# Discontinuous Feedback in Nonlinear Control: Stabilization Under Disturbances and Optimization

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# Eduardo Sontag - 60 years

# Discontinuous Stabilizing Feedback - 15 years



## Asymptotic Controllability Implies Feedback Stabilization

Francis H. Clarke, Yuri S. Ledyaev, Eduardo D. Sontag, *Fellow, IEEE*, and Andrei I. Subbotin

***Abstract*— It is shown that every asymptotically controllable system can be globally stabilized by means of some (discontinuous) feedback law. The stabilizing strategy is based on pointwise optimization of a smoothed version of a control-Lyapunov function, iteratively sending trajectories into smaller and smaller neighborhoods of a desired equilibrium. A major technical problem, and one of the contributions of the present paper, concerns the precise meaning of "solution" when using a discontinuous controller.**

***Index Terms*— Control-Lyapunov functions, feedback, nonsmooth analysis, stabilization.**



Discontinuous Stabilizing Feedback - 15 years

### I. Introduction

A LONGSTANDING open question in nonlinear control theory concerns the relationship between asymptotic controllability to the origin in $\mathbb{R}^n$ of a nonlinear system

$$\dot{x} = f(x,u) \quad (1)$$

by an "open-loop" control $u : [0,+\infty) \to \mathbb{U}$ and the existence of a feedback control $k : \mathbb{R}^n \to \mathbb{U}$ which stabilizes trajectories of the system

$$\dot{x} = f(x,k(x)) \quad (2)$$

with respect to the origin.

For the special case of linear control systems $\dot{x} = Ax + Bu$, this relationship is well understood: asymptotic controllability is equivalent to the existence of a continuous (even linear) stabilizing feedback law. But it is well known that continuous

value $u$ so that $xf(x,u) < 0$," but it is easy to construct examples of functions $f$, even analytic, for which this property is satisfied but for which no possible continuous section $k : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ exists so that $xf(x,k(x)) < 0$ for all nonzero $x$. General results regarding the nonexistence of continuous feedback were presented in the paper [3], where techniques from topological degree theory were used (an exposition is given in [25]).

These negative results led to the search for feedback laws which are not necessarily of the form $u = k(x)$, $k$ a continuous function. One possible approach consists of looking for dynamical feedback laws, where additional "memory" variables are introduced into a controller, and as a very special case, time-varying (even periodic) continuous feedback $u = k(t,x)$. Such time-varying laws were shown in [27] to be always possible in the case of one-dimensional systems, and in the major work [9] (see also [10]) it was shown that they are also always possible when the original system is completely controllable and has "no drift," meaning essentially that $f(x,0) = 0$ for all states (see also [26] for numerical algorithms and an alternative proof of the time-varying result for analytic systems). However, for the general case of asymptotically controllable systems with drift, no dynamic or time-varying solutions are known. Thus, it is natural to ask about the existence of *discontinuous* feedback laws $u = k(x)$. Such feedbacks are often obtained when solving optimal-control problems, for example, so it is interesting to search for general theorems ensuring their existence. Unfortunately, allowing

# STABILIZATION

# Linear Control Systems: "Output Regulation"

Linear system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

$x(t)$ - state vector, $u(t)$ - input (control) vector

system is *controllable* ⇒ system is stabilizable

Namely, $\exists$ linear feedback control $u = Kx$ such that

closed-loop system $\dot{x} = Ax + BKx$ is stable

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Namely, $\exists$ linear feedback control $u = Kx$ such that

closed-loop system $\dot{x} = Ax + BKx$ is stable

Only output $y(t) = Cx(t)$ is available for measurement

input/output system is *observable* ➡ $\exists$ dynamic observer

$$\dot{z} = (A - LC)z + Bu(t) + Ly(t)$$

dynamic observer with output injection tracks $x(t)$

$$z(t) - x(t) \to 0$$

# Linear Control Systems: “Output Regulation”

## MAIN CONCLUSION:

Let linear control system

$$\dot{x} = Ax + Bu, \quad y(t) = Cx(t)$$

be *controllable* and *observable*

# Linear Control Systems: “Output Regulation”

## MAIN CONCLUSION:

Let linear control system

$$\dot{x} = Ax + Bu, \quad y(t) = Cx(t)$$

be *controllable* and *observable* then $\exists$ *dynamic observer* with output injection

$$\dot{z} = (A - LC)z + Bu(t) + Ly(t)$$

(**REMINDER:** $z(t)$ tracks $x(t)$ as $t \to +\infty$)
and *dynamic feedback* control $u(t) = Kz(t)$ such that

$$\dot{x} = Ax + BKz(t), \quad \dot{z} = (A - LC)z + BKz(t) + Ly(t)$$

is *asymptotically stable*

# Nonlinear Control Systems: “Output Regulation” Program

For linear control system

$$\dot{x} = Ax + Bu, \quad y(t) = Cx(t)$$

*controllability+observability* ⇨ $\exists$ stabilizing dynamic feedback

For nonlinear control system

$$\dot{x} = f(x, u), \quad y(t) = h(x(t))$$

**QUESTION:**

*controllability+observability* ⇨ $\exists$ stabilizing dynamic feedback **?**

# Nonlinear Control Systems: "Output Regulation" Program

For nonlinear control system

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**QUESTION:**

*controllability+observability* ➡ $\exists$ stabilizing dynamic feedback **?**

**REMINDER:** *Dynamic feedback controller*

Dynamic observer with output injection

$$\dot{z} = g(z, y(t)), \quad y(t) = h(x(t))$$

Closed-loop system

$$\dot{x} = f(x, k(z, y(t)))$$

for feedback $u(t) = k(z(t), y(t))$ such that

$$x(t) \to S \text{ as } t \to +\infty$$

# Nonlinear Control Systems: “Output Regulation” Program

Nonlinear control system under persistent disturbances

$$\dot{x} = f(x, u, d), \quad y(t) = h(x(t))$$

$d(t) \in \mathbb{D}$ - persistent disturbance

**QUESTION:**

*controllability+observability* ➡ $\exists$ stabilizing dynamic feedback **?**

Dynamic observer with output injection

$$\dot{z} = g(z, y(t)), \quad y(t) = h(x(t))$$

Closed-loop system

$$\dot{x} = f(x, k(z, y(t)), d(t))$$

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**APPLICATIONS OF OUTPUT REGULATION**

- General methods of design of output feedback controllers
- General theory of adaptive control (control under uncertainty)

# Nonlinear Control Systems: “Output Regulation” Program

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*controllability*+*observability* ⇒ $\exists$ stabilizing dynamic feedback **?**

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Important contributions by

**Coron**, **Isidori et al.**, **Praly**, **Teel**

# Why is "Output Regulation" Problem Difficult? Example:

Control system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{U}$$

Asymptotic controllability: for any initial point $x_0$ there exists control $u(\cdot) \in \mathcal{U}$

$$x(t; x_0, u) \to 0 \quad \text{as} \quad t \to +\infty$$

in some uniform manner

Stabilizing feedback control $k : \mathbb{R}^n \to \mathbb{U}$

$$\dot{x} = f(x, k(x))$$

is asymptotically stable

# Why is “Output Regulation” Problem Difficult? Example:

Relation between asymptotic controllability (AC) and feedback stabilization (FS):
Obvious $\dot{x} = f(x, k(x))$ is AS then $\dot{x} = f(x, u)$ is AC

$\exists$ feedback stabilizer $\Rightarrow$ asymptotic controllability

Long standing question: Is it true?

asymptotic controllability $\Rightarrow$ $\exists$ feedback stabilizer

# Why is "Output Regulation" Problem Difficult? Example:

$\exists$ feedback stabilizer $\Rightarrow$ asymptotic controllability

Long standing question: Is it true?

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Topological obstacles to existence of continuous feedback stabilizers:

- **Sontag&Sussmann 1980** one-dimensional example
- **Brockett 1982** general covering condition (topological obstacles), *nonholonomic integrator* example
- **Artstein 1983** - smooth control Lyapunov functions and continuous feedback
- **Coron 1990** stabilization of non-drift affine control systems

# Why is "Output Regulation" Problem Difficult? Example:

DISCONTINUOUS stabilizing feedback $k(x)$

$$\dot{x} = f(x, k(x))$$

Filippov (or more meaningful Krasovskii) solutions for discont.feedback

$$\dot{x} \in F(x) := \cap_{\delta > 0} \operatorname{co} f(x, k(x + \delta B))$$

– the **same topological obstacles**

# Why is “Output Regulation” Problem Difficult? Example:

**Clarke, Ledyaev, Sontag and Subbotin 1996**

**THEOREM:**

Asymptotic Controllability ⟺ IFF $\exists$ Feedback Stabilizer

**IMPORTANT:** New concept of *DISCONTINUOUS FEEDBACK* of *“sample-and-hold"* type (but different from traditional engineering “sample-and-hold" approach)

**PRECISE and NATURAL** mathematical model of digital computer control

# Nonlinear Control Systems under Persistent Disturbances

‘Output Regulation" Program: definition of *asymptotic controllability*

Control system

$$\dot{x} = f(x, u, d) \quad u \in \mathbb{U}, \quad d \in \mathbb{D}$$

$u(t)$ - control, $d(t)$ -disturbance

# Nonlinear Control Systems under Persistent Disturbances

‘Output Regulation" Program: definition of *asymptotic controllability*
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$$\dot{x} = f(x, u, d) \quad u \in \mathbb{U}, \quad d \in \mathbb{D}$$

$u(t)$ - control, $d(t)$ -disturbance
$d_t$ a restriction of function $d(\cdot)$ on the interval $[0, t]$
Non-anticipating strategy : operator $\mathcal{F}$ defining control $u(t)$

$$u(t) = \mathcal{F}(t, d_t)$$

# Nonlinear Control Systems under Persistent Disturbances

"Output Regulation" Program: definition of *asymptotic controllability*

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Non-anticipating strategy : operator $\mathcal{F}$ defining control $u(t)$

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Asymptotic Controllability (AC): $\forall$ initial point $x_0$ $\exists$ a strategy $\mathcal{F}(t, d_t)$

$$x(t; x_0, u(\cdot), d(\cdot)) \to 0 \quad \text{as} \quad t \to +\infty$$

in some uniform manner (with respect to $d(\cdot)$ and $x_0$)

# Nonlinear Control Systems under Persistent Disturbances

Feedback stabilizing controller; $k : \mathbb{R}^n \to \mathbb{U}$

$$\dot{x} = f(x, k(x), d(t)), \quad x(0) = x_0$$

for any $d(\cdot)$

$$x(t; x_0, d(\cdot)) \to 0 \quad \text{as} \quad t \to +\infty$$

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Why do we need feedback **?**

# Nonlinear Control Systems under Persistent Disturbances

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uniformly with respect to $d(\cdot)$ (and $x_0$ in some sense)

Robustness with respect to errors and perturbations!

# Nonlinear Control Systems under Persistent Disturbances

Original system

$$\dot{x} = f(x, k(x), d(t))$$

Perturbed system

$$\dot{x}(t) = f(x(t), k(x(t) + e(t)) + a(t), d(t)) + w(t)$$

- $e(t)$ – measurement error
- $a(t)$ – actuator error
- $w(t)$ – external disturbance

If $k(x)$ is CONTINUOUS then robustness follows from classical results on structural robustness of AS property ( **Krasovskii mid-1950s**)

$$\dot{x} = f(x, k(x)) + w(t) \quad \|w(t)\| \le \Delta(x(t))$$

What happens when $k(x)$ is DISCONTINUOUS?

# Main Results

Control system under persistent disturbances

$$\dot{x} = f(x, u(t), d(t))$$

Closed-loop system for feedback $k(x)$

$$\dot{x} = f(x, k(x), d(t))$$

**Ledyaev and Vinter 2005, 2010**

**THEOREM:**

Asymptotic Controllability ⟺ IFF $\exists$ Feedback Stabilizer

**THEOREM:**

Discontinuous Feedback Stabilizer is Robust w.r.t.Small Errors

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# Main Results

Meaning of these results

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**THEOREM:**

Asymptotic Controllability ⟺ (IFF) $\exists$ Feedback Stabilizer

Asymptotic Controllability: for any $x_0$ $\exists\, \mathcal{F}$ s.t. using complete perfect INFINITE MEMORY information $d_t$ at each moment $t$

$$u(t) = \mathcal{F}(t, d(\cdot)_t)$$

we can drive to the origin as $t \to +\infty$
Theorem claims: NO NEED to use infinite memory information (NO infinite-dimensional *information states*) to drive to the origin
Only use updated values of FINITE-DIMENSIONAL state vector $x(t)$

# Precise Definitions and Statements

**Main Assumptions:**

**A1.** Sets $\mathbb{U}$, $\mathbb{D}$ are compact, function $f : \mathbb{R}^n \times \mathbb{U} \times \mathbb{D} \to \mathbb{R}^n$ is continuous and is loc. Lipschitz on $x$ on compact subsets of $\mathbb{R}^n \times \mathbb{U} \times \mathbb{D}$.

**A2.** ( **Isaacs 1965** condition) For any $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\max_{d \in \mathbb{D}} \min_{u \in \mathbb{U}} \langle p, f(x, u, d) \rangle = \min_{u \in \mathbb{U}} \max_{d \in \mathbb{D}} \langle p, f(x, u, d) \rangle$$

**REMARK.** NO growth condition on $f$.

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Set $\mathcal{D}$ of all meas. func. $d : \mathbb{R}_+ \to \mathbb{D}$ (called *disturbances*)
Set $\mathcal{M}_{\mathbb{U}}$ of all *relaxed controls* (weakly meas. functions)
$\mu : \mathbb{R}_+ \to \text{prm}(\mathbb{U})$ ( $\text{prm}(\mathbb{U})$ – set of all probab. Radon measures on $\mathbb{U}$)
$N : \mathcal{D} \to \mathcal{M}_{\mathbb{U}}$ – non-anticipating strategy if $\forall\, d^1$, $d^2 \in \mathcal{D}$ s.t. for some $t \in \mathbb{R}_+$ $\quad d^1_t = d^2_t$ we have $N(d^1)_t = N(d^2)_t$.

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**Varaiya-Lin, Kalton-Elliot 1970s, Chentsov 1980s, Gusyatnikov ...**

# Precise Definitions and Statements

For $\forall\ \ d(\cdot) \in \mathcal{D}$ and a strategy $N$ consider relaxed control

$$\nu := N(d(\cdot))$$

$x(t; x_0, N, d)$ – is a solution (locally exists)

$$\dot{x}(t) = \hat{f}(x(t), \nu(t), d(t)), \quad x(t_0) = x_0$$

where

$$\hat{f}(x, \nu.d) := \int_{\mathbb{U}} f(x, u, d)\nu(du)$$

# Precise Definitions and Statements

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where

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REMEMBER

$$\boxed{x(t; x_0, N, d)}$$

# Precise Definitions and Statements

DISCONTINUOUS feedback $k : \mathbb{R}^n \to \mathbb{U}$ and diff.equation with discontinuous right-hand side

$$\dot{x} = f(x, k(x), d(t)), \quad x(0) = x_0$$

Concept of solution : $\pi$-trajectory (from positional differential games theory **Krasovskii & Subbotin 1970s**)
Partition $\pi = \{t_i\}_{i \geq 0}$ of $[0, +\infty)$, $\lim_{i\to\infty} t_i = +\infty$
Diameter of partition: $d(\pi) := \sup_i (t_{i+1} - t_i)$
$\pi$-trajectory $x_\pi(t) := x(t)$

$$\dot{x}(t) \; = \; f(x(t), k(x(t_i)), d(t)) \, , \; t \in [t_i, t_{i+1}]$$

Natural model of computer digital control (*"sampling"*)

# Precise Definitions and Statements

**DEFINITION: ASYMPTOTIC CONTROLLABILITY** $\dot{x} = f(x, u, d)$

$\forall\ x_0 \in \mathbb{R}^n$ there exists a non-anticipating strategy $N$ such that

- (**ATTRACTIVENESS**) For any disturbance $d \in \mathcal{D}$ a trajectory $x(t; x_0, N, d)$ is defined on the entire interval $\mathbb{R}_+$ and $x(t; x_0, N, d) \to 0$ as $t \to +\infty$ uniformly with respect to disturbances $d \in \mathcal{D}$;
- (**UNIFORM BOUNDEDNESS**)

$$\sup_{d \in \mathcal{D}} \sup_{t \geq 0} \|x(t; x_0, N, d)\| < +\infty$$

- (**LYAPUNOV STABILITY**) $\forall\ \varepsilon > 0\ \ \exists\ \delta > 0$ s.t. $\forall\ x_0$ satisfying $\|x_0\| < \delta\ \exists$ non-anticipating strategy $N$ s.t. $\forall d \in \mathcal{D}$

$$\|x(t; x_0, N, d)\| < \varepsilon \quad \forall\ t \geq 0$$

# Precise Definitions and Statements

**DEFINITION: STABILIZING FEEDBACK** $\dot{x} = f(x, k(x), d)$

For any $0 < r < R \; \exists \; M = M(R) > 0$, $\delta = \delta(r, R) > 0$, and $T = T(r, R) > 0$ s.t. $\forall \, \pi$ with $d(\pi) < \delta$ and $\forall \, x_0$ such that $\|x_0\| \leq R$ and $\forall$ disturbance $d \in \mathcal{D}$, the $\pi$-trajectory $x(\cdot)$, $x(0) = x_0$ is defined on $[0, +\infty)$ and

- (**UNIFORM ATTRACTIVENESS)**

$$\|x(t)\| \leq r \quad \forall \, t \geq T$$

- (**OVERSHOOT BOUNDEDNESS**)

$$\|x(t)\| \leq M(R) \quad \forall \, t \geq 0$$

- (**LYAPUNOV STABILITY**)

$$\lim_{R \downarrow 0} M(R) = 0$$

# Precise Definitions and Statements

**Ledyaev and Vinter 2005, 2010**

**THEOREM:** Under Assumptions A1 and A2 we have

Asymptotic Controllability $\Longleftrightarrow$ IFF $\exists$ Feedback Stabilizer

# Precise Definitions and Statements

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**THEOREM:** Under Assumptions A1 and A2 we have

Asymptotic Controllability ⟺ (IFF) $\exists$ Feedback Stabilizer

Even more, we can prove existence of continuous functions $\delta : \mathbb{R}^n \backslash \{0\} \to (0, +\infty)$, $\beta : [0, +\infty) \times [0, +\infty) \to (0, +\infty)$ of class $\mathcal{KL}$: $\beta(t, r)$ - monot. decreasing in $t$, increasing in $r$, $\lim_{t \to +\infty} \beta(t, r) = 0$, $\lim_{r \to 0} \beta(t, r) = 0$.

for discontinuous stabilizing feedback $k(x)$ and any $\pi = \{t_i\}_{i \geq 0}$ s.t. $0 < t_{i+1} - t_i \leq \delta(x(t_i))$ we have the next *decay estimate*

$$\|x(t)\| \leq \beta(t, \|x(0)\|) \quad \forall\, t \geq 0$$

# Proof: Control Lyapunov Functions

Control Lyapunov function (CLF) pair $(V(x), W(x))$

- (**POSITIVENESS**)

$$V(x) \geq 0, \; V(x) = 0 \Leftrightarrow x = 0, \quad W(x) > 0 \quad \forall\, x \neq 0$$

- (**PROPERNESS**)

$$V(x) \to +\infty \quad \text{as } \|x\| \to +\infty$$

- (**INFINITESIMAL DECREASE**)

$$\min_{u \in \mathbb{U}} \max_{d \in \mathbb{D}} \langle \nabla V(x), f(x,u,d) \rangle \leq -W(x) \quad \forall\, x \in \mathbb{R}^n \backslash \{0\}$$

**Kokotovic & Freeman 1990s robust control Lyapunov function**

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We assumed that $V$ is $C^1$ and $\exists$ continuous $k : \mathbb{R}^n \to \mathbb{U}$ s.t.

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Then solutions $x(t)$ of the closed-loop system

$$\dot{x} = f(x, k(x), d(t)), \quad x(0) = x_0$$

are well-defined and we have a *decay estimate*

$$\boxed{\|x(t)\| \leq \beta(t, \|x(0)\|) \quad \forall\, t \geq 0}$$

Thus, existence of $C^1$ CLF $V$ and continuous (or DISCONTINUOUS) $k(x)$ satisfying ($\star$) ⇒ AC (asymptotic controllability)

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Is inverse valid?

AC (asymptotic controllability) ⇒ existence of $C^1$ CLF $V$

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**Ledyaev and Vinter 2005, 2010**

**THEOREM:** Under Assumptions A1 and A2

Asymptotic Controllability $\overset{\text{IFF}}{\Longleftrightarrow}$ $\exists$ lower semicont. CLF $V$

CLF pair $(V, W)$: $V$ is lower semicontinuous ($\liminf_{x \to x_0} V(x) \geq V(x_0)$),

$W$ – continuous

- (**POSITIVENESS**)

$$V(x) \geq 0, \; V(x) = 0 \Leftrightarrow x = 0, \quad W(x) > 0 \quad \forall \, x \neq 0$$

- (**PROPERNESS**)

$$V(x) \to +\infty \quad \text{as } \|x\| \to +\infty$$

- (**INFINITESIMAL DECREASE**)

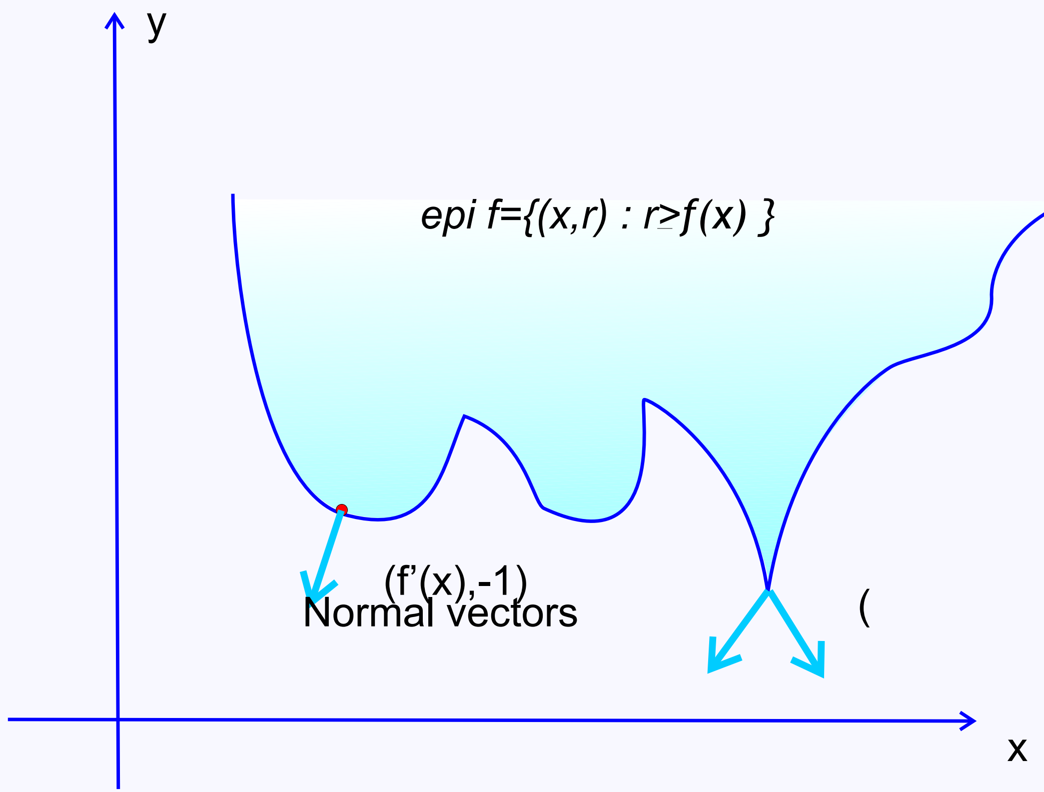
$$\min_{u \in \mathbb{U}} \max_{d \in \mathbb{D}} \langle \zeta, f(x, u, d) \rangle \leq -W(x) \quad \forall \, \zeta \in \partial_P V(x), \; x \in \mathbb{R}^n \backslash \{0\}$$

# Proof: Control Lyapunov Functions

**NONSMOOTH ANALYSIS: proximal subgradients**

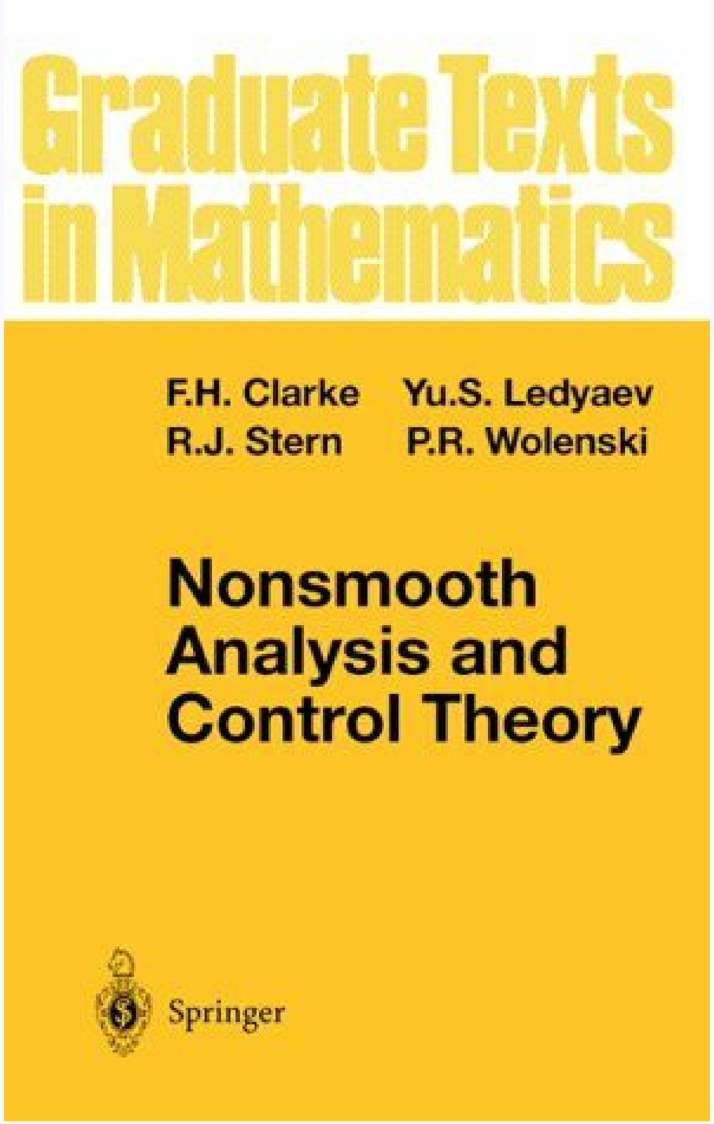
$\zeta \in \partial_P f(x)$ if $\exists\ \sigma > 0$

$$\langle \zeta, z - x \rangle - \sigma \|z - x\|^2 \leq f(z) - f(x) \quad \forall\ z \text{ near } x$$



# Proof: Control Lyapunov Functions

Reference on Nonsmooth Analysis (proximal calculus) and its applications



# Proof: Control Lyapunov Functions

Proof of the existence of l.s.c. CLF $V$ for AC system

$$V(x) := \inf_{N} \sup_{d \in \mathcal{D}} \int_0^{+\infty} W(x(t; x, N, d))dt$$

It is analogous to proofs of inverse Lyapunov function theorems for diff.equations:
asymptotic stability ➡ existence of Lyapunov function

# Proof: Control Lyapunov Functions

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It is analogous to proofs of inverse Lyapunov function theorems for diff.equations:

asymptotic stability ⇒ existence of smooth Lyapunov functions

**Massera 1949, Krasovskii 1950s,Kurzweil 1955, ...**

For control systems (AC ⇒ continuous CLF) **Sontag 1983**

?

**OPEN QUESTION:** Does CONTINUOUS CLF exist for AC control system under persistent disturbances?

# Design of (Dis-)Continuous Feedback Stabilizer via CLF

Let $(V, W)$ be a Control Lyapunov Function (CLF) pair
$V(x)$ is lower semicontinuous, $V(x) > 0$ iff $x \neq 0$, $V(x) \to +\infty$ as $\|x\| \to +\infty$ and infinitesimal decrease condition holds

$$H(x, \zeta) := \min_{u \in \mathbb{U}} \max_{d \in \mathbb{D}} \langle \zeta, f(x, u, d) \rangle \leq -W(x) \ \ \forall \, \zeta \in \partial_P V(x), \forall \, x \in \mathbb{R}^n \backslash \{0\}$$

Note, if $V \in C^1$ then $\partial_P V(x) \subset \{\nabla V(x)\}$
In the case $V$ continuous, the stabilizing feedback construction is contained in **Clarke,Ledyaev,Sontag&Subbotin 1996**

*Asymptotic Controllability Implies Feedback Stabilization*

How to handle a lower semicontinuous CLF $V$?

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*Asymptotic Controllability Implies Feedback Stabilization*

How to handle a lower semicontinuous CLF $V$?
Use method of **Clarke,Ledyaev and Subbotin 1997**

*The synthesis of universal feedback pursuit strategies in differential games* SIAM J.Control and Optimization

# Design of (Dis-)Continuous Feedback Stabilizer via CLF

Method

- Kruzhkov transform ($\kappa$ - some constant)

$$v(x) := 1 - \exp(-\kappa V(x)) > 0, \quad v(x) = 0 \Longleftrightarrow x = 0$$

- For any $x \in \mathbb{R}^n$ and $\zeta \in \partial_P v(x)$

$$H(x, \zeta) \leq \kappa W(x)(v(x) - 1)$$

$$\zeta \in \partial_P v(x) \Leftrightarrow \zeta \in \kappa \exp(-\kappa V(x)) \partial_P V(x)$$

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$$v_\alpha(x) := \min_y [v(y) + \frac{1}{2\alpha^2} \|y - x\|^2]$$

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- "Taylor expansion" formula: $\forall\, f \in \mathbb{R}^n$

$$v_\alpha(x+\tau f) \;\le\; v_\alpha(x) + \tau\langle \zeta_\alpha(x), f\rangle + \frac{\tau^2\|f\|^2}{2\alpha^2}\,.$$

$$\zeta_\alpha(x) \;:=\; \frac{x-y_\alpha(x)}{\alpha^2} \in \partial_P v(y_\alpha(x))$$

$y_\alpha(x)$ an arbitrary minimizer $y \to v(y) + \frac{1}{2\alpha^2}\|y-x\|^2$

# Design of (Dis-)Continuous Feedback Stabilizer via CLF

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$y_\alpha(x)$ an arbitrary minimizer $y \to v(y) + \frac{1}{2\alpha^2}\|y-x\|^2$
Compare traditional one-sided Taylor expansion formula for $\varphi \in C^2$:

$$\varphi(x+\tau f) \leq \varphi(x) + \tau\langle \varphi'(x), f\rangle + C\tau^2\|f\|^2$$

We have some analogue for $v_\alpha$ ($v$ is only l.s.c.) ($\bigstar$)
magic of proximal calculus!

# Design of (Dis-)Continuous Feedback Stabilizer via CLF

Definition of the stabilizing feedback $k(x)$

$$\max_{d\in\mathbb{D}}\langle\zeta_\alpha(x), f(x,k(x),d)\rangle \;=\; \min_{u\in\mathbb{U}}\max_{d\in\mathbb{D}}\langle\zeta_\alpha(x), f(x,u,d)\rangle = H(x,\zeta_\alpha(x))$$

Then

$$\max_{d\in\mathbb{D}}\langle\zeta_\alpha(x), f(x,k(x),d)\rangle \leq H(x,\zeta_\alpha(x)) \leq -\kappa W(y_\alpha(x))(1-v(y_\alpha(x)))$$

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⇒ $v_\alpha(x(t)) \le v_\alpha(x(t_i))$ (invariance of level sets) and also $v_\alpha(x(t))$ is monotonic.decreasing

# Robustness of Discontinuous Feedback I

Original closed-loop system

$$\dot{x} = f(x, k(x), d(t))$$

Perturbed system

$$\dot{x} = f(x, k(x + e(t)) + a(t), d(t)) + w(t)$$

- $e(t)$ – measurement error
- $a(t)$ – actuator error
- $w(t)$ – external disturbance

# Robustness of Discontinuous Feedback I

Perturbed system

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- $e(t)$ – measurement error
- $a(t)$ – actuator error
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Structural assumption

$$a(t) = a_1(t) + a_2(t), \quad w(t) = w_1(t) + w_2(t)$$

Small errors means

small magnitude but unbounded impulse

$$\|e(\cdot)\|_\infty < \varepsilon, \quad \|a_1(\cdot)\|_\infty < \varepsilon, \quad \|w_1(\cdot)\|_\infty < \varepsilon$$

small impulse but unbounded magnitude

$$\|a_2(\cdot)\|_1 < \varepsilon, \quad \|w_2(\cdot)\|_1 < \varepsilon$$

# Robustness of Discontinuous Feedback I

It follows from the design of discontinuous feedback $k(x)$ that it is robust with respect to small actuator errors and external disturbances...

What about measurement errors?
Instead of $x(t_i)$ we use corrupted data

$$x'(t_i) := x(t_i) + e(t_i) \quad \Rightarrow \quad k(x'(t_i))$$

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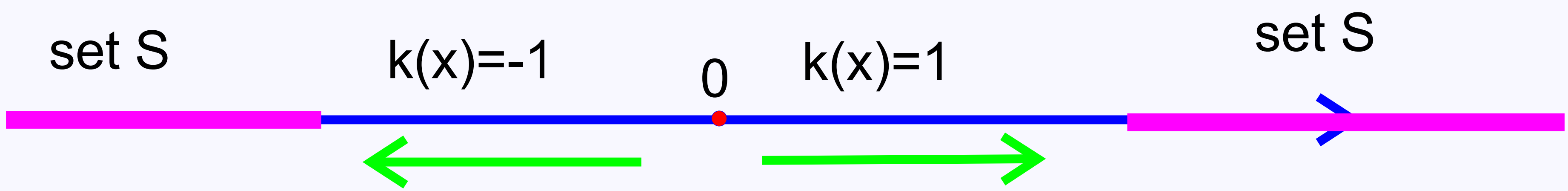
**Control Problem:** Drive $x(t)$ to $S := (-\infty, -1| \cup [1, +\infty)$

$$\dot{x} = u, \quad x \in \mathbb{R}, \quad u \in \mathbb{U} := \{-1, 1\}$$

Feedback

$$k(x) = \begin{cases} +1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$



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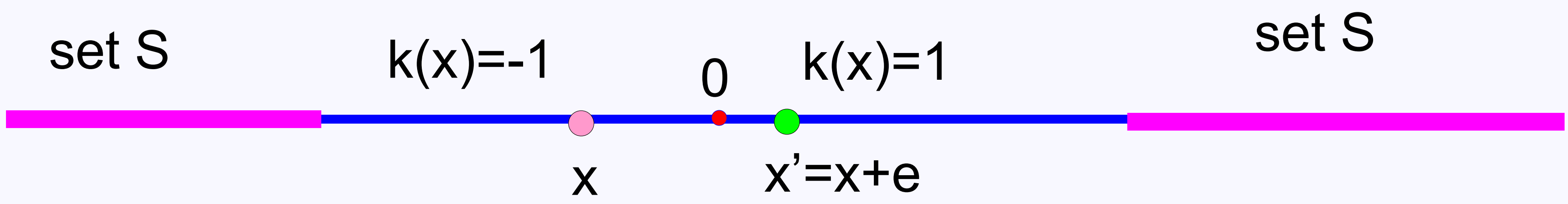

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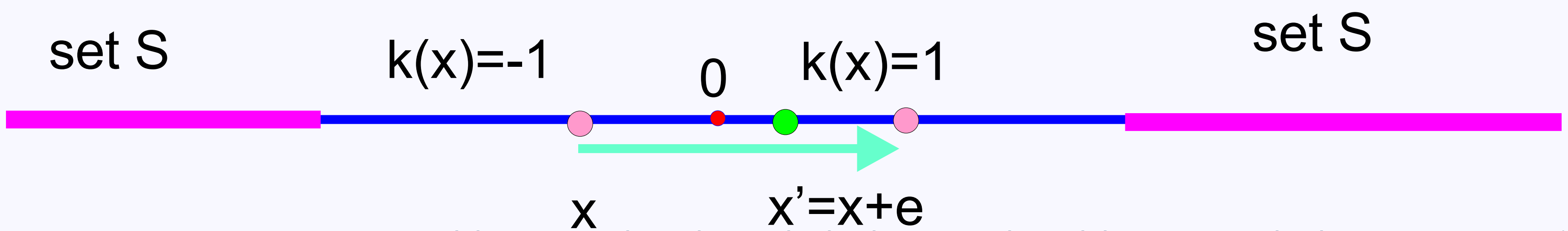

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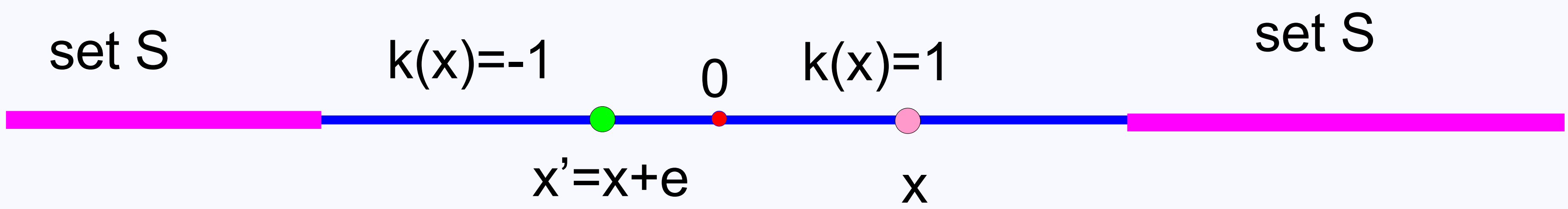

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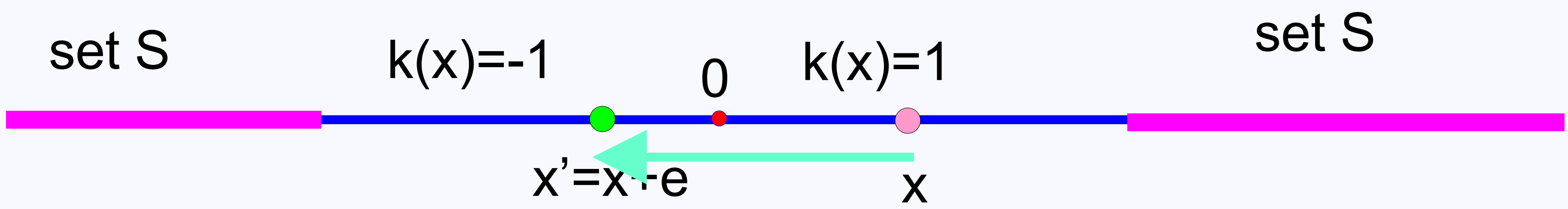


# Robustness of Discontinuous Feedback II

**FIRST REMEDY: Control with Guide Procedure Krasovskii & Subbotin**

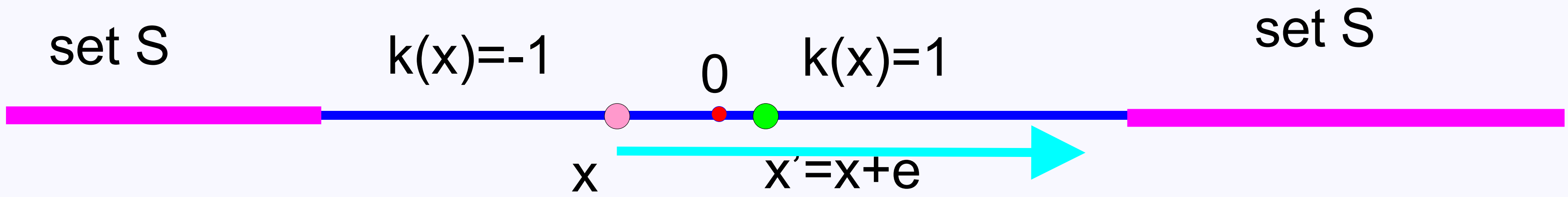
**begin. 1970s** use of a computational model of closed-loop system

In the context of stabilization problems **Ledyaev&Sontag 1997**

**SECOND REMEDY: Restrict a sampling rate** $\nu := \sup \frac{1}{t_{i+1} - t_i}$ **from above**

➡ $t_{i+1} - t_i \geq 1/\nu$ **and let us assume that**

small measurement error: $\|e(t)\| < 1/2\nu \leq \frac{1}{2}(t_{i+1} - t_i)$



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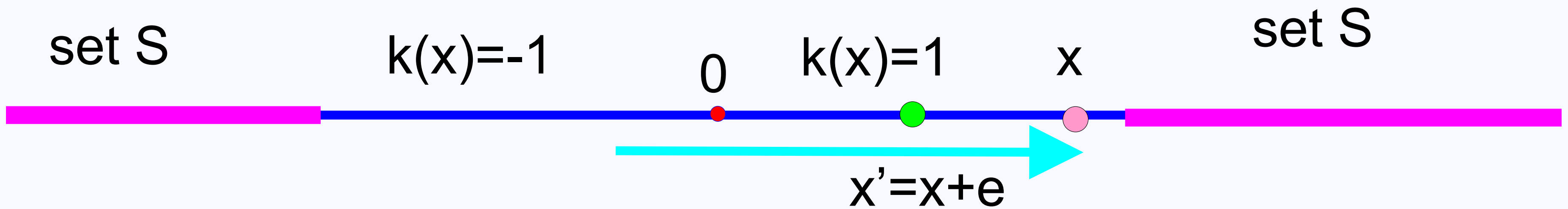

# Robustness of Discontinuous Feedback II

**PRESCRIPTION in GENERAL CASE: Keep sampling interval $t_{i+1} - t_i$ bounded from below, then $k$ is also robust with respect to small measurement errors**

In the case of stabilization of control system **Clarke,Ledyaev, Rifford and Stern, 2000**

*Lyapunov functions and feedback stabilization*
SIAM J.Control Optimiz.

In the case of stabilization of control system under persistent disturbances **Ledyaev and Vinter 2005, 2010**

# Robustness of Discontinuous Feedback II

**DEFINITION:** Feedback $k : \mathbb{R}^n \to \mathbb{U}$ is *robust stabilizing* if $\forall\ 0 < r < R\ \exists\ M = M(R) > 0$, $\delta = \delta(r,R) > 0$, $T = T(r,R) > 0$ and $b_j = b_j(r,R), j = 1,2,3,$ s.t. $\forall$ partition $\pi$ with

$$\frac{1}{2}\delta < t_{i+1} - t_i < \delta$$

$\forall$ initial state $x_0$: $\|x_0\| \leq R$, for any disturb. $d \in \mathcal{D}$, any external disturb. $w(t)$, actuator errors $a(t)$ and measurement errors $e(t)$ satisfying

$$\|w(t)\| < b_1, \quad \|a(t)\| < b_2, \quad \|e(t)\| < b_3 \quad \forall\ t \geq 0$$

the $\pi$-trajectory $x(\cdot)$ starting from $x_0$ is well-defined and it holds:

- (**UNIFORM ATTRACTIVENESS**) $\|x(t)\| \leq r \quad \forall t \geq T$;
- (**OVERSHOOT BOUNDEDNESS**) $\|x(t)\| \leq M(R) \quad \forall t \geq 0$;
- (**LYAPUNOV STABILITY**) $\lim_{R\to 0} M(R) = 0$.

# Robustness of Discontinuous Feedback II

**Ledyaev and Vinter 2005, 2010**

**THEOREM:**
Under Assumptions A1 and A2 we have
the stabilizing feedback $k(x)$ is robust stabilizing

# Robustness of Discontinuous Feedback II

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**APPLICATION:** Quantization of values $x$: find a net $\{y_j\}$ such that $\|y_i - y_j\| < \sup\|e(t)\|/2 < b_3/2(r, R)$ then we can use only values of control

$$k(y_j) \quad \text{if} \quad \|x' - y_j\| < b_3/2$$

**ANOTHER APPLICATION:** existence of piece-wise constant robust stabilizing feedback

# Robustness of Discontinuous Feedback II

General Principle for Robust Feedback

**Ledyaev 1999** in **Ledyaev&Rifford 1999**

**THEOREM:** *Integral Decrease Principle:*

$V(x)$ contin. or loc.Lipschitz $\exists\, k : \mathbb{R}^n \to \mathbb{U}$ and $\delta(x) > 0$ such that

$$V(x + \tau f) - V(x) \leq -\tau W(x) \quad \forall\, f \in \mathrm{co}\, f(x, k(x), D), 0 \leq \tau \leq \delta(x)$$

Then $k(x)$ is robust stabilizing

$v_\alpha(x)$ can be chosen as $V(x)$ in our case

Analogous principle for differential games **Ledyaev 2002**

# Robustness of Stabilizing Feedback for Any Sampling Rate

Let (dis)-continuous $k(x)$ be *robustly sampling-stabilizing* (permitting arbitrary large sampling rate) if $\forall\ 0 < r < R\ \exists\ \ T = T(r, R)$, $\delta = \delta(r, R)$, $\eta = \eta(r, R)$, and $M(R)$ s.t. for any disturb. $d \in \mathcal{D}$ measurement errors $e(t)$ and external disturbances $w(t)$ for which

$$\|e(t)\| \le \eta \quad \forall t \ge 0, \qquad \|w(\cdot)\|_\infty \le \eta$$

and any partition $\pi$ with $d(\pi) \le \delta$:

$$0 < t_{i+1} - t_i < \delta,$$

every $\pi$-trajectory with $\|x(0)\| \le R$ does not blow-up and satisfies the following relations:

- **(UNIFORM ATTRACTIVITY)** $\|x(t)\| \le r \quad \forall t \ge T$;
- **(BOUNDED OVERSHOOT)** $\|x(t)\| \le M(R)\ \ \forall t \ge 0$;
- **(LYAPUNOV STABILITY)** $\lim_{R \downarrow 0} M(R) = 0$.

# Robustness of Stabilizing Feedback for Any Sampling Rate

**Ledyaev&Sontag, 1998**

**THEOREM:**
$\exists$ robust sampl.-stabiliz. feedback $k(x)$ $\overset{\text{IFF}}{\Longleftrightarrow}$ $\exists\ C^\infty$ CLF $V(x)$

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**THEOREM:**
$\exists$ robust sampl.-stabiliz. feedback $k(x)$ $\Longleftrightarrow$ IFF $\exists$ $C^\infty$ CLF $V(x)$

Let $V(x)$ be $C^\infty$ control Lyapunov function
Then ANY $k(x)$ s.t.

$$\max_{d \in \mathbb{D}} \langle \nabla V(x), f(x, k(x), d) \rangle \leq -W(x)$$

is ROBUST STABILIZING for any high enough sampling rate

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**Artstein 1983** for affine-control systems:

$\exists$ SMOOTH control Lyapunov function IFF $\exists$ continuous stabilizing feedback

# Robustness of Stabilizing Feedback for Any Sampling Rate

**PROOF** is based on the inverse Lyapunov function theorem for differential inclusion

$$\dot{x} \in F(x)$$

$F(x)$ upper semicontinuous multifunction

**Clarke,Ledyaev&Stern 1999**

| |
|---|
| **THEOREM:**<br>Diff. inclusion $\dot{x} \in F$ is *strongly* AS ⟺ IFF ⟺ $\exists\, C^{\infty}\ V(x)$ |

Proof is based on structural robustness of AS of diff.inclusions

$$\dot{x} \in \mathrm{co}\, F(x + \Delta(x)B) + \Delta(x)B$$

# Robustness of Stabilizing Feedback for Any Sampling Rate

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**THEOREM:**
Diff. inclusion $\dot{x} \in F$ is *strongly* AS ⟺ IFF $\exists\, C^\infty\ V(x)$

**APPLICATION** Criteria for AS of Filippov or Krasovskii solutions in terms of $C^\infty$ Lyapunov function $V$

$$\dot{x} \in \cap_{\varepsilon>0} \mathrm{co}\, f(x, k(x + \varepsilon B), \mathbb{D})$$

Limits of trajectories of perturbed system are solutions of this differential inclusion

# Underwater Vehicle Example:

Lyapunov function $V(x) = x_1^2 + x_2^2 + x_3^2$

$$\begin{aligned} \dot{x}_1 &= u_2 u_3 \\ \dot{x}_2 &= u_1 u_3 \\ \dot{x}_3 &= u_1 u_2 \end{aligned} \qquad \mathbb{U} := \{(u_1, u_2, u_3) : |u_i| \leq 1, i = 1, 2, 3\}$$

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discontinuous ROBUST stabilizer

$$u_{j(x)} := -\mathrm{s}ign(x_{i(x)}), \quad u_{l(x)} := 1$$

$$u_{i(x)} := -\mathrm{s}ign\left(x_{j(x)} u_{l(x)} + x_{l(x)} u_{j(x)}\right)$$

$$i(x) := \max\{i : |x_i| = \max |x_l|\}, \; j(x) := i(x) + 1$$

$$l(x) := i(x) + 2$$

# Robust Stabilization of Nonholonomic Integrator

**Brockett's example (nonholonomic integrator) 1982**

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_1u_2 - x_2u_1\end{aligned} \qquad \mathbb{U} := \{(u_1, u_2) : |u_i| \le 1, i = 1, 2\}$$

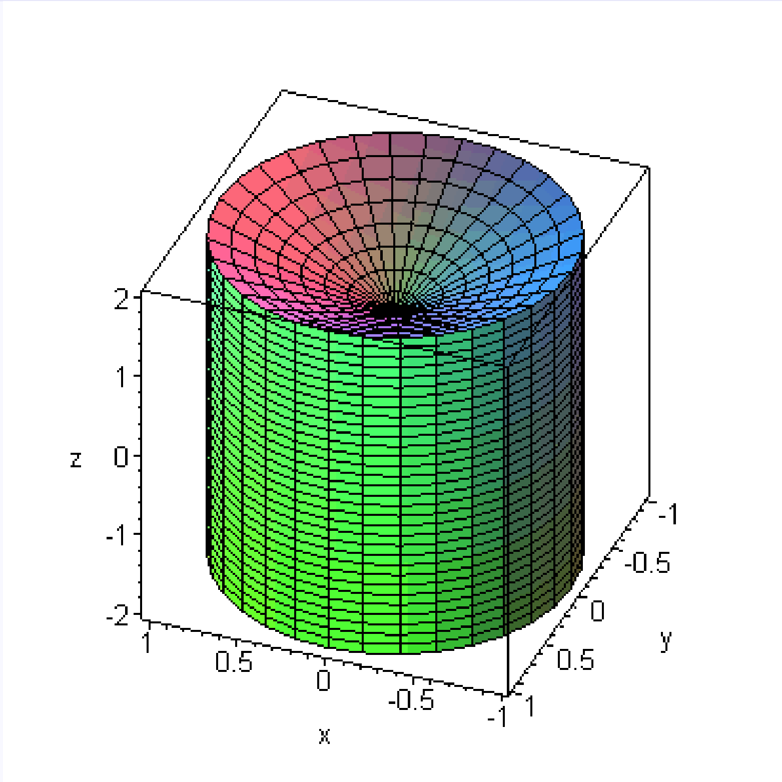
**Ledyaev&Rifford 1999**

design of robust discontinuous stabilizing feedback based on nonsmooth control Lyapunov functions

$$V(x) = \max\{\sqrt{x_1^2 + x_2^2}, |x_3| - \sqrt{x_1^2 + x_2^2}\}$$

Known results: **Bloch&Drakunov 1994, Astolfi 1995** - no robustness results

# Robust Stabilization of Nonholonomic Integrator




## Stabilization of nonholonomic integrator: pictures

Cylindrical coordinates: $r = \sqrt{x_1^2 + x_2^2},\ z = x_3$

$$\dot{r} = v_1, \quad \dot{z} = rv_2$$

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$$u(t, y_t, d_t)$$

such that for the system

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**CONJECTURE:** $\exists$ dynamic stabilizing feedback $k(z, y)$, $g(z, y)$ such that

$$\dot{x}(t) = f(x(t), k(z(t), y(t)), d(t)), \ \dot{z}(t) = g(z(t), y(t)), \ y(t) = h(x(t))$$

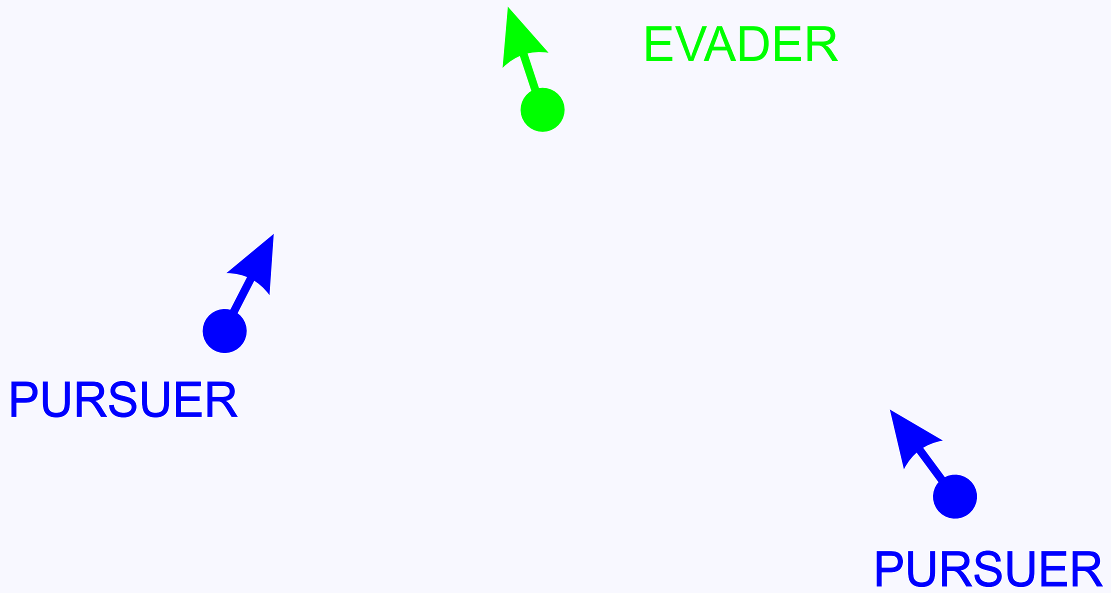
is robustly stabilizing: $x(t) \to S$ as $t \to +\infty$

# OPTIMIZATION

# Discontinuous Feedback and Team Optimal Control

We discuss mathematical techniques for deriving optimal solution of some coordinated control problem

Differential Game of Team Pursuit




Examples of Team Pursuit

# Discontinuous Feedback and Team Optimal Control

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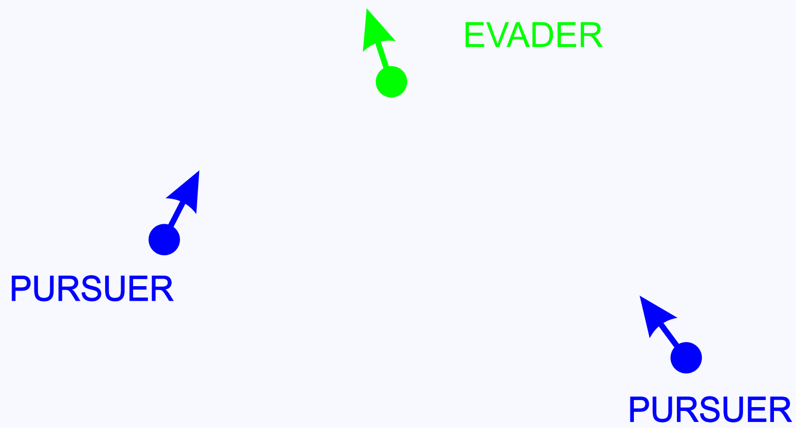


Consider objects $x_0, x_1, \ldots, x_m$ in $\mathbb{R}^n$ with *"simple"* dynamics

$$\dot{x}_0 = u_0, \quad \dot{x}_1 = u_1, \quad \ldots, \quad \dot{x}_m = u_m$$

Controls $u_0(t), u_1(t), \ldots, u_m(t)$ are subject to constraints

$$\|u_0\| \leq \sigma_0, \quad \|u_1\| \leq \sigma_1, \quad \ldots \quad , \|u_m\| \leq \sigma_m$$

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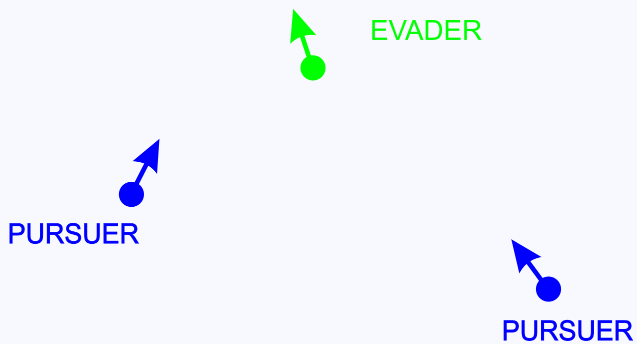


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The object $x_0$ is an **EVADER** (it tries to avoid a capture by one of the objects $x_1, \ldots, x_m$ ). Objects $x_1, \ldots, x_m$ are **PURSUERS**(they try to capture the object $x_0$ ), The pursuit is over at some moment $T$ if

$$\|x_0(T) - x_i(T)\| \leq l_i$$
for some $i \in I := \{1, 2, \ldots, m\}$

# Discontinuous Feedback and Team Optimal Control

**IMPORTANT POINT: PURSUERS and EVADER** can use only *closed-loop* control (or feedback control)

$$u_i(t) = k_i(x(t)), \quad i \in I$$

where $x := [x_0, x_1, \ldots, x_m]$.

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**Optimal pursuit time** $w(x)$ for initial point $x$ is a value function of the differential game of pursuit.

If $w(x)$ is smooth (differentiable) then it satisfies the *eikonal* equation

$$H(x, \nabla w(x)) = -1, \qquad w(x)|_M = 0$$

where Hamiltonian $H$ is defined as follows

$$H(x, \nabla w(x)) = \min_{p \in P} \max_{q \in Q} \langle \nabla w(x), f(x, p, q) \rangle$$

for the differential game of pursuit with the terminal set $M$ and dynamics

$$\dot{x} = f(x, p, q), \qquad p \in P, \quad q \in Q$$

# Discontinuous Feedback and Team Optimal Control

In general, $w(x)$ is *nonsmooth* (lower semicontinuous) function, optimal feedback controls $k_p(x), k_q(x)$ are discontinuous
For lower semicontinuous value function $w(x)$ relation

$$H(x, \nabla w(x)) = -1, \quad w(x)|_M = 0$$

is replaced by two inequalities in terms of subgradients of $w(x)$
One of them

$$H(x, \zeta) \leq -1, \quad \forall\, \zeta \in \partial_P w(x),\ x \notin M$$

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The synthesis of universal feedback pursuit strategies in differential games

**THEOREM: Clarke,Ledyaev,Subbotin 1997 :**
Let $D \subset \bar{G}$ be a compact set such that $w$ is bounded on $D$, then for any $\varepsilon > 0$ there exists $\delta > 0$ and a feedback control $k$ such that for any $x_0 \in D$ and $\Delta$, diam $(\Delta) < \delta$ we have

$$\theta^{\varepsilon}(x_0, k_p, \Delta) < w(x_0) + \varepsilon$$

where $\theta^{\varepsilon}(x_0, k_p, \Delta)$ is a pursuit guaranteed time for feedback $k_p$and sampling partition $\Delta$ to drive $x$ into set $M^{\varepsilon}$ ($\varepsilon$-neighbourhood of $M$)

# Team Optimal Pursuit

Dynamics of **EVADER** $x_0$ and **PURSUERS** $x_1, \ldots, x_m$ in $\mathbb{R}^n$

$$\dot{x}_0 = u_0, \quad \dot{x}_1 = u_1, \quad \ldots, \quad \dot{x}_m = u_m$$

Controls $u_0(t), u_1(t), \ldots, u_m(t)$ are subject to constraints

$$\|u_0\| \leq \sigma_0, \quad \|u_1\| \leq \sigma_1, \quad \ldots \quad , \|u_m\| \leq \sigma_m$$

Terminal set

$$M := \{x = [x_0, x_1, \ldots, x_m] \; : \; \min_{1 \leq i \leq m} (\|x_0 - x_i\| - l_i) \leq 0\}$$

**ASSUMPTION:** $\boldsymbol{m \leq n, \; \sigma_i \geq \sigma_0}$ and $\boldsymbol{\sigma_i + l_i > \sigma_0, \; i = 1, \ldots, m}$

# Team Optimal Pursuit

**ASSUMPTION:** $\boldsymbol{m \leq n}$ and $\boldsymbol{\sigma_i \geq \sigma_0, \;\; \sigma_i + l_i > \sigma_0, \; i = 1, \ldots, m}$
Consider sets for $i \in I := \{1, \ldots, m\}$

$$Y_i(x) := \{y \in R^n \; : \; \Phi_i(y, x_i) \leq 0\}, \quad i \in I$$

where

$$\Phi_i(y, x_i) := \frac{\|y - x_0\|}{\sigma_0} - \frac{\|y - x_i\| - l_i}{\sigma_i}$$

Nonsmooth function (value (*marginal*) function for mathematical programming problem)

$$w(x) := \sup \{\frac{\|y - x_0\|}{\sigma_0} \; : \; y \in Y(x)\}$$

$$Y(x) := \cap_{i \in I} Y_i(x)$$

# Team Optimal Pursuit

Nonsmooth function (value (*marginal*) function for mathematical programming problem)

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$$Y(x) := \left\{ y \ : \ \frac{\|y - x_0\|}{\sigma_0} - \frac{\|y - x_i\| - l_i}{\sigma_i} \leq 0 \ \forall i \in I \right\}$$

If $w(x) < +\infty$ then define

$$Y_{opt}(x) := \left\{ y \in Y(x) \ : \ \frac{\|y - x_0\|}{\sigma_0} = w(x) \right\}$$

# Team Optimal Pursuit

$$w(x) := \sup \{ \frac{\|y - x_0\|}{\sigma_0} \ : \ y \in Y(x) \}$$

**PURSUERS**' feedback controls

$$k_i(x) := \sigma_i \frac{y - x_i}{\|y - x_i\|}, \text{ where } y \in Y_{opt}(x), \ i \in I$$

**EVADER**'s feedback control

$$k_0(x) := \sigma_0 \frac{y - x_0}{\|y - x_0\|}, \text{ where } y \in Y_{opt}(x),$$

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**THEOREM: Ivanov & Ledyaev 1980**

Under Assumptions A the nonsmooth function $w(x)$ is the value function of the team pursuit problem

# Team Optimal Pursuit

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**THEOREM: Ledyaev 2007**

Under Assumptions A the discontinuous feedbacks $k_1, \ldots, k_m$ are optimal universal robust pursuit feedback controls, $k_0$ is optimal universal robust evader's feedback for the team pursuit problem

# Team Optimal Pursuit

Meaning of the set $Y(x)$

$$Y(x) := \{y \ : \ \frac{\|y - x_0\|}{\sigma_0} - \frac{\|y - x_i\| - l_i}{\sigma_i} \leq 0, \ \forall \ i \in I\}$$

At any point $y \in Y(x)$ **EVADER** comes before interception by **EACH** **PURSUER** ⇒ **EVADER** can avoid interception on the time interval $[0, w(x))$

EXAMPLE: the set $Y_1(x) \cup Y_2(x) \cup Y_3(x)$

EXAMPLE: the set $Y(x)$

EXAMPLE: the set $Y_{opt}(x)$

# Team Optimal Pursuit

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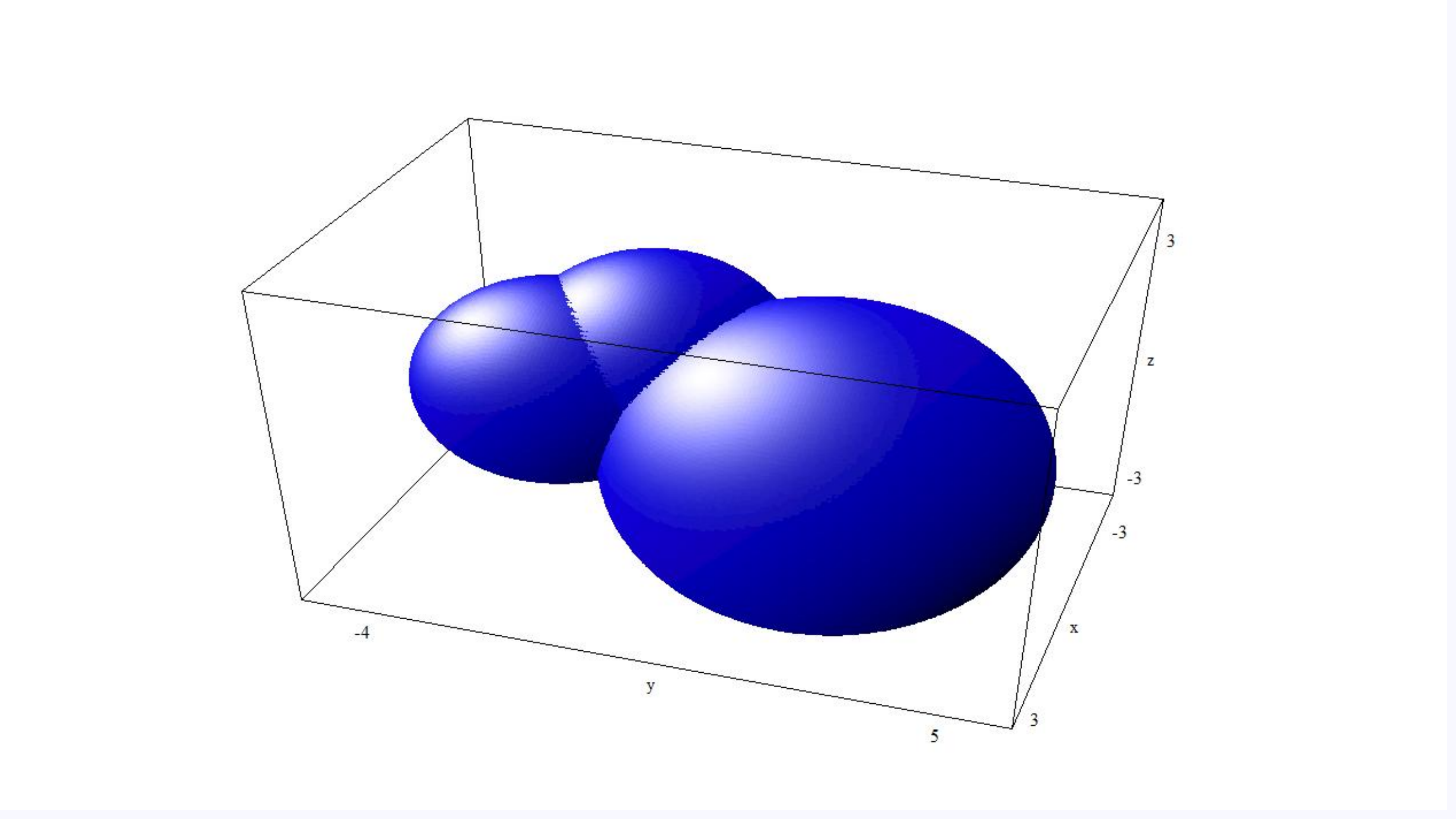

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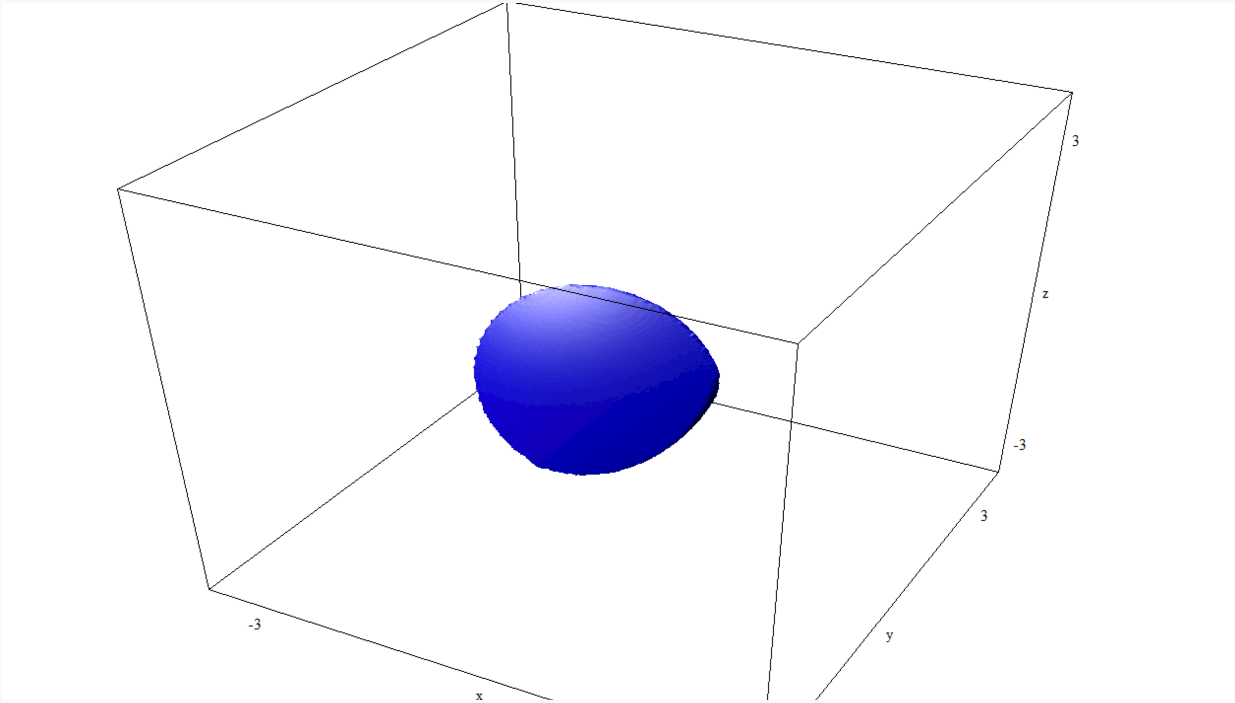

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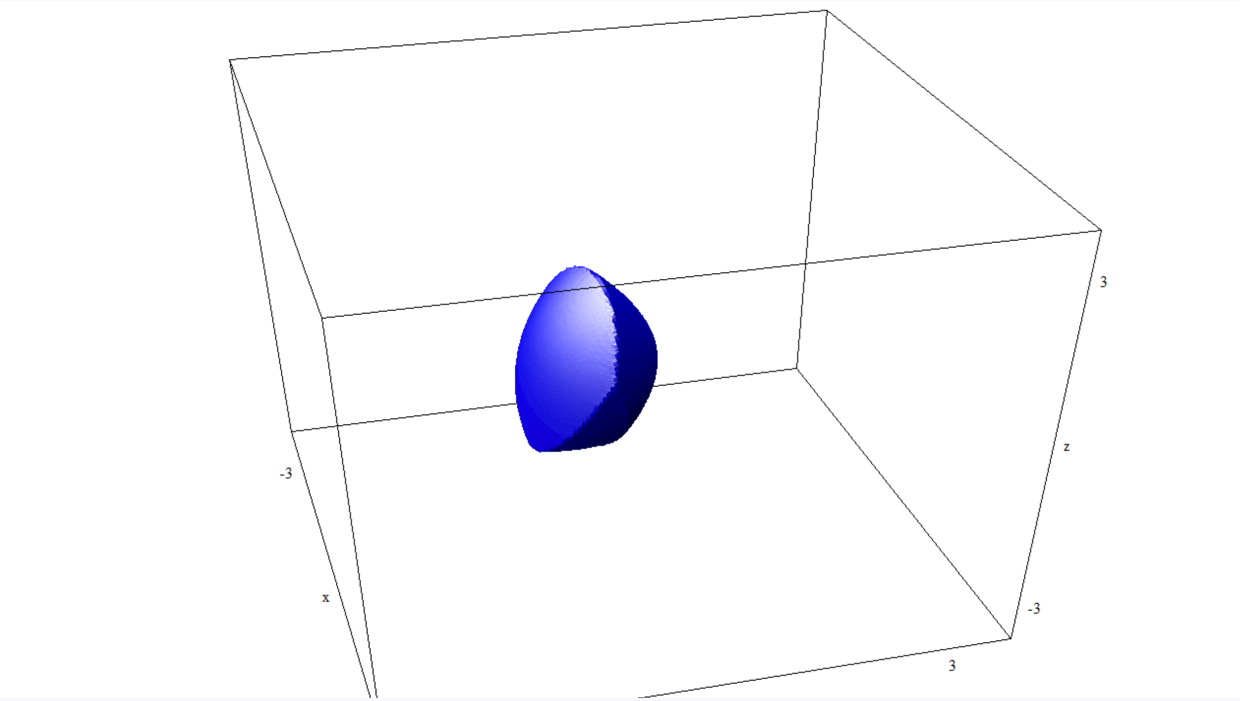

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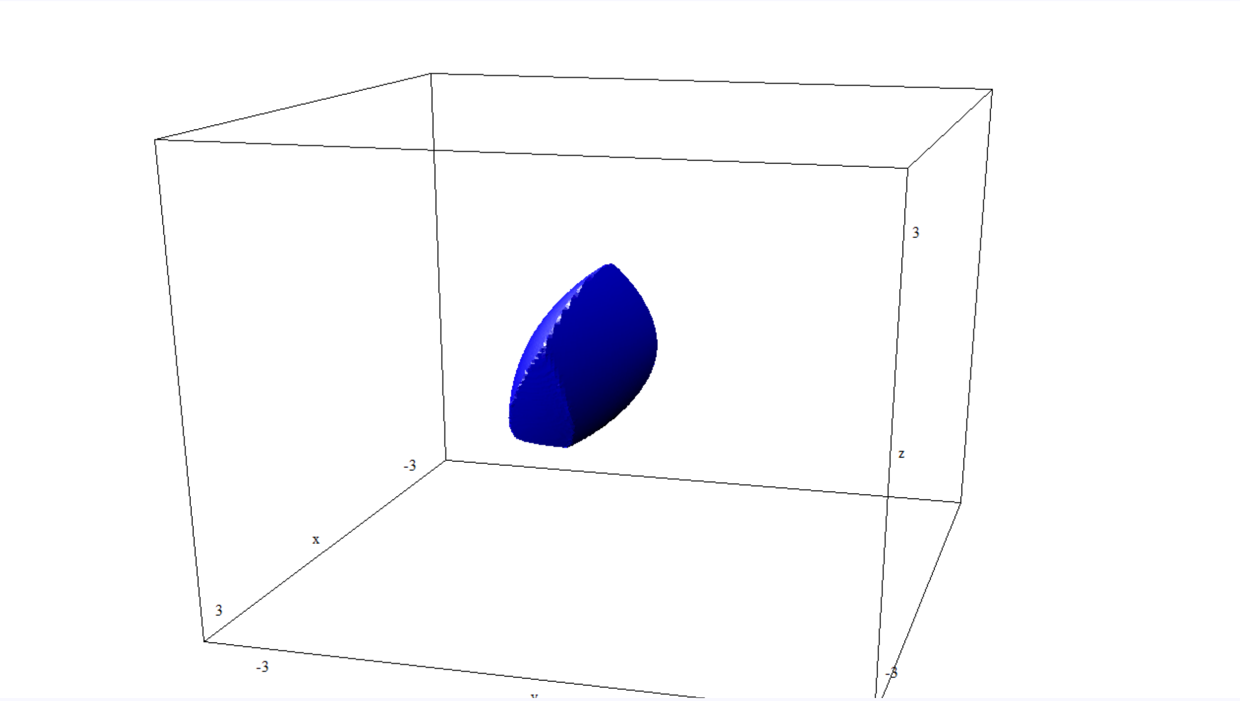
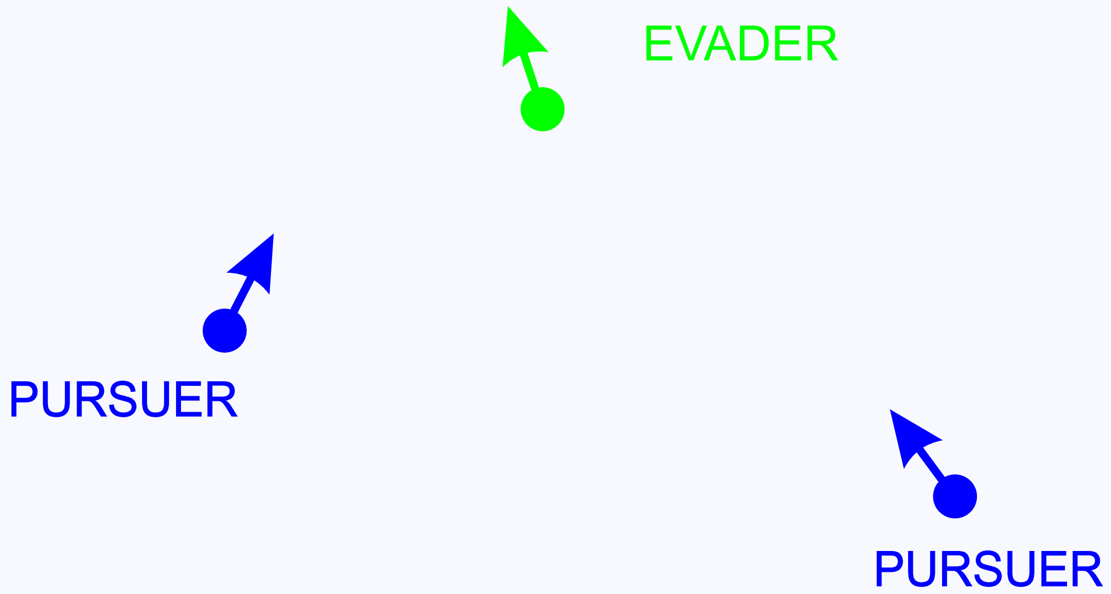


# Team Optimal Pursuit

Differential Game of Team Pursuit




Examples of Team Pursuit

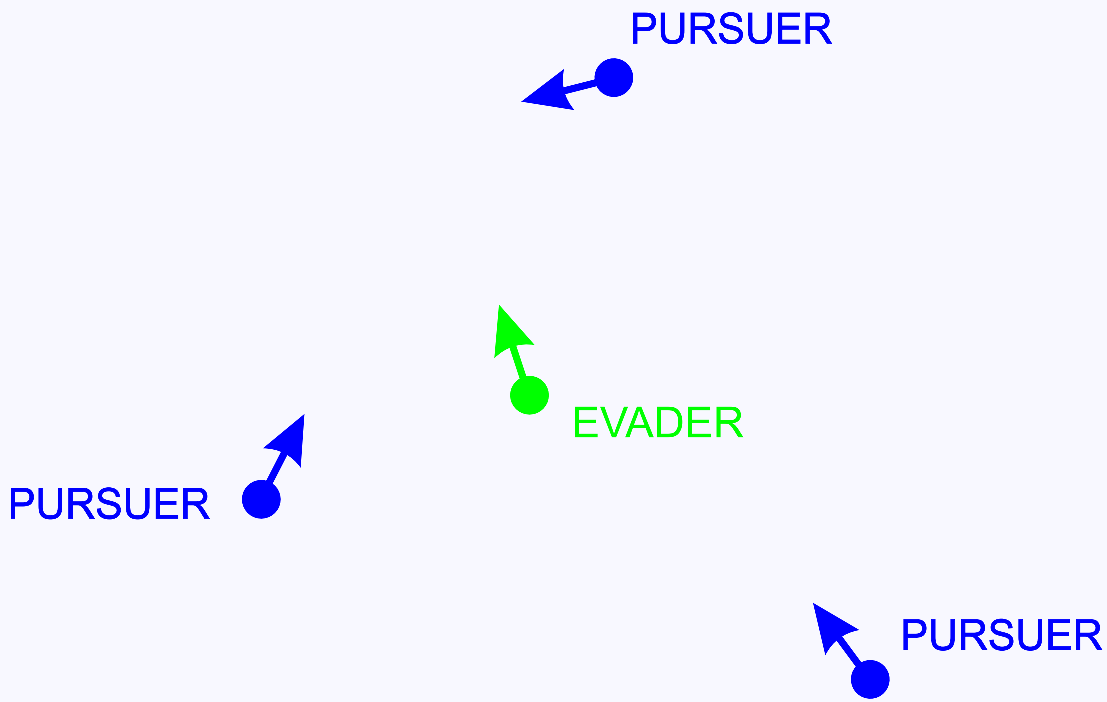
# Open Problems

Unsolved pursuit problems for games with simple motions

Progress in solving one of them should help to solve another

# Open Problems

Pursuit in the case $m > n$ and $x_0 \in \text{conv}\,\{x_1, \ldots, x_m\}$



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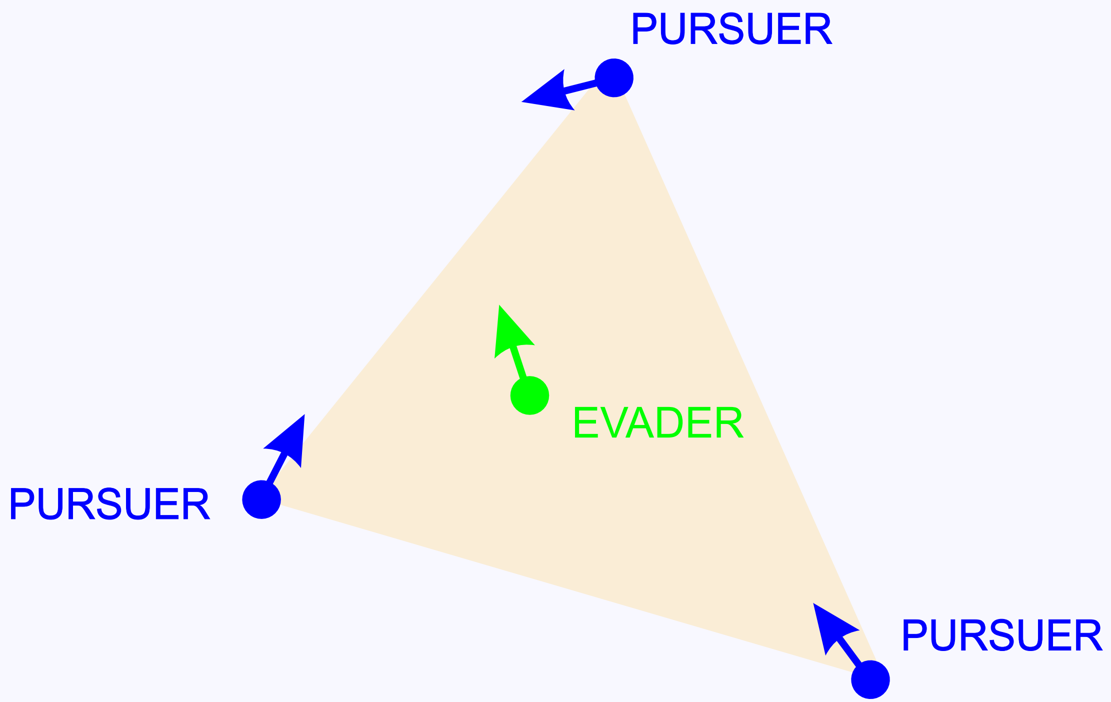

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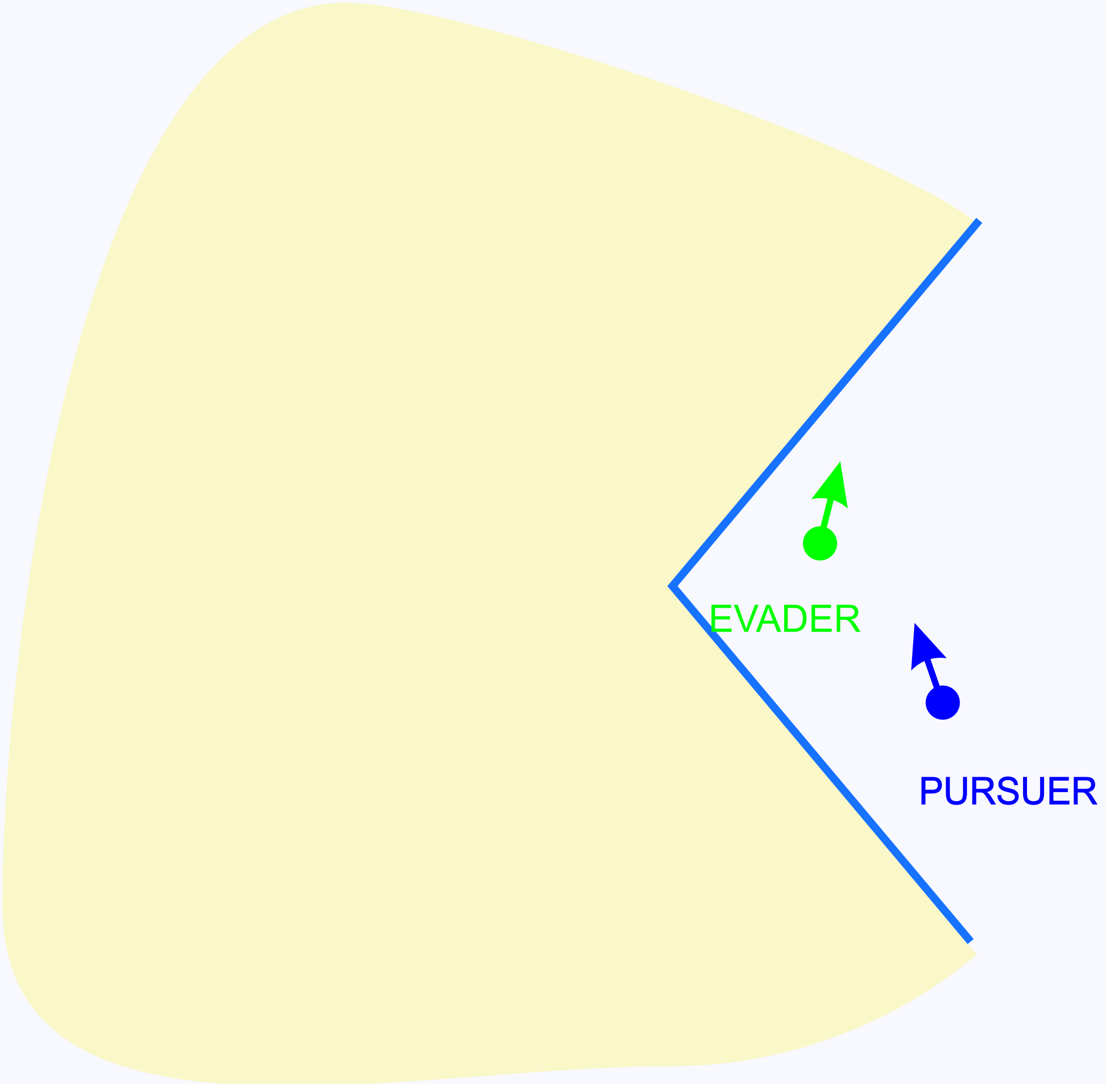
Pursuit inside a "corner"



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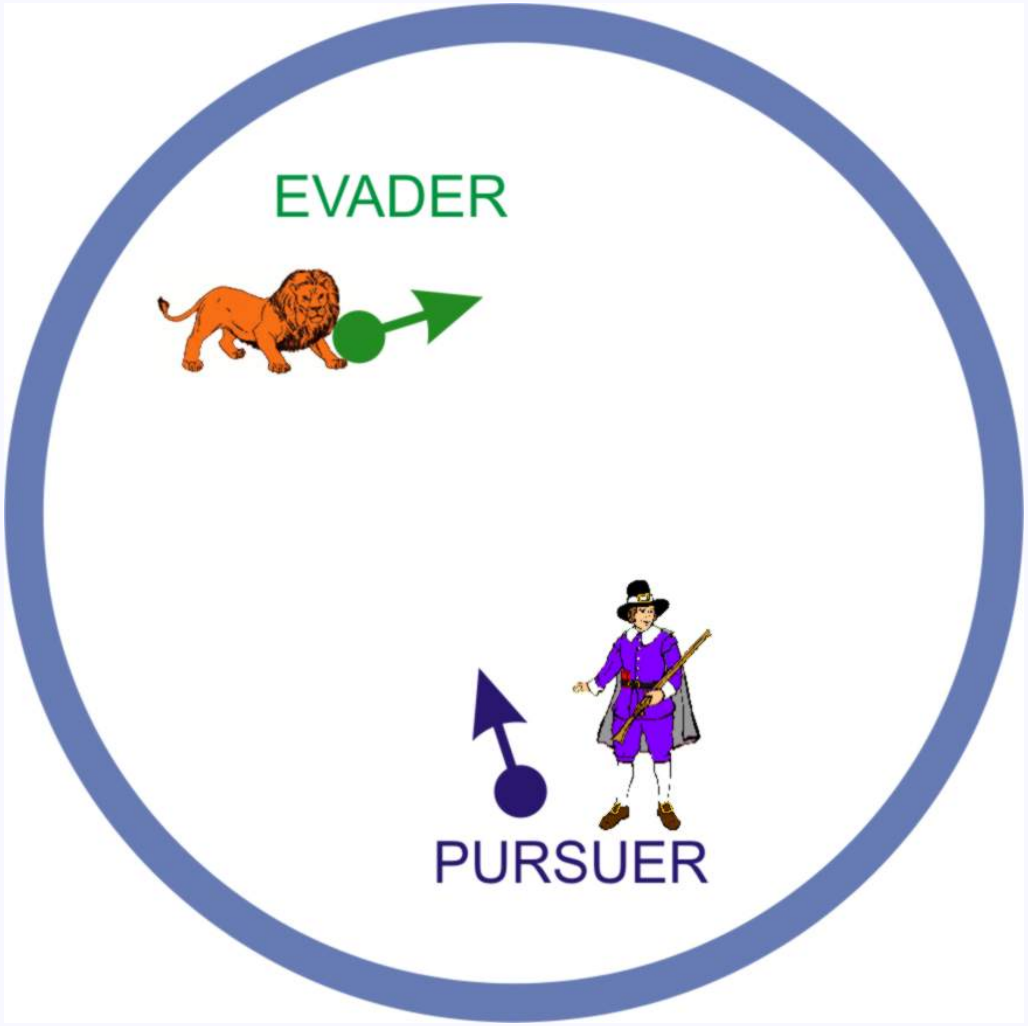
Pursuit inside a circular arena **(Rado 1925)** : *Lion and Man* have equal maximal velocities



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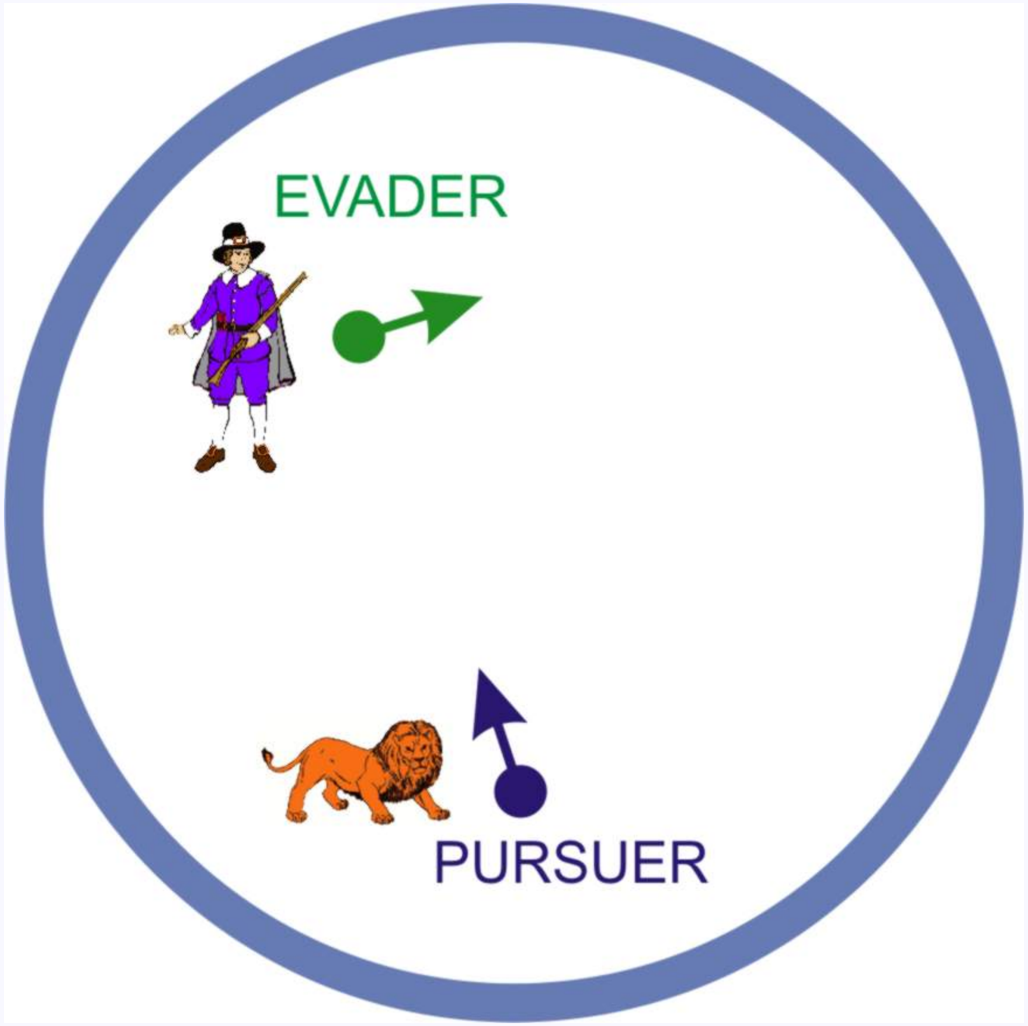

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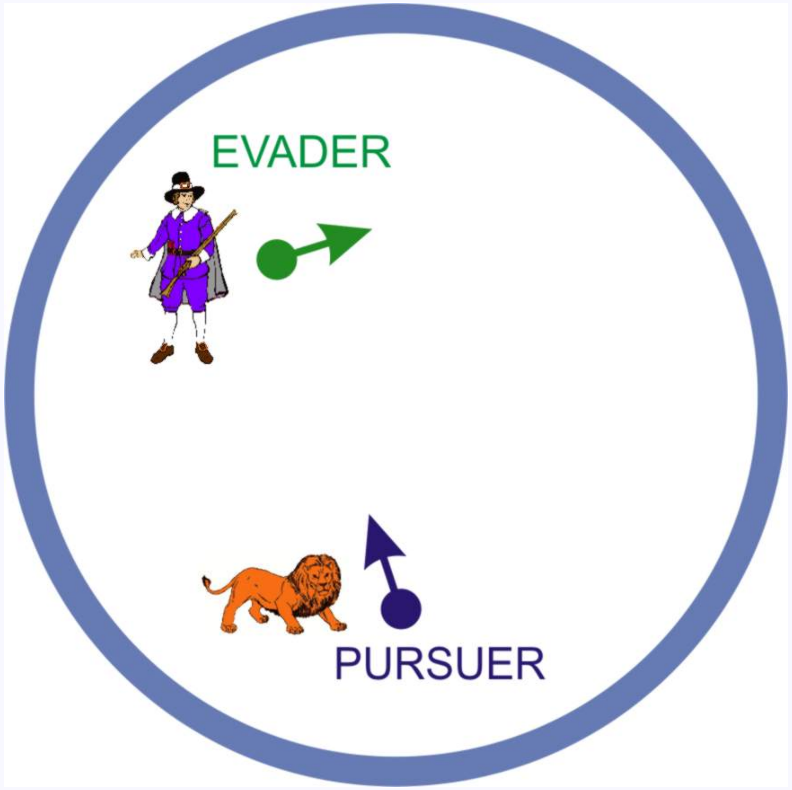


**Besicovitch** $\exists$ evader's strategy such that $\|x_L(t) - x_M(t)\| > 0, \forall\, t \geq 0$

**Ivanov & Ledyaev 1980** $\forall \ell > 0 \; \exists$ pursuers's strategy such that $\exists\; \theta = \theta(x_L(0), x_M(0))$ such that $\|x_L(\tau) - x_M(\tau)\| \leq \ell$ for some $\tau \leq \theta$

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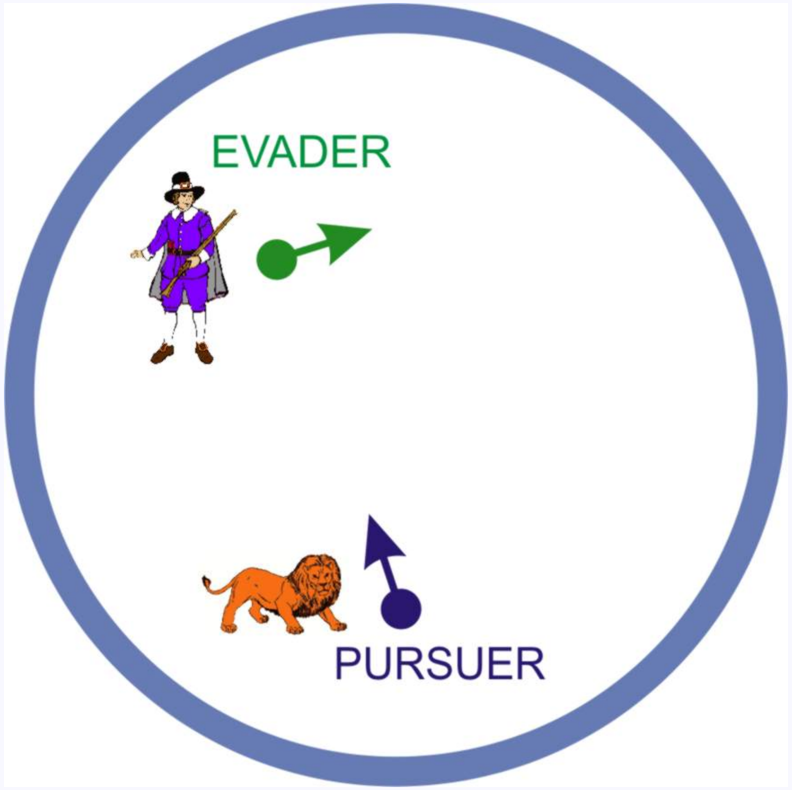


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**QUESTION:** Optimal pursuit time $\theta$ and optimal strategies?

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- Concept of **DISCONTINUOUS FEEDBACK CONTROL** - precise mathematical model of digital computer-aided control.
- Applications to stabilization and optimal control problems. New approach to **OUTPUT REGULATION** problem.
- Robustness of stabilizing and optimal feedback by restricting a sampling rate.
- If there exists smooth CLF then stabilizing $k$ is robust for any highly enough sampling rate (analogous result for optimal feedback in differential game).
- Nonsmooth control Lyapunov and value functions and analytical techniques for working with them.

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