LIE BRACKETS AND STABILITY OF SWITCHED SYSTEMS

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Happy 60th birthday, Eduardo!

SWITCHED SYSTEMS

Switched system:

$$\dot{x} = f_\sigma(x)$$

- $\dot{x} = f_p(x), \ \ p \in \mathcal{P}$ is a family of systems
- $\sigma : [0, \infty) \to \mathcal{P}$ is a switching signal

Switching can be:

- State-dependent or time-dependent
- Autonomous or controlled

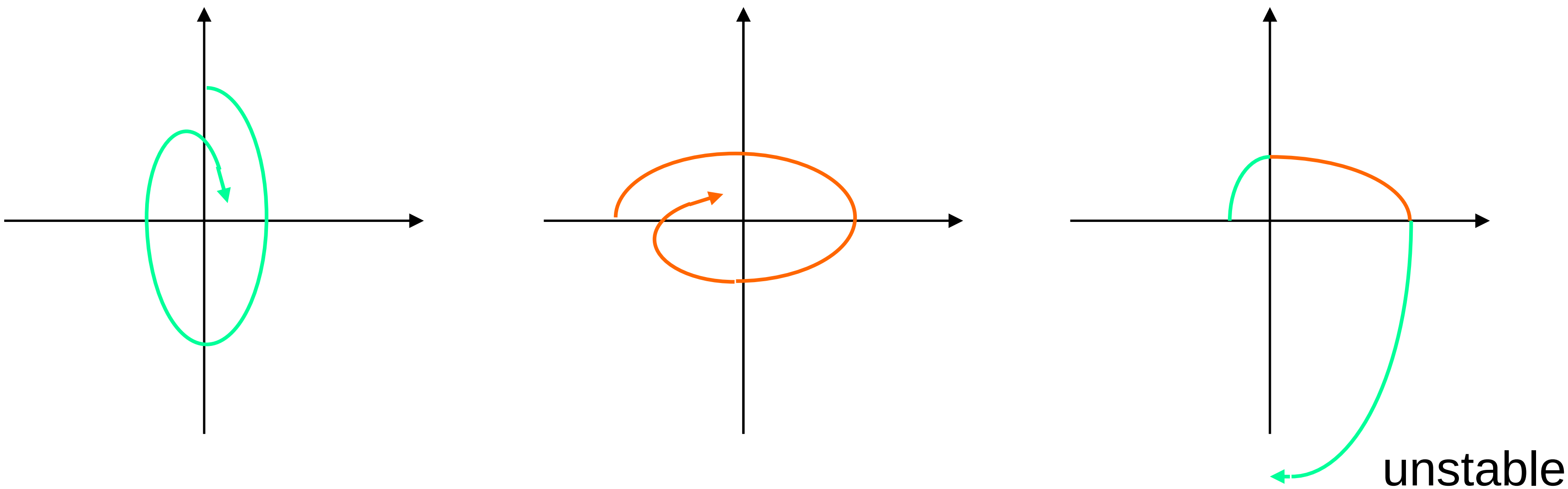
Details of discrete behavior are “abstracted away”

Discrete dynamics $\longrightarrow$ classes of switching signals

Properties of the continuous state x: stability

STABILITY ISSUE

$\dot{x} = f_1(x)$ $\dot{x} = f_2(x)$ $\dot{x} = f_\sigma(x)$



Asymptotic stability of each subsystem is

not sufficient for stability

GLOBAL UNIFORM ASYMPTOTIC STABILITY

GUAS is Lyapunov stability

$$\forall \varepsilon \;\; \exists \delta \;\; |x(0)| \le \delta \Rightarrow |x(t)| \le \varepsilon \;\; \forall t \ge 0, \forall \sigma$$

plus asymptotic convergence

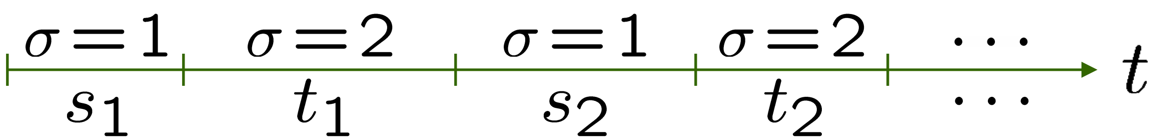
$$\forall \varepsilon, \delta \;\; \exists T \;\; |x(0)| \le \delta \Rightarrow |x(t)| \le \varepsilon \;\; \forall t \ge T, \forall \sigma$$

GUES: $\quad |x(t)| \le c e^{-\lambda t} |x(0)| \quad \forall t \ge 0, \; \forall \sigma$

COMMUTING STABLE MATRICES => GUES

$$\mathcal{P} = \{1, 2\}, \ A_1 A_2 = A_2 A_1$$

(commuting Hurwitz matrices)



$$x(t) = e^{A_2 t_k} e^{A_1 s_k} \dots e^{A_2 t_1} e^{A_1 s_1} x(0)$$

$$= e^{A_2(t_k + \dots + t_1)} e^{A_1(s_k + \dots + s_1)} x(0) \longrightarrow 0$$

For > 2 subsystems – similarly

COMMUTING STABLE MATRICES => GUES

Alternative proof:

$\exists$ quadratic common Lyapunov function

[Narendra–Balakrishnan '94]

$$P_1 A_1 + A_1^T P_1 = -I$$

$$P_2 A_2 + A_2^T P_2 = -P_1$$

$$\vdots$$

$$P_m A_m + A_m^T P_m = -P_{m-1}$$

$x^T P_m x$ is a common Lyapunov function

LIE ALGEBRAS and STABILITY

Lie algebra: $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{\mathsf{LA}}$

Lie bracket: $[A_1, A_2] = A_1 A_2 - A_2 A_1$

$\mathfrak{g}^1 = \mathfrak{g}, \quad \mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k] \subset \mathfrak{g}^k$ $\quad$ $\mathfrak{g}$ is nilpotent if $\exists\, k$ s.t. $\mathfrak{g}^k = 0$

$\| \qquad \cup \qquad \Downarrow$

$\mathfrak{g}^{(1)} = \mathfrak{g}, \quad \mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(k)}$ $\quad$ $\mathfrak{g}$ is solvable if $\exists\, k$ s.t. $\mathfrak{g}^{(k)} = 0$

Nilpotent means suff. high-order Lie brackets are 0

e.g. $[A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$

Nilpotent $\Rightarrow$ GUES [Gurvits '95]

SOLVABLE LIE ALGEBRA => GUES

Lie's Theorem: $\mathfrak{g}$ is solvable $\Rightarrow$ triangular form

$$A_p = \begin{pmatrix} \lambda_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Example:

$$A_1 = \begin{pmatrix} -a_1 & b_1 \\ 0 & -c_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -a_2 & b_2 \\ 0 & -c_2 \end{pmatrix}$$

$\dot{x}_2 = -c_\sigma x_2 \Rightarrow x_2 \to 0$ exponentially fast

$\dot{x}_1 = -a_\sigma x_1 + b_\sigma \underset{\downarrow \, 0}{x_2} \Rightarrow x_1 \to 0$ exp fast

$\exists$ quadratic common Lyap fcn $x^T D x,\ D$ diagonal

[L–Hespanha–Morse '99], see also [Kutepov '82]

MORE GENERAL LIE ALGEBRAS

Levi decomposition: $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$

↑ radical (max solvable ideal)

- $\mathfrak{s}$ is compact (purely imaginary eigenvalues) $\Rightarrow$ GUES, quadratic common Lyap fcn
- $\mathfrak{s}$ is not compact $\Rightarrow$ not enough info in Lie algebra:

There exists one set of stable generators for $\mathfrak{g}$ which gives rise to a GUES switched system, and another which gives an unstable one

[Agrachev–L '01]

SUMMARY: LINEAR CASE

Lie algebra $\{A_p, p \in \mathcal{P}\}_{\mathsf{LA}}$ w.r.t. $[A_1, A_2] = A_1 A_2 - A_2 A_1$

Assuming GES of all modes, GUES is guaranteed for:

- commuting subsystems: $[A_p, A_q] = 0 \;\; \forall p, q \in \mathcal{P}$

$\bigcap$

- nilpotent Lie algebras (suff. high-order Lie brackets are 0)

$\bigcap$ e.g. $[A_1, [A_1, A_2]] = [A_2, [A_1, A_2]] = 0$

- solvable Lie algebras (triangular up to coord. transf.)

$\bigcap$

- solvable + compact (purely imaginary eigenvalues)

Quadratic common Lyapunov function exists in all these cases

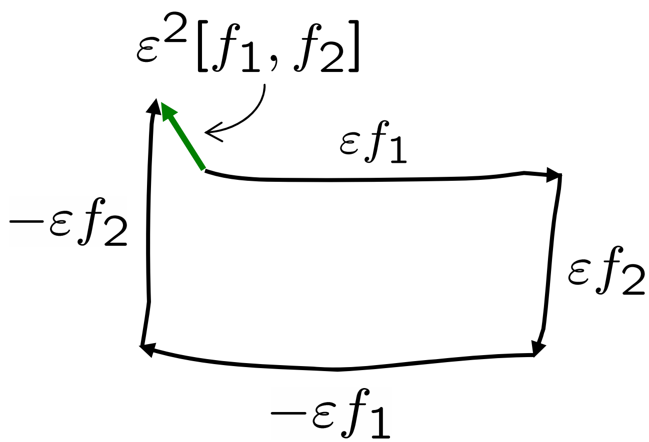
Extension based only on the Lie algebra is not possible

SWITCHED NONLINEAR SYSTEMS

Lie bracket of nonlinear vector fields:

$$[f_1, f_2] := \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2$$

Reduces to earlier notion for linear vector fields (modulo the sign)



SWITCHED NONLINEAR SYSTEMS

- Commuting systems

$$[f_p, f_q] = 0 \;\Rightarrow\; \text{GUAS}$$

Can prove by trajectory analysis [Mancilla-Aguilar '00]
or common Lyapunov function [Shim et al. '98, Vu–L '05]

- Linearization (Lyapunov's indirect method)

$$A_p = \frac{\partial f_p}{\partial x}(0), \;\; p \in \mathcal{P}$$

- Global results beyond commuting case – ?

[Unsolved Problems in Math. Systems and Control Theory, '04]

SPECIAL CASE

f_1, f_2 globally asymptotically stable

$$[f_1,[f_1,f_2]] = [f_2,[f_1,f_2]] = 0$$

Want to show: $\dot{x} = f_\sigma(x),\ \sigma \in \{1,2\}$ is GUAS

Will show: differential inclusion

$$\dot{x} \in \mathrm{co}\{f_1(x), f_2(x)\}$$

is GAS

OPTIMAL CONTROL APPROACH

Associated control system:

$$\dot{x} = f(x) + g(x)u$$

where $f := f_1, \quad g := f_2 - f_1, \quad u \in [0,1]$

(original switched system $\leftrightarrow u \in \{0,1\}$)

Worst-case control law [Pyatnitskiy, Rapoport, Boscain, Margaliot]:

fix x_0 and small enough t_f

$$|x(t_f)|^2 \to \max_u$$

MAXIMUM PRINCIPLE

$$H(x,u,\lambda) = \lambda^T f(x) + \underbrace{\lambda^T g(x)}_{\varphi} u$$

φ (along optimal trajectory)

Optimal control:

$u(t) = 0$ if $\varphi(t) < 0$, $u(t) = 1$ if $\varphi(t) > 0$

$$\dot{\varphi} = \lambda^T [f,g], \quad \ddot{\varphi} = \lambda^T [f,[f,g]] + \lambda^T [g,[f,g]]u = 0$$

$$\Downarrow$$

φ is linear in t

$\Downarrow$ (unless $\varphi \equiv 0$)

at most 1 switch

$$\Downarrow$$

GAS

GENERAL CASE

$$\dot{x} = f(x) + \sum_{k=1}^{m} g_k(x) u_k$$

$$\varphi_{ij} := \lambda^T (g_i(x) - g_j(x))$$

Want: φ_{ij} polynomial of degree $< r$

$\Downarrow$ (proof – by induction on m)

bang-bang with $(r+1)^m - 1$ switches

$\Downarrow$

GAS

See [Margaliot–L '06] for details; also [Sharon–Margaliot '07]

REMARKS on LIE-ALGEBRAIC CRITERIA

• Checkable conditions

• In terms of the original data

• Independent of representation

• Not robust to small perturbations

In any neighborhood of any pair of $n \times n$ matrices there exists a pair of matrices generating the entire Lie algebra $gl(n, \mathbb{R})$ [Agrachev–L '01]

How to capture closeness to a "nice" Lie algebra?

ROBUST CONDITIONS [Agrachev–Baryshnikov–L '10]

$\dot{x}(t) = A_{\sigma(t)}x(t)$ $\{A_p : p \in \mathcal{P}\}$ compact set of Hurwitz matrices

GUES:

$|x(t)| \le ce^{-\lambda t}|x_0| \ \ \forall\, t, x_0, \sigma(\cdot)$

Lie algebra:

$\mathfrak{g} := \{A_p : p \in \mathcal{P}\}_{LA}$

Levi decomposition: $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ ($\mathfrak{r}$ solvable, $\mathfrak{s}$ semisimple)

$$A_p = R_p + S_p, \quad R_p \in \mathfrak{r}, \quad S_p \in \mathfrak{s} \quad \forall\, p \in \mathcal{P}$$

Switched transition matrix splits as $\Phi(t) = \Phi_S(t)\Phi_R(t)$ where

$\dot{\Phi}_S(t) = S_{\sigma(t)}\Phi_S(t)$ and $\dot{\Phi}_R(t) = \big(\Phi_S^{-1}(t)R_{\sigma(t)}\Phi_S(t)\big)\Phi_R(t)$

Let $\bar{\lambda}_R := \max\limits_{p\in\mathcal{P}} \operatorname{Re}\lambda(R_p)$ and $\lambda_S^* := \limsup\limits_{t\to\infty} \frac{1}{t}\log\|\Phi_S(t)\|$

$$\boxed{\bar{\lambda}_R + \lambda_S^* < 0 \implies \text{GUES}}$$

robust condition but
not constructive

ROBUST CONDITIONS [Agrachev–Baryshnikov–L '10]

$\dot{x}(t) = A_{\sigma(t)}x(t)$ $\{A_p : p \in \mathcal{P}\}$ compact set of Hurwitz matrices

GUES:

$|x(t)| \le ce^{-\lambda t}|x_0| \;\; \forall\, t, x_0, \sigma(\cdot)$

Lie algebra:

$\mathfrak{g} := \{A_p : p \in \mathcal{P}\}_{LA}$

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Let $\bar{\lambda}_R := \max_{p \in \mathcal{P}} \mathrm{Re}\,\lambda(R_p)$ and $\hat{\lambda}_S := \max\{\|S_p\| : p \in \mathcal{P}\}$

$$\boxed{\bar{\lambda}_R + \hat{\lambda}_S < 0 \implies \text{GUES}}$$

more conservative but easier to verify

There are also intermediate conditions

ROBUST CONDITIONS [Agrachev–Baryshnikov–L '10]

Levi decomposition:

$$A_p = R_p + S_p, \quad R_p \in \mathfrak{r}, \quad S_p \in \mathfrak{s} \quad \forall p \in \mathcal{P}$$

Switched transition matrix splits as $\Phi(t) = \Phi_S(t)\Phi_R(t)$

Previous slide: $\|S_p\|$ small $\Longrightarrow$ GUES

But we also know: $\mathfrak{s}$ compact Lie algebra (not nec. small) $\Longrightarrow$ GUES

Cartan decomposition: $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{p}$ ($\mathfrak{k}$ compact subalgebra)

$$S_p = K_p + P_p, \quad K_p \in \mathfrak{k}, \quad P_p \in \mathfrak{p} \quad \forall p \in \mathcal{P}$$

Transition matrix Φ_S further splits: $\Phi_S(t) = \Phi_K(t)\Phi_P(t)$ where
$\dot{\Phi}_K(t) = K_{\sigma(t)}\Phi_K(t)$ and $\dot{\Phi}_P(t) = \Big(\Phi_K^{-1}(t)P_{\sigma(t)}\Phi_K(t)\Big)\Phi_P(t)$

Let $\hat{\lambda}_P := \max\left\{\left\|e^{-K}P_pe^{K}\right\| : K \in \mathfrak{k},\, p \in \mathcal{P}\right\}$

$$\boxed{\bar{\lambda}_R + \hat{\lambda}_P < 0 \Longrightarrow \text{GUES}}$$

ROBUST CONDITIONS [Agrachev–Baryshnikov–L '10]

Levi decomposition: $A_p = R_p + S_p, \quad R_p \in \mathfrak{r}, \quad S_p \in \mathfrak{s}$

Cartan decomposition: $S_p = K_p + P_p, \quad K_p \in \mathfrak{k}, \quad P_p \in \mathfrak{p}$

$$\bar{\lambda}_R = \max_{p \in \mathcal{P}} \mathrm{Re}\, \lambda(R_p), \ \ \hat{\lambda}_P = \max \left\{ \left\| e^{-K} P_p e^{K} \right\| : K \in \mathfrak{k},\ p \in \mathcal{P} \right\}$$

$$\boxed{\bar{\lambda}_R + \hat{\lambda}_P < 0 \implies \text{GUES}}$$

Example:

$$A_p = \begin{pmatrix} \lambda_p & \alpha_p + \delta_p \\ -\alpha_p + \delta_p & \lambda_p \end{pmatrix}, \quad \lambda_p < 0$$

$$\mathfrak{g} = gl(2), \ \mathfrak{r} = \mathbb{R} I_{2\times 2}, \ \mathfrak{s} = sl(2), \ \mathfrak{k} = so(2)$$

$$R_p = \begin{pmatrix} \lambda_p & 0 \\ 0 & \lambda_p \end{pmatrix}, \ K_p = \begin{pmatrix} 0 & \alpha_p \\ -\alpha_p & 0 \end{pmatrix}, \ P_p = \begin{pmatrix} 0 & \delta_p \\ \delta_p & 0 \end{pmatrix}$$

$$\bar{\lambda}_R = \max_{p \in \mathcal{P}} \lambda_p, \qquad \hat{\lambda}_P = \max_{p \in \mathcal{P}} |\delta_p|$$

CONCLUSIONS

- Discussed a link between Lie algebra structure and stability under arbitrary switching
- Linear story is rather complete, nonlinear results are still preliminary
- Focus of current work is on stability conditions robust to perturbations of system data