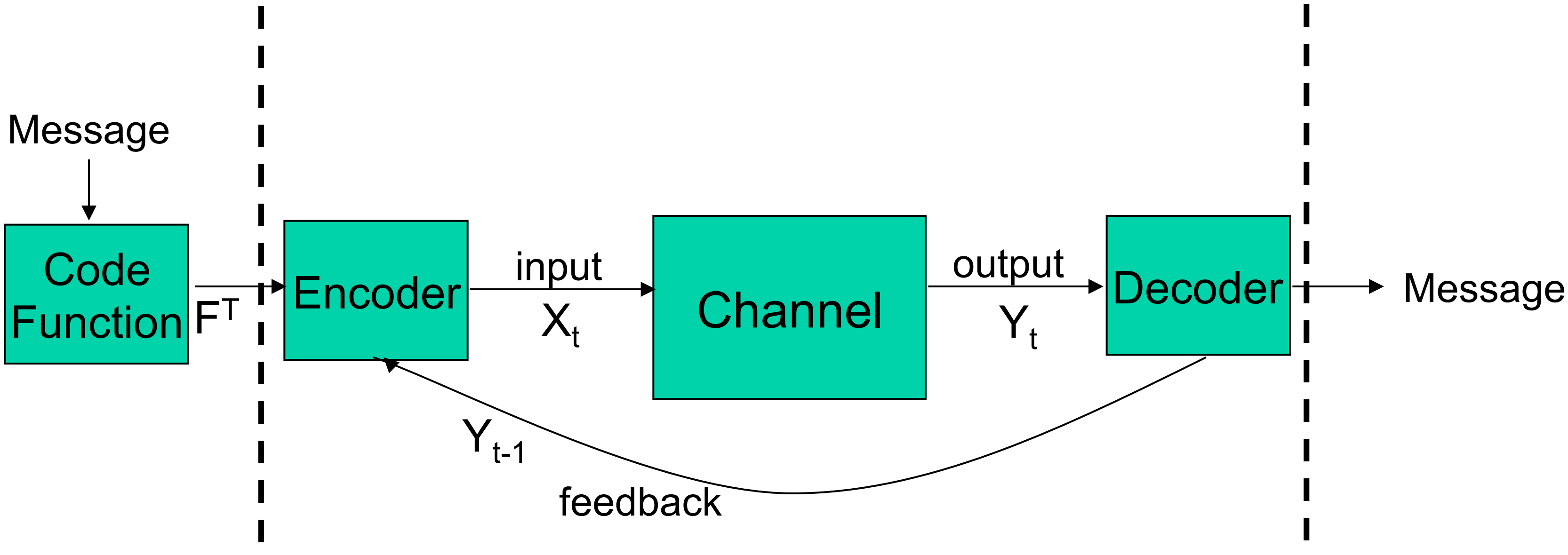


Converting a Feedback Channel to a Non-Feedback Channel



- A general *causal* channel has the form: $P(Y_t \mid X^t, Y^{t-1})$
- $P(Y_t \mid F^t, Y^{t-1}) = P(Y_t \mid X_1 = F_1, X_2 = F_2(Y_1), \ldots, X_t = F_t(Y^{t-1}), Y^{t-1})$
- The "F – Y" channel has no feedback

Converting a Feedback Channel to a Non-Feedback Channel – Part 2

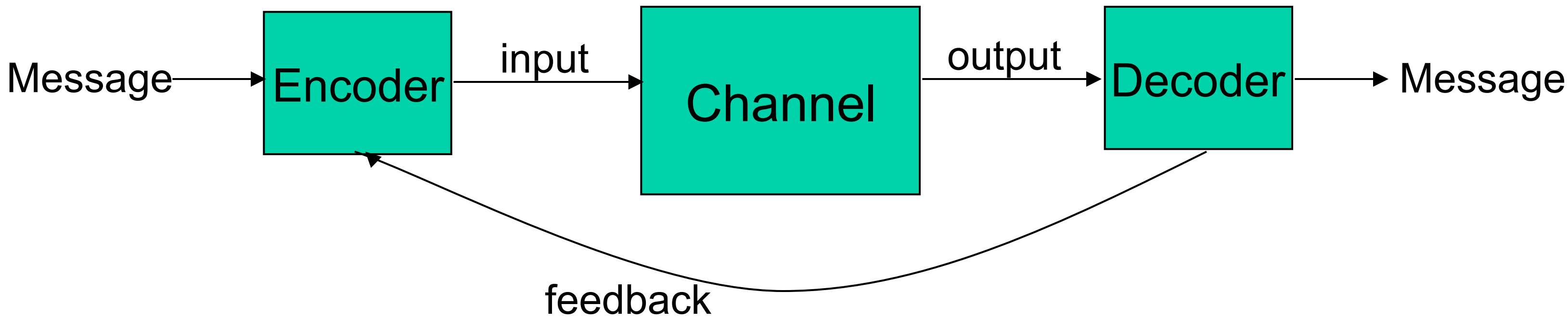


- Without feedback: $C = \lim_{T \to \infty} \max_{P(X^T)} 1/T \; I(X^T; Y^T)$

  With feedback: $C = \lim_{T \to \infty} \max_{P(F^T)} 1/T \; I(F^T; Y^T)$
  (both modulo information stability issues)

- Difficult optimization which does not give us much insight

Directed Information



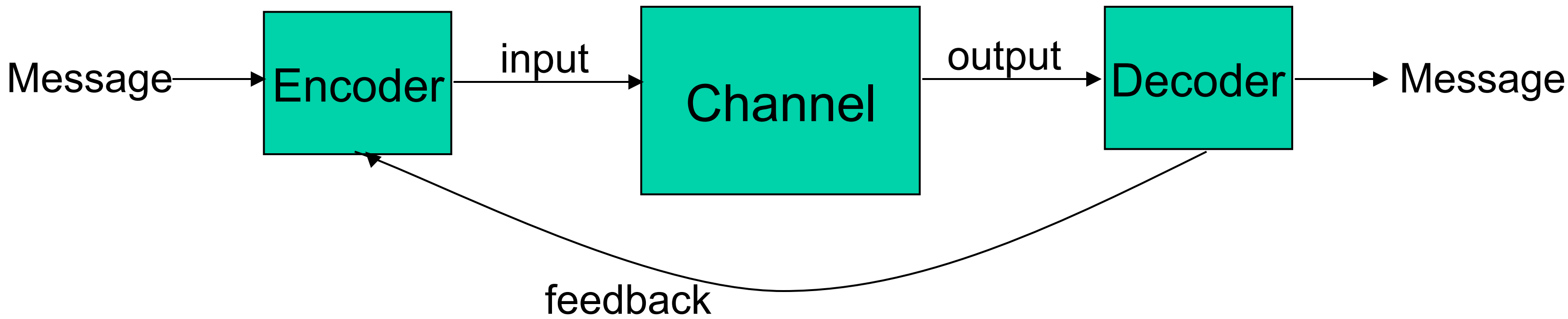
- $I(X^T; Y^T) = H(Y^T) - \sum_t H(Y_t \mid X^T, Y^{t-1}) = \sum_t I(X^T; Y_t \mid Y^{t-1})$

  - note dependence on future X's
  - Massey: "statistical dependence, unlike causality, has no inherent directivity."

- Massey's directed information, further developed by Kramer:

$$I(X^T \rightarrow Y^T) = H(Y^T) - \sum_t H(Y_t \mid X^t, Y^{t-1}) = \sum_t I(X^t; Y_t \mid Y^{t-1})$$

Directed Information – Part 2

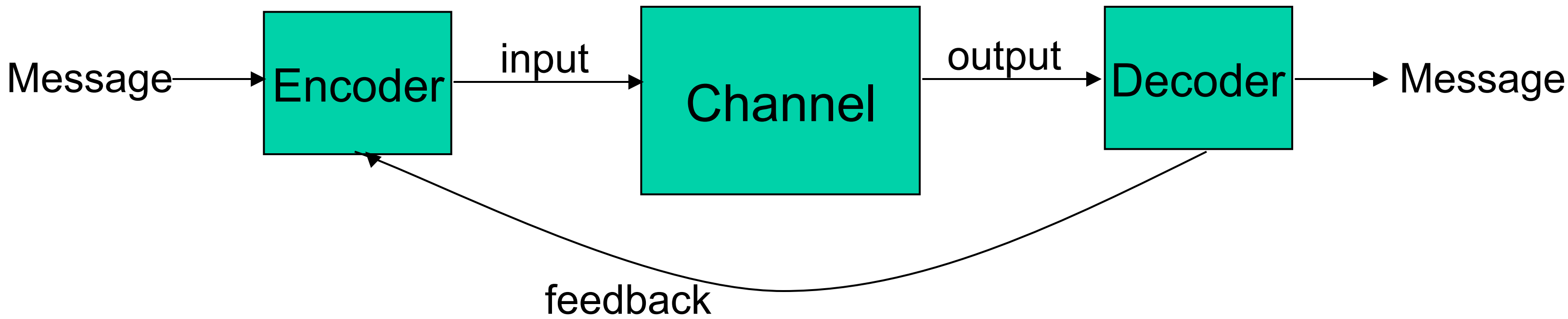


- $I(X^T \rightarrow Y^T) = H(Y^T) - \sum_t H(Y_t \mid X^t, Y^{t-1})$

- Consider example from before: $P(Y_t \mid X_t) = P(Y_t)$
  with feedback coding $X_t = Y_{t-1}$
  then $I(X^T \rightarrow Y^T) = H(Y^T) - \sum_t H(Y_t \mid X^t, Y^{t-1})$
  $= H(Y^T) - \sum_t H(Y_t)$
  $= 0$

- By DPI: $I(M; Y^T) = I(F^T; Y^T) \leq I(X^T; Y^T)$
  Message is not in one-to-one relation with X^T

Directed Information – Part 3



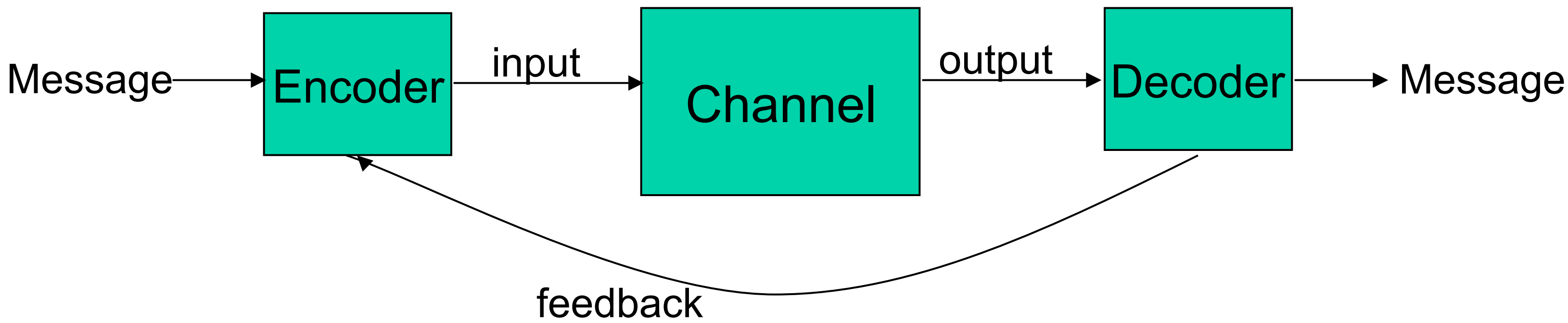
Massey showed:

- $I(X^T; Y^T) = I(X^T \to Y^T) + I(Y^T \to X^T)$
  conservation of information

- $I(X^T; Y^T) \geq I(X^T \to T^T)$
  with equality if and only if there is no feedback

- Directed information preserves causality, it is not symmetric

Information Spectrum Approach



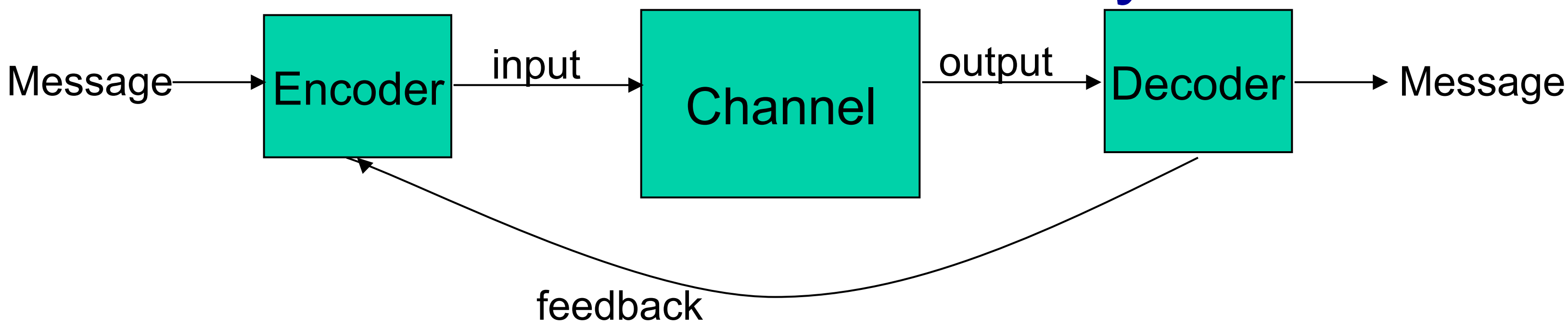
- $I(F^T; Y^T) = I(F^T \to Y^T)$
  $= H(Y^T) - \sum_t H(Y_t \mid F^t, Y^{t-1})$
  $= H(Y^T) - \sum_t H(Y_t \mid X^t, Y^{t-1})$
  $= I(X^T \to Y^T)$

- Coding theorem [Tat00] (based on Verdu/Han):

$$C = \sup_{\{P(X_t \mid X^{t-1}, Y^{t-1})\}} \text{ liminf in prob } 1/T\; i(X^T \to Y^T)$$

- Full-fledged coding theorem: both direct and converse; explicit construction of code-function distribution; and error exponent analysis. Can allow for arbitrary causal, deterministic feedback.

Information Stability

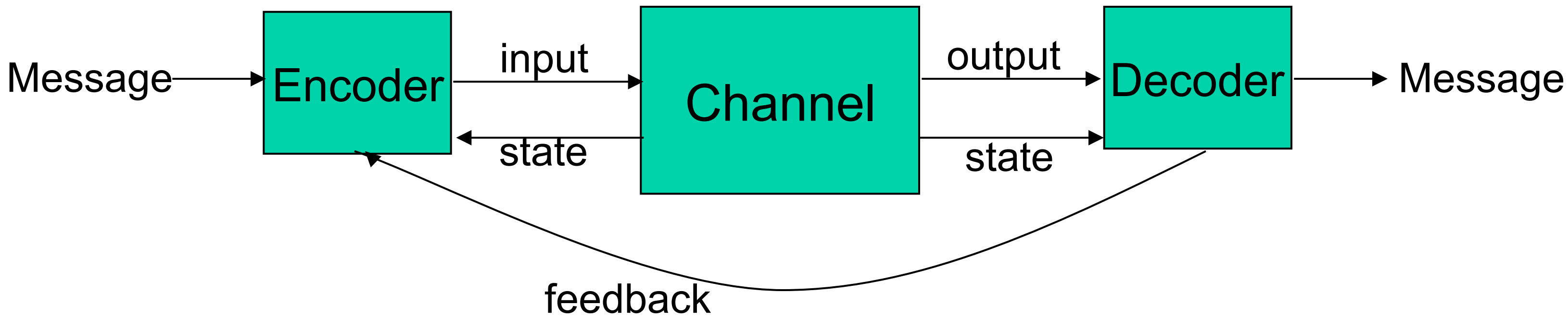


- How to compute!
- Two problems:
  - not information stable
  - not a single-letter characterization

- [Tat00] proposed an approach that *simultaneously* insured information stability and a single-letter characterization.

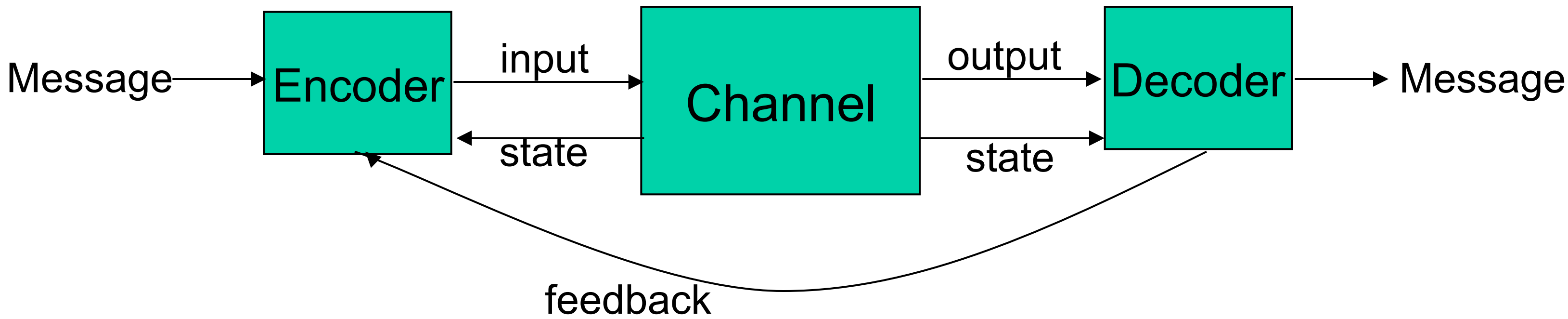
- Idea: Formulate optimization as an infinite horizon average cost Markov decision control problem.

Channels with Memory



- How to model memory in channels?
- General causal channel: $P(Y_t \mid X^t, Y^{t-1})$
- Need a compact representation
- Need stationarity (if we hope to calculate anything)

Markov Channel



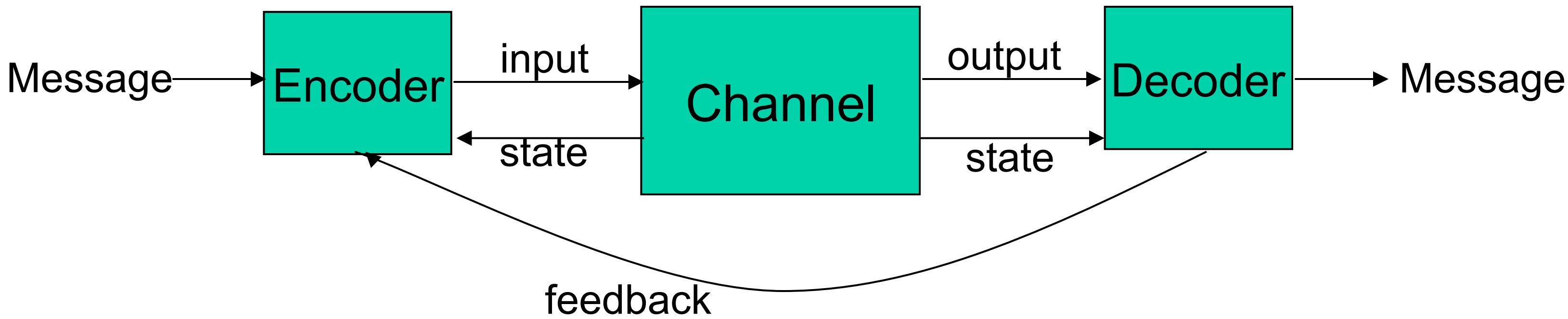
- A Markov Channel consists of: $P(S_1)$

  (1) state transition: $P(S_{t+1} \mid S_t, X_t)$

  (2) channel output: $P(Y_t \mid S_t, X_t)$

- If $P(S_{t+1} \mid S_t, X_t)$ is independent of X_t then non-ISI channel
- Assume time invariant (stationary)
- Time order: $M, S_1, X_1, Y_1, S_2, X_2, Y_2, \ldots, S_T, X_T, Y_T, M$

Side-Information versus Feedback



- Time order: M, S_1, X_1, Y_1, S_2, X_2, Y_2, …, S_T, X_T, Y_T, M

- Information pattern:
  - Tx: subset of (S^t, X^{t-1}, Y^{t-1})
  - Rx: subset of (S^t, Y^{t-1})
  - Nested information patterns

- Code-functions: $X_t = f_t(S^t, Y^{t-1})$
  (don't need X^{t-1})

- What is the difference between side-information and feedback?

A Quick Review of Dynamic Programming

- Sequential optimization.
  - State: S_t
  - Action: U_t
  - Dynamics: $P(S_{t+1} \mid S_t, U_t)$
  - Action:
  - Policy: $S_t \mapsto U_t$
  - Running cost: $c(S_t, U_t)$

- Infinite horizon average cost problem:

$$\sup \liminf 1/T \, E[\textstyle\sum_t c(S_t, U_t)]$$

- ACOE: Find J and w(s) that solve:

$$J + w(s) = \max_u c(s, u) + \textstyle\sum_{s+} P(s_+ \mid s,u) \, w(s_+)$$

then J is optimal cost and u*(s) is optimal policy.

Dynamic Programming Formulation

- ISI Markov Channel: $P(S_1)$, $P(S_{t+1} \mid S_t, X_t)$, $P(Y_t \mid S_t, X_t)$
- $I(X^T \rightarrow Y^T, S^T) = \sum_t I(X_t; Y_t, S_{t+1} \mid S_t)$
- $I(X_t; Y_t, S_{t+1} \mid S_t) = I(X_t; Y_t \mid S_t) + I(X_t; S_{t+1} \mid Y_t, S_t)$

- Dynamic programming framework:
  - state: S_t
  - action: $P(X_t)$
  - policy: $S_t \mapsto P(X_t)$, i.e. $P(X_t \mid S_t)$
  - running cost: $I(X_t; Y_t, S_{t+1} \mid S_t = s_t) = c(\, P(X_t), s_t)$

- Infinite horizon average cost problem:

$$\sup \liminf 1/T \, E[\, \textstyle\sum_t c(\, P(X_t), S_t \,)]$$

DP Formulation – Part 2

- Markov Channel: $P(S_1)$, $P(S_{t+1} \mid S_t, X_t)$, $P(Y_t \mid S_t, X_t)$
- Running cost: $c(P(X_t), s_t) = I(X_t; Y_t, S_{t+1} \mid S_t = s_t)$
- Infinite horizon average cost problem:

$$\sup \liminf 1/T\, E[\textstyle\sum_t c(P(X_t), S_t)]$$

- ACOE [Tat00]: If there exists a C and a w(S) such that $\forall$ s:

$$C + w(s) = \max_{P(X)}\{ I(X; Y, S_+ \mid s) + \textstyle\sum_{x,s_+} w(s_+)\, P(s_+ \mid x, s)\, P(x) \}$$

then C is the capacity of the ISI Markov channel. P(X|S) optimal input distribution.

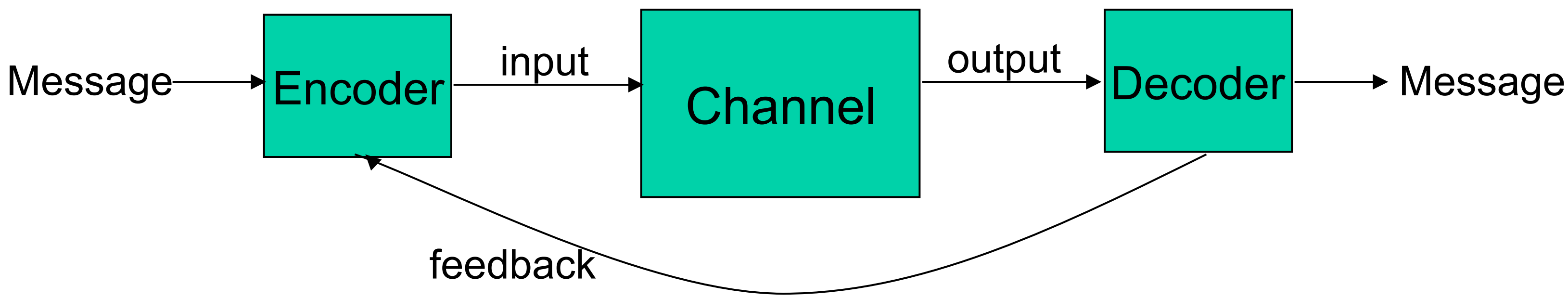
- Remarks:
  - Implicit single-letter characterization
  - If no ISI, ACOE becomes trivial
  - Multiplex between different codebooks indexed by S

DP Formulation – Part 3

- $C + w(s) = \max_{P(X)}\{ I(X; Y, S_+ \mid s) + \sum_{x,s_+} w(s_+) P(s_+ \mid x, s) P(x \mid s) \}$
- When does a solution to the ACOE exist? We need to insure ergodicity under the optimal policy.
- One sufficient condition is:
  $\| P(S_+ \mid S=s_1, X=x_1) - P(S_+ \mid S=s_2, X=x_2) \|_{TV} < 1 \quad \forall\, s_1, s_2, x_1, x_2$
- example: $P(s_+ \mid s, x) > 0 \quad \forall\, s, x, s_+$
- Related to Gallager's indecomposability:
  $\exists\, t$ s.t. $\| P(S_{t+1} \mid X^t=x^t, S_1=a) - P(S_{t+1} \mid X^t=x^t, S_1 = b) \|_{TV} < 1 \;\forall\, x^t, a, b$
  (in our setting this depends on the policy)
- Our sufficient condition insures that under any $P(X|S)$ the closed loop dynamic has a unique ergodic measure.

$$P(S_{t+1} \mid S_t) = \sum_x P(S_{t+1} \mid S_t, x) P(x \mid S_t)$$

Markov Channel with Output Feedback



- Markov channel: $P(S_1)$, $P(S_{t+1} \mid S_t, X_t)$, $P(Y_t \mid S_t, X_t)$.

- Now assume the state is not observed by either Tx or Rx. There is only output feedback. (Recall if state is known output feedback will not increase capacity.)

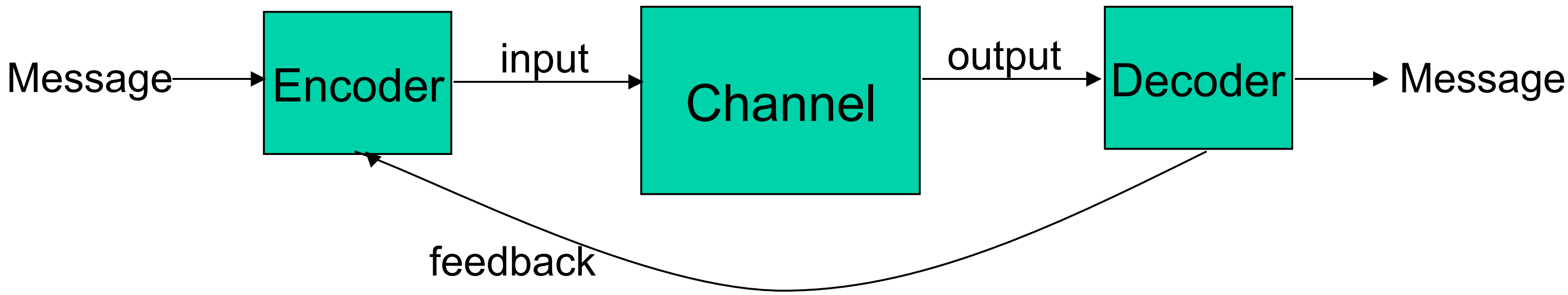
- At the beginning of the *t*-th epoch the

  Tx knows (X^{t-1}, Y^{t-1}) and Rx knows Y^{t-1}

  Note that the Rx's information pattern is nested in the Tx's information pattern. Find sufficient statistics (before it was S_t.)

- Input distribution has the form: $P(X_t \mid X^{t-1}, Y^{t-1})$

Encoder's Estimate: Π



- Use output feedback to estimate state at the encoder:

$$\Pi_t[X^{t-1}, Y^{t-1}] = P(S_t \mid X^{t-1}, Y^{t-1})$$

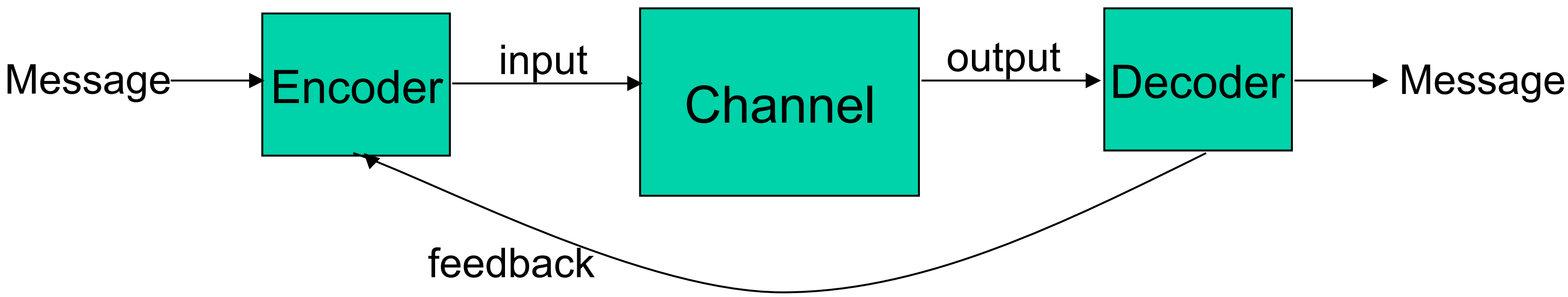
- There exists a policy independent function Φ_Π such that

$$\Pi_{t+1} = \Phi_\Pi(\Pi_t, X_t, Y_t).$$

This can be computed recursively at the Tx.

- Note that the statistic Π depends on information from both the Tx and the Rx

The Π Process and ISI



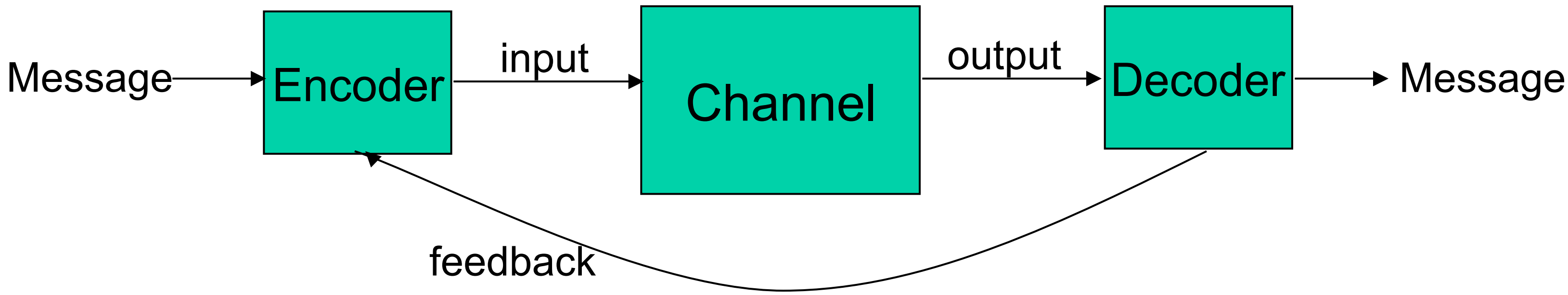
- $I(X^T \to Y^T) = H(Y^T) - \sum_t H(Y_t \mid X^t, Y^{t-1})$
  $= H(Y^T) - \sum_t H(Y_t \mid X_t, \Pi_t[X^{t-1}, Y^{t-1}])$
  $= \sum_t I(X_t, \Pi_t; Y_t \mid Y^{t-1})$

- Think of the pair (X_t, Π_t) as the input. The Rx does not know Π_t.

- Issue of dual effect. Even if underlying channel, $P(S_{t+1} \mid S_t)$, does not have ISI it is generically the case that the corresponding Π_t process *does* depend on the inputs:

$$P(\Pi_{t+1} \mid \Pi_t, X_t) = \sum_{s,y} \{\Pi_{t+1} = \Phi_\Pi(\Pi_t, X_t, y)\} P(y \mid s, X_t) \Pi_t(s)$$

Decoder's Estimate: Γ



- Tx: $(X^{t-1}, Y^{t-1}, \Pi_t)$, Rx: Y^{t-1}
- Goal: supremize $1/T \sum_t I(X_t, \Pi_t ; Y_t \mid Y^{t-1})$
- Rx needs estimate of Tx's estimate of the state:

$$\Gamma_t[Y^{t-1}] = P(\Pi_t \mid Y^{t-1})$$

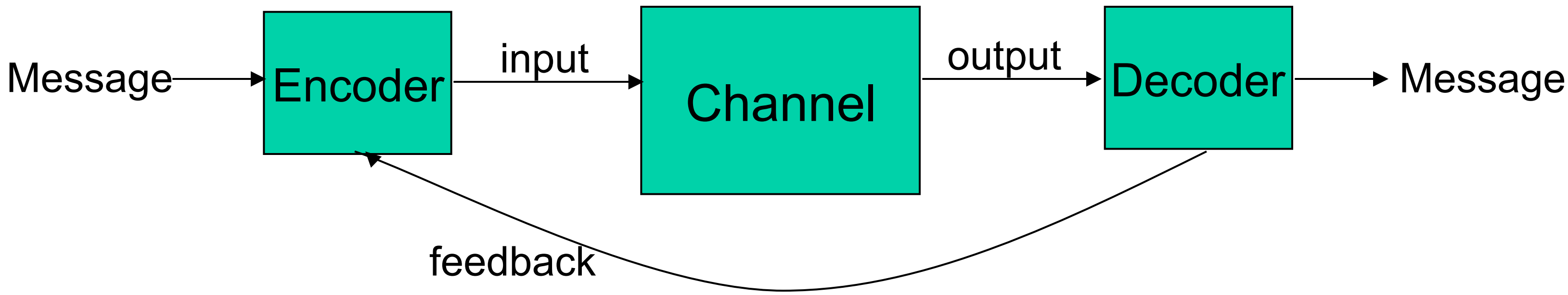
- note: not $P(S_t \mid Y^{t-1})$

- There exists a policy independent function Φ_Γ such that

$$\Gamma_{t+1} = \Phi_\Gamma(\Gamma_t, Y_t).$$

This can be computed at both the Tx and Rx.

Sufficient Statistics at Tx and Rx



- Before: Tx: (X^{t-1}, Y^{t-1}), Rx: Y^{t-1}

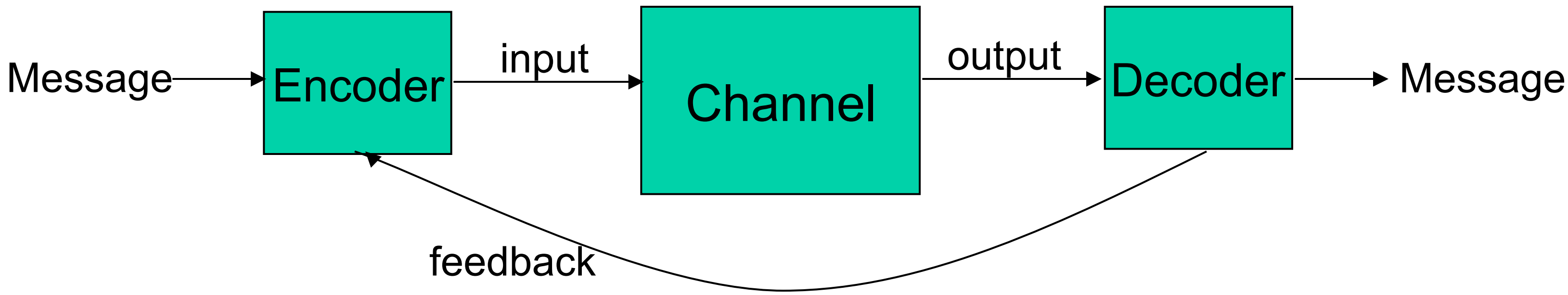
  Now: Tx: (Π_t, Γ_t), Rx: Γ_t

- Before: supremize $1/T \sum_t I(X^t ; Y_t \mid Y^{t-1})$

  Now: supremize $1/T \sum_t I(X_t, \Pi_t; Y_t \mid \Gamma_t)$

- Separation structure between estimation and coding.
Great simplification (though still complicated….)

ACOE



• Theorem [Tat05]: If there exists a bounded number C, a bounded function w: $\Gamma \mapsto R$, and a policy achieving the supremum for each $\Gamma = \gamma$ in the following ACOE:

$$C + w(\gamma) = \sup_{P(X,\,\Pi)} \left(I(X, \Pi; Y \mid \gamma) + \int w(\Gamma_+)\, P(d\Gamma_+ \mid \gamma, P(X, \Pi)) \right)$$

Then C is the capacity.

• Verification Theorem

In control problems with extensive sensing (vision sensor in feedback loop), control needs to act on “information” instead of signals.

Natural role for coding and decoding.