Nonlinear stabilization when delay is a function of state

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The Sontag Army **(the ISS/CLF Corps)**

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...

Outline

- LTI systems with time-varying delay
- nonlinear systems with state-dependent delay
- happy birthday slide

LTI Systems w/ Constant Delay

$$\dot{X}(t) = AX(t) + BU(t - D)$$

A - possibly unstable; D - arbitrarily large

Assume: (A, B) controllable and matrix K found such that $A + BK$ is Hurwitz.

LTI Systems w/ Constant Delay

$$\dot{X}(t) = AX(t) + BU(t-D)$$

Predictor-based control law:

$$U(t) = K \underbrace{\left[e^{AD} X(t) + \int_{t-D}^{t} e^{A(t-\theta)} B U(\theta) d\theta \right]}_{X(t+D) \triangleq P(t)}$$

Time-Varying Input Delay

Basic idea introduced by Artstein (TAC, 1982), but only conceptually (nor explicitly), for LT**V** systems with **TV** delays.

Explicit design for LTI plants presented by Nihtila (CDC, 1991), but no analysis of stability or of feasibility of the controller.

Time-Varying Input Delay

$$\dot{X}(t) = AX(t) + BU(\phi(t))$$

$$\phi(t) = t - D(t) := \text{“delayed time”}$$

Predictor feedback

$$U(t) = K\left[e^{A\left(\phi^{-1}(t)-t\right)}X(t) + \int_{\phi(t)}^{t} e^{A\left(\phi^{-1}(t)-\phi^{-1}(\theta)\right)}B\frac{U(\theta)}{\phi'\left(\phi^{-1}(\theta)\right)}d\theta\right]$$

Need a Lyapunov functional.

Construct one with a backstepping transformation of the **actuator state**:

$$W(\theta) = U(\theta) - K\left[\overbrace{e^{A\left(\phi^{-1}(\theta)-t\right)}X(t) + \int_{\phi(t)}^{\theta} e^{A\left(\phi^{-1}(\theta)-\phi^{-1}(\sigma)\right)}B\frac{U(\sigma)}{\phi'\left(\phi^{-1}(\sigma)\right)}d\sigma}^{X(\phi^{-1}(\theta)) \triangleq P(\theta)}\right]$$

$$\phi(t) \leq \theta \leq t$$

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$$W(\theta) = U(\theta) - K\Bigg[\overbrace{e^{A\left(\phi^{-1}(\theta)-t\right)}X(t) + \int_{\phi(t)}^{\theta} e^{A\left(\phi^{-1}(\theta)-\phi^{-1}(\sigma)\right)}B\frac{U(\sigma)}{\phi'\left(\phi^{-1}(\sigma)\right)}d\sigma}^{X(\phi^{-1}(\theta)\triangleq P(\theta)}\Bigg]$$

$$\phi(t) \le \theta \le t$$

$$V(t) = X(t)^T P X(t) + a\int_{\phi(t)}^{t} \frac{e^{b\frac{\phi^{-1}(\theta)-t}{\phi^{-1}(t)-t}}}{\left(\phi^{-1}(t)-t\right)\phi'\left(\phi^{-1}(\theta)\right)} W(\theta)^2 d\theta$$

Theorem 1 *$\exists G, g > 0$ s.t.*

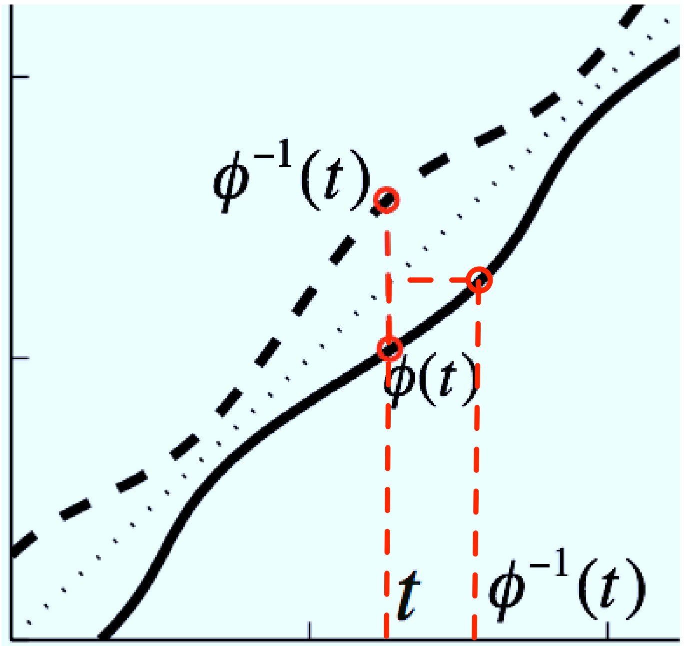
$$|X(t)|^2 + \int_{t-D(t)}^{t} U^2(\theta)d\theta \leq Ge^{-gt}\left(|X_0|^2 + \int_{-D(0)}^{0} U^2(\theta)d\theta\right), \quad \forall t \geq 0,$$

where G (but not g) depends on the function $D(\cdot)$.

Conditions on the delay function $D(t) = t - \phi(t)$:

- $D(t) \geq 0$ (causality);
- $D(t)$ is uniformly bounded from above (all inputs applied to the plant eventually reach the plant);
- $D'(t) < 1$ (plant never feels input values that are older than the ones it has already felt— input signal direction never reversed);
- $D'(t)$ is uniformly bounded from below (delay cannot disappear instantaneously, but only gradually).

Achilles heel: $\phi^{-1}(t) > t > \phi(t)$



$D(t)$ needs to be known sufficiently far in advance

$\Rightarrow$ method appears not to be usable for state-dependent delays

Nonlinear systems with state-dependent delay

(with Nikolaos Bekiaris-Liberis)

- Control over networks
- Driver reaction delay
- Milling processes
- Rolling mills
- Engine cooling systems
- Population dynamics

Nonlinear Systems with State-Dependent Input Delay

$$\dot{X}(t) = f\left(X(t), U\left(t - D(X(t))\right)\right)$$

Challenge:

$$\begin{aligned} P(t) &= \text{value of the state at the time when the control applied at } t \text{ reaches the system} \\ &= X\left(t + D(P(t))\right) \end{aligned}$$

Nonlinear Systems with State-Dependent Input Delay

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Challenge:

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$$P(\theta) = X(t) + \int_{t-D(X(t))}^{\theta} \frac{f(P(s), U(s))}{1 - \nabla D(P(s)) f(P(s), U(s))} ds, \qquad t - D(X(t)) \leq \theta \leq t$$

Controller (possibly time-varying)

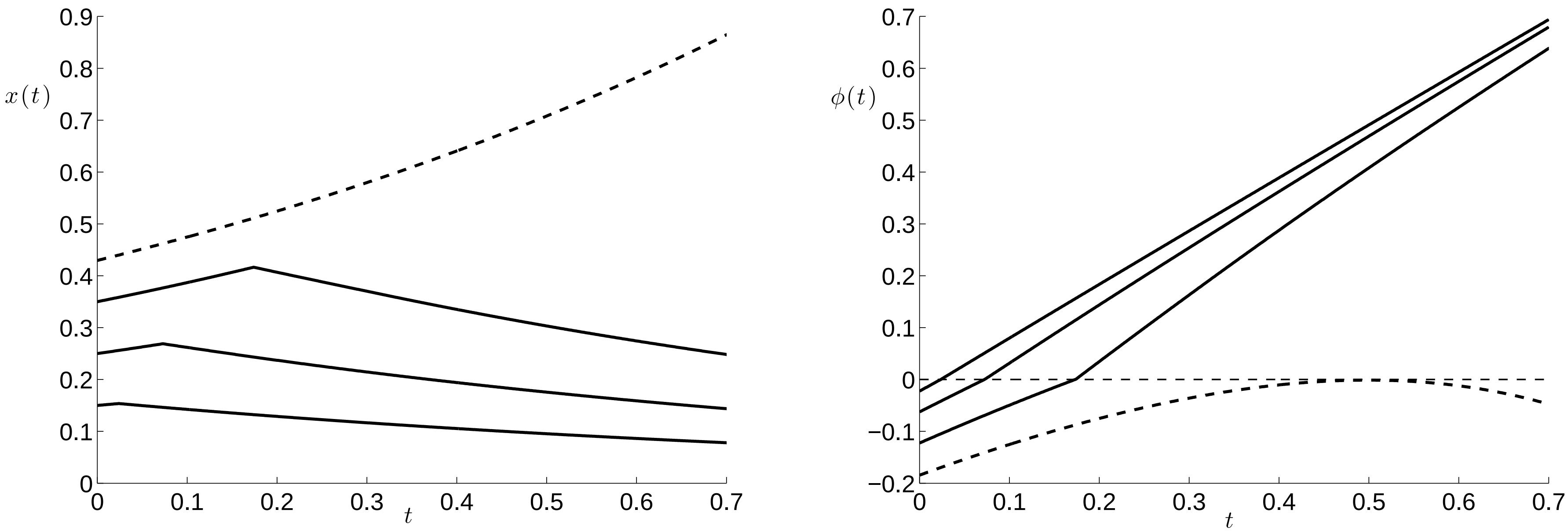
$$U(t) = \kappa\left(\left(t + D\left(P(t)\right)\right), P(t)\right)$$

Example 1 (stabilizing, but not global even for linear systems)

$$\dot{X}(t) = X(t) + U\left(t - X(t)^2\right)$$

Simulations with input initial conditions $U(\theta) = 0, -X(0)^2 \leq \theta \leq 0.$

For $X(0) \geq X^* = \frac{1}{\sqrt{2e}} = 0.43$, the controller never “kicks in” (dashed)



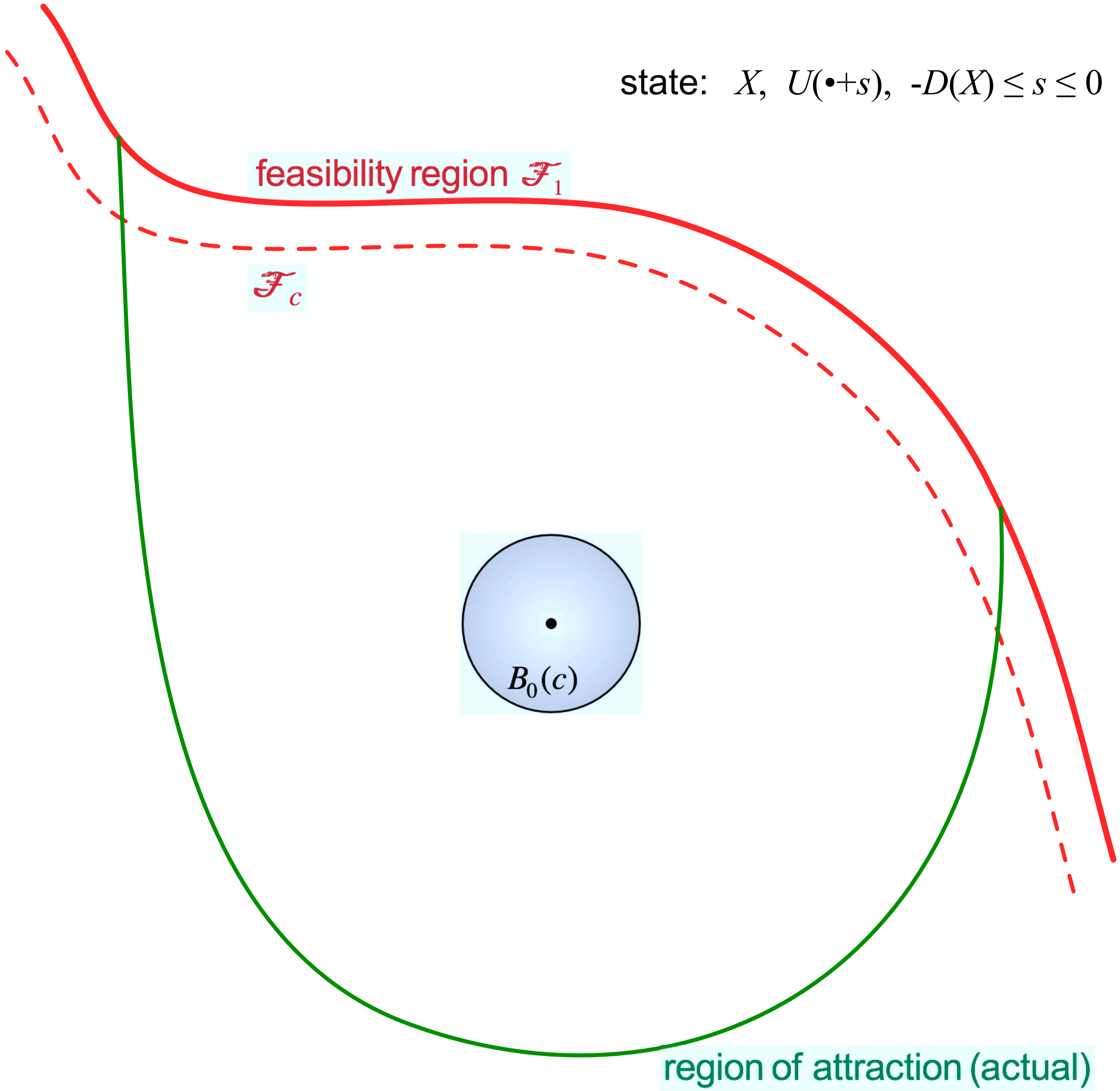
Result not global because of feasibility condition "delay rate < 1"

To keep the prediction horizon finite and control bounded, the initial conditions and solutions must satisfy

$$\mathcal{F}_c: \qquad \nabla D\left(P(\theta)\right) f\left(P(\theta), U(\theta)\right) < c, \quad \text{for all } \theta \geq -D\left(X(0)\right),$$

for some $c \in (0, 1]$.

We refer to $\mathcal{F}_1$ as the *feasibility condition* of the controller.



state: X, $U(\bullet+s)$, $-D(X) \leq s \leq 0$
feasibility region $\mathscr{F}_1$
$\mathscr{F}_c$
$B_0(c)$
region of attraction (actual)

Theorem 2 (local u.a.s. in sup-norm of U)

$\exists \psi_{RoA} \in \mathcal{K}$, $\rho \in \mathcal{KC}$, *and* $\beta \in \mathcal{KL}$ *s.t.* $\forall$ *initial cond. that satisfy*

$$B_0(c): \qquad |X(0)| + \sup_{-D(X(0)) \leq \theta \leq 0} |U(\theta)| < \psi_{RoA}(c)$$

for some $0 < c < 1$,

$$|X(t)| + \sup_{t-D(X(t)) \leq \theta \leq t} |U(\theta)| \leq \beta\left(\rho\left(|X(0)| + \sup_{-D(X(0)) \leq \theta \leq 0} |U(\theta)|, c\right), t\right), \quad \forall t \geq 0.$$

If U *is locally Lipschitz on the interval* $[-D(X(0)), 0)$, *there exists a unique solution to the closed-loop system with* X *Lipschitz on* $[0, \infty)$, U *Lipschitz on* $(0, \infty)$

Assumption 1 $D \in C^1(\mathbb{R}^n; \mathbb{R}_+)$

Assumption 2 $\dot{X} = f(X, \omega)$ *is forward complete*

Assumption 3 $\dot{X} = f(X, \kappa(t, X))$ *is g.u.a.s.*

Lemma 1 (infinite-dimensional backstepping transformation of the actuator state)

$$\boxed{W(\theta) = U(\theta) - \kappa(\sigma(\theta), P(\theta)),} \qquad t - D(X(t)) \leq \theta \leq t,$$

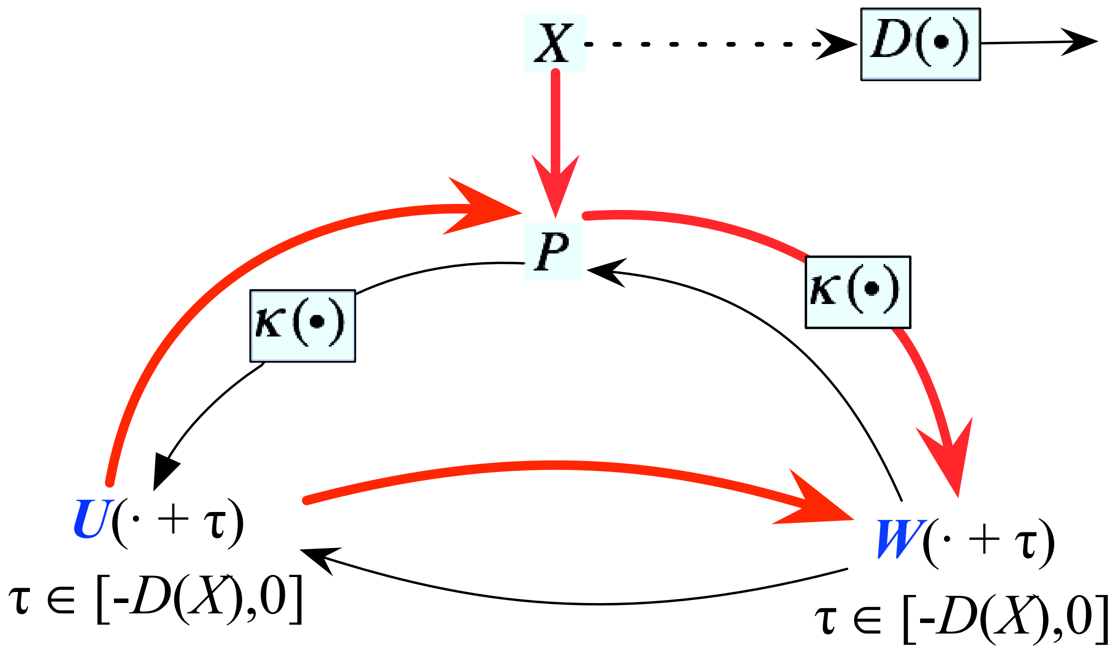
transforms the closed-loop system into the "target system"

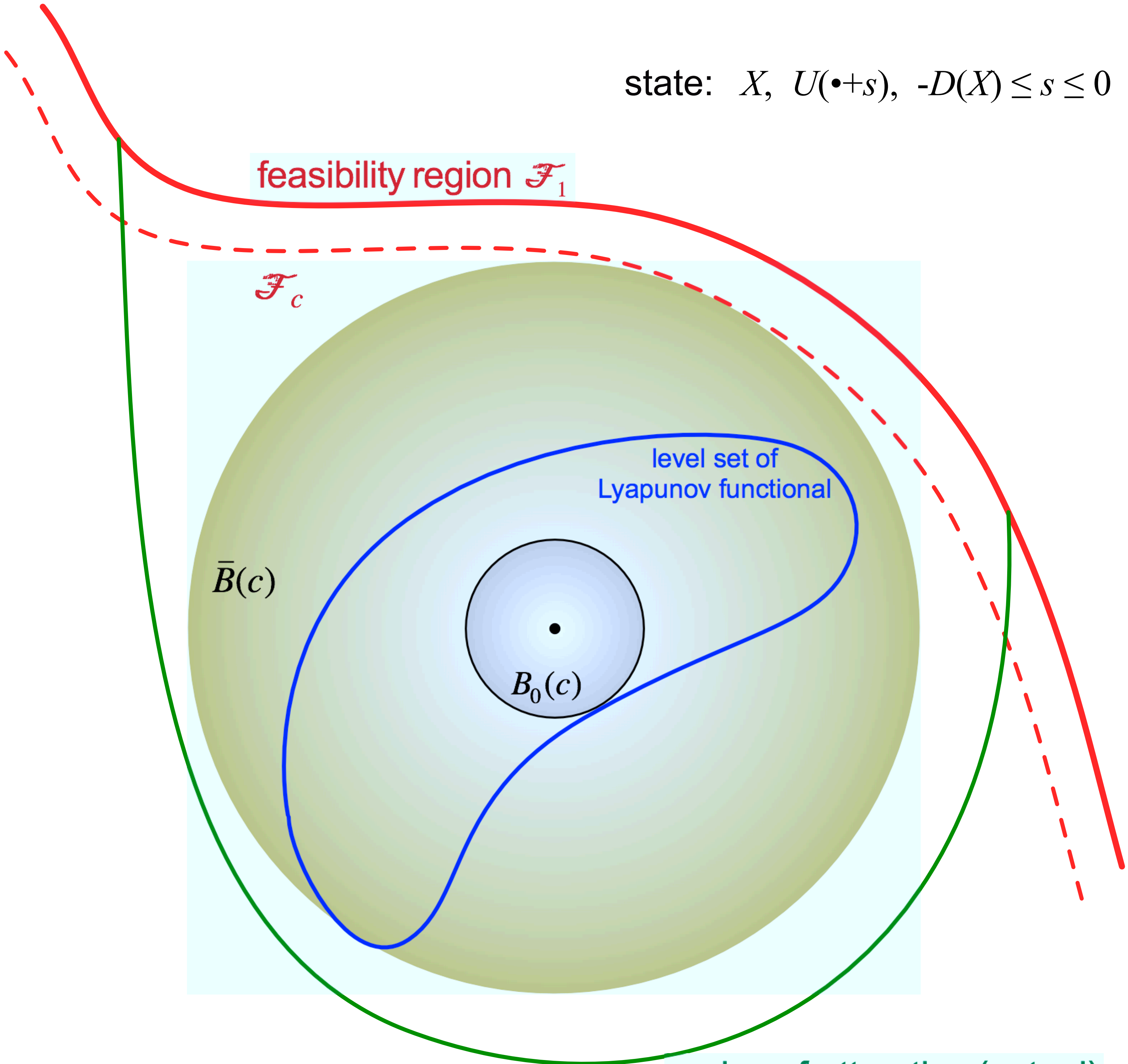
$$\begin{aligned}
\dot{X}(t) &= f(X(t), \kappa(t, X(t)) + W(t - D(X(t)))) \\
W(t) &= 0, \quad \forall t \geq 0.
\end{aligned}$$

Lemma 2 (u.a.s. of target system)

$\exists \rho_* \in \mathcal{KC}$, $\beta_2 \in \mathcal{KL}$ *s.t., for all solutions satisfying* $\mathcal{F}_c$ *for* $0 < c < 1$,

$$|X(t)| + \sup_{t - D(X(t)) \leq \theta \leq t} |W(\theta)| \leq \beta_2\left(\rho_*\left(|X(0)| + \sup_{-D(X(0)) \leq \theta \leq 0} |W(\theta)|, c\right), t\right),$$


X
D(•)
P
κ(•)
κ(•)
U(· + τ)
τ ∈ [-D(X),0]
W(· + τ)
τ ∈ [-D(X),0]



state: X, $U(\bullet+s)$, $-D(X) \le s \le 0$
feasibility region $\mathcal{F}_1$
$\mathcal{F}_c$
level set of
Lyapunov functional
$\bar{B}(c)$
$B_0(c)$
region of attraction (actual)

Lemma 3 (norm equivalence between the original system and target system)

$\exists \rho_2 \in \mathcal{KC}_\infty$, $\alpha_9 \in \mathcal{K}_\infty$ s.t., for all solutions satisfying $\mathcal{F}_c$ for $0 < c < 1$,

$$|X(t)| + \sup_{t-D(X(t))\le\theta\le t} |U(\theta)| \le \alpha_9^{-1}\left(|X(t)| + \sup_{t-D(X(t))\le\theta\le t} |W(\theta)|\right)$$

$$|X(t)| + \sup_{t-D(X(t))\le\theta\le t} |W(\theta)| \le \rho_2\left(|X(t)| + \sup_{t-D(X(t))\le\theta\le t} |U(\theta)|, c\right)$$

Lemma 4 (finding a ball $\bar{B}$ around the origin and within the feasibility region)

$\exists \bar{\rho}_c \in \mathcal{KC}_\infty$ s.t. $\mathcal{F}_c$ $(0 < c < 1)$ is satisfied by all solutions that satisfy

$$\boxed{\bar{B}(c): \quad |X(t)| + \sup_{t-D(X(t))\le\theta\le t} |U(\theta)| < \bar{\rho}_c(c, c)} \qquad \forall t \ge 0.$$

Lemma 5 (finding a ball B_0 of initial conditions s.t. all solutions are confined in $\bar{B} \subset \mathcal{F}_c$)

$\exists \psi_{\mathrm{RoA}} \in \mathcal{K}$ s.t. for all initial conditions in $B_0(c)$, the solutions remain in $\bar{B}(c) \subset \mathcal{F}_c$ for some $0 < c < 1$.

Examples

Example 2 Non-holonomic unicycle with $D(x,y) = x^2 + y^2$

A predictor-based version of Pomet's (1992) time-varying controller:

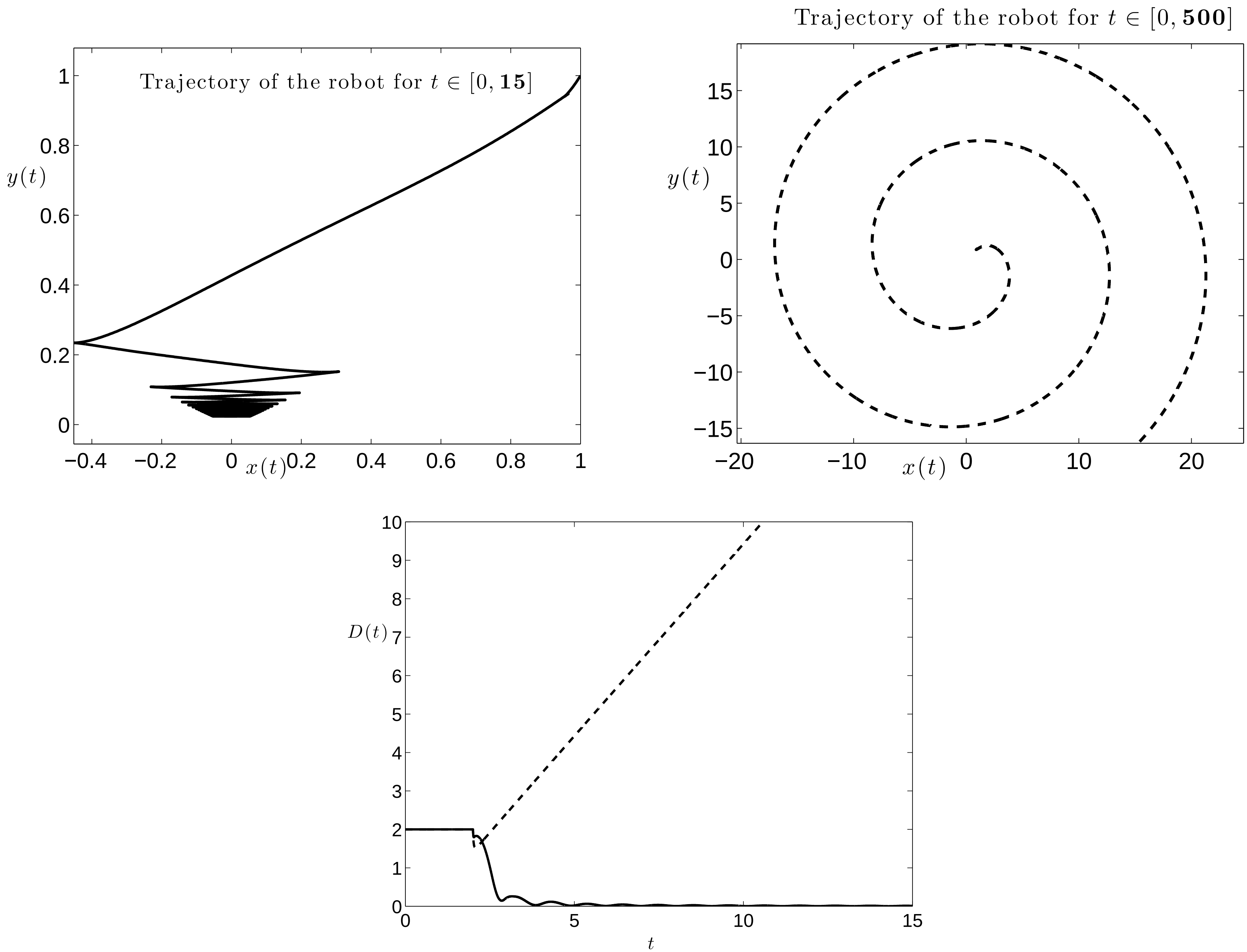
$$\begin{aligned} \omega &= -5P^2 \cos(3\sigma(t)) - pq\left(1 + 25\cos(3\sigma(t))^2\right) - \Theta \\ v &= -P + 5Q(\sin(3\sigma(t)) - \cos(3\sigma(t))) + Q\omega, \end{aligned}$$

where

$$\begin{aligned} P &= X\cos(\Theta) + Y\sin(\Theta) \\ Q &= X\sin(\Theta) - Y\cos(\Theta), \end{aligned}$$

and the predictor is given by

$$\begin{aligned} X(t) &= x(t) + \int_{t-D(x(t),y(t))}^{t} \dot{\sigma}(s)v(s)\cos(\Theta(s))\,ds \\ Y(t) &= y(t) + \int_{t-D(x(t),y(t))}^{t} \dot{\sigma}(s)v(s)\sin(\Theta(s))\,ds \\ \Theta(t) &= \theta(t) + \int_{t-D(x(t),y(t))}^{t} \dot{\sigma}(s)\omega(s)ds \\ \sigma(t) &= t + D(X(t),Y(t)) \\ \dot{\sigma}(s) &= \frac{1}{1 - 2\left(X(s)v(s)\cos(\Theta(s)) + Y(s)v(s)\sin(\Theta(s))\right)} \end{aligned}$$

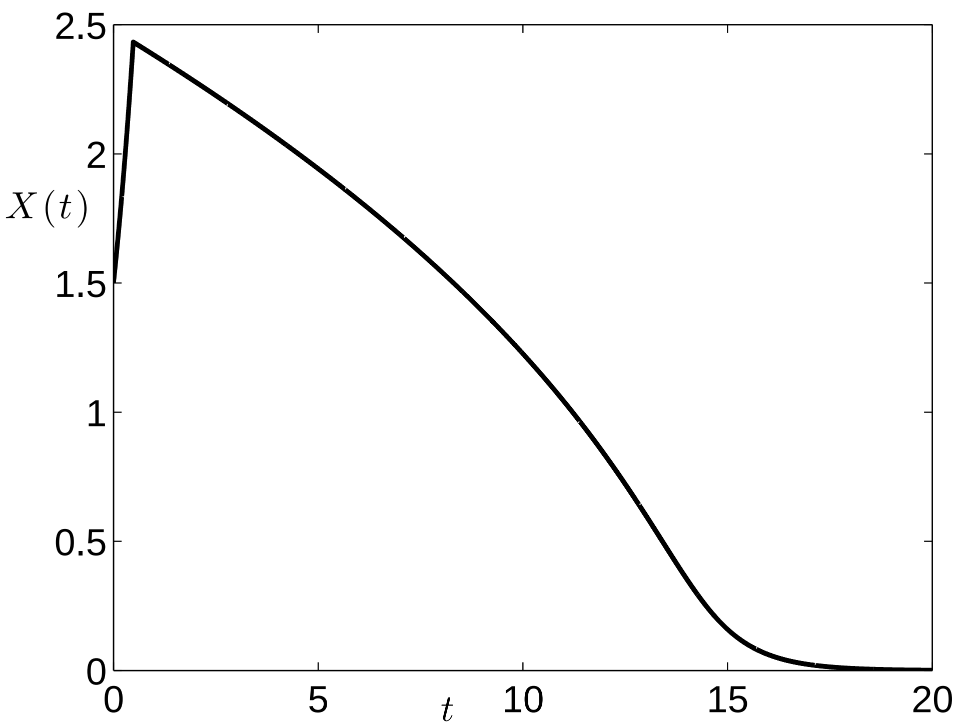


Solid: with delay compensation; dashed: without.

Example 3 (global stabilization—not with D unif. bdd but with $\dot{D} = \nabla D f < 1$)

$$\dot{X}(t) = \frac{X(t) + U(t - D(X(t)))}{1 + U(t - D(X(t)))^2}, \qquad D(X) = \frac{1}{4}\log\left(1 + X^2\right).$$

In the delay-free case, the controller $U = -2X$ yields the closed-loop system $\dot{X} = -\frac{X}{1+4X^2}$.



1. $X(t)$ grows exponentially,
2. $X(t)$ decays as "backwards" square root,
3. $X(t)$ decays exponentially.

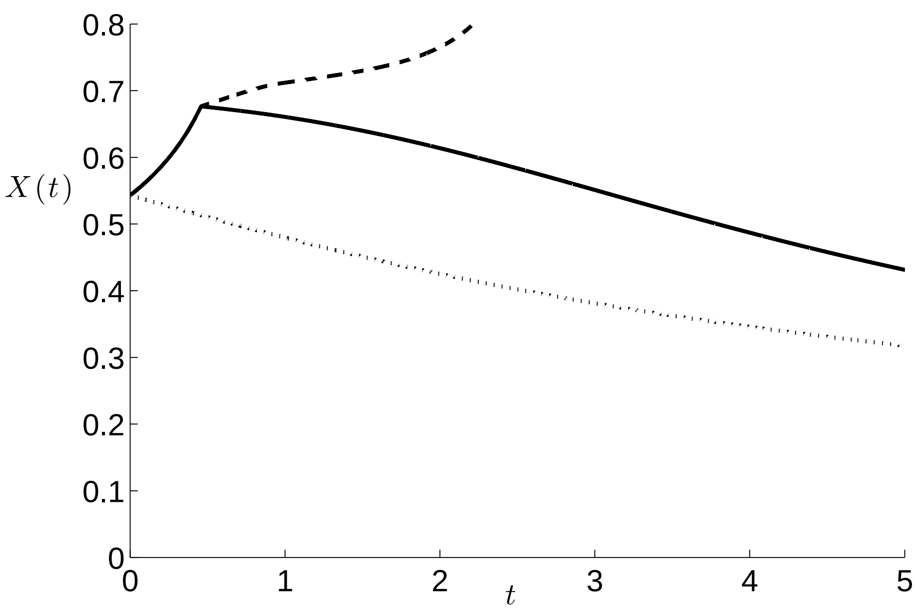
Example 4 (forward completeness not needed for local stabilization)

$$\dot{X}(t) = X^4(t) + 2X^5(t) + X^2(t)(1 + X(t))U(t - X^2(t)).$$

Origin not reachable for $X_0 < -1$, hence not glob. stabilizable.

Origin not loc. exp. stabilizable.

Delay-free controller $U = -X$ yields $\dot{X} = -X^3 + 2X^5$, with RoA $= \frac{1}{\sqrt{2}} \approx 0.7$.



Solid: controller with delay compensation.
Dotted: delay-free case.
Dashed: $U = -X$ applied to the plant with delay.

The initial condition $X_0 = 0.54$ is large. The state $X(\sigma^*)$ is almost at $R = \frac{1}{\sqrt{2}}$ when control kicks in.

Theorem 3 (loc. asymp. stabilization of ODE $\Rightarrow$ loc. asymp. stabilization $\forall$ delay fcn)

If in the absence of delay there exist $R > 0$ *and* $\beta_1 \in \mathcal{KL}$ *s.t.* $\forall t \geq 0$,

$$|X(0)| < R \qquad \Rightarrow \qquad |X(t)| \leq \beta_1(|X(0)|, t),$$

then there exist $\delta > 0$ *and* $\beta_2 \in \mathcal{KL}$ *s.t.* $\forall t \geq 0$,

$$|X(0)| + \sup_{-D(X(0)) \leq \theta \leq 0} |U(\theta)| < \delta$$

$$\Downarrow$$

$$|X(t)| + \sup_{t - D(X(t)) \leq \theta \leq t} |U(\theta)| \leq \beta_2\left(|X(0)| + \sup_{-D(X(0)) \leq \theta \leq 0} |U(\theta)|, t \right).$$

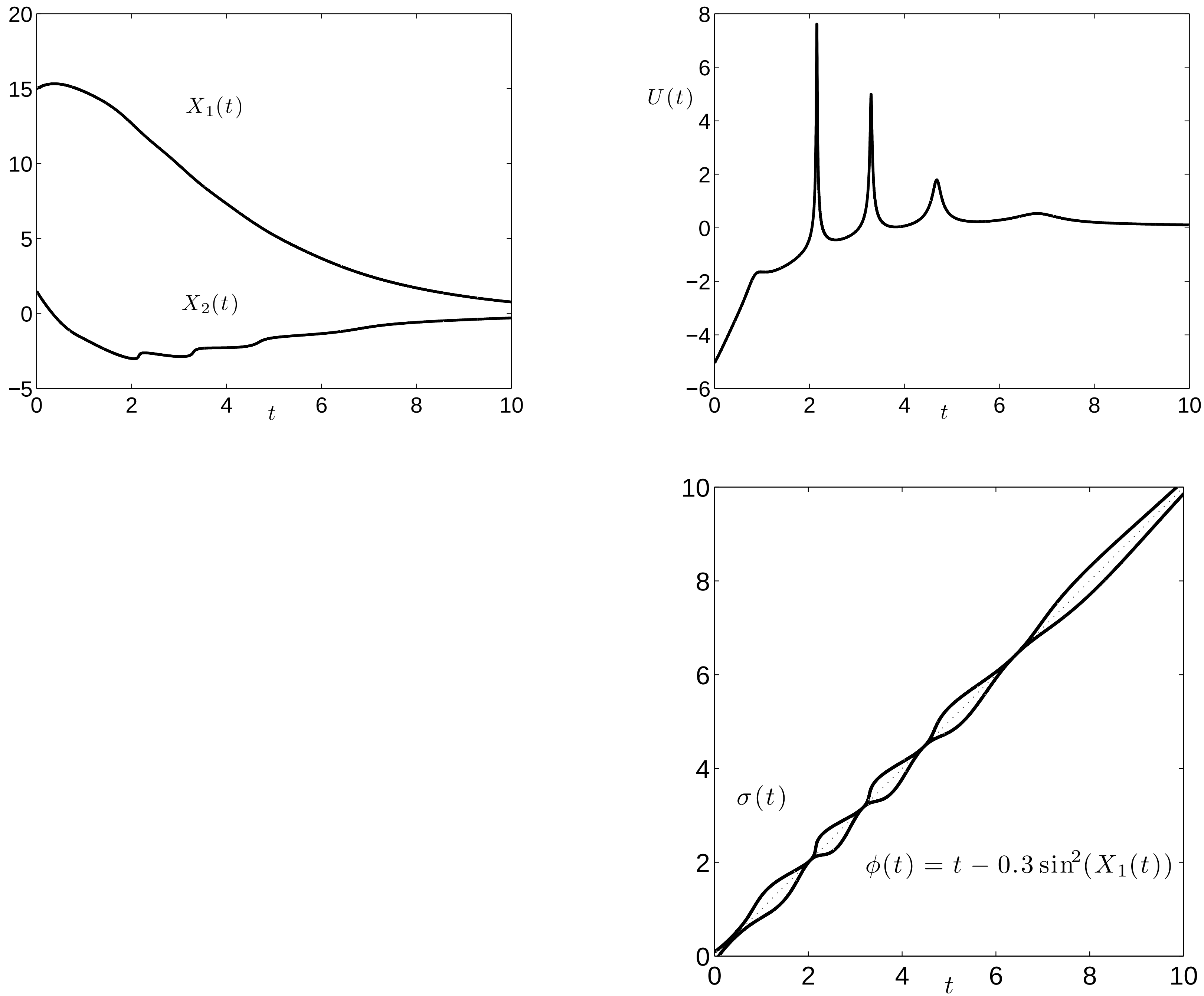
Extra challenge: Make δ so small that, when control kicks in, $|X| < R$.

(Estimate the time the control kicks in from a fixed pt. problem on a delay bound, which is a contraction for sufficiently small initial condition.)

Example 5 (state-dependent delay on **state**)

$$\begin{aligned}\dot{X}_1(t) &= X_2\left(t - a\sin^2 X_1(t)\right), \qquad a \geq 0 \\ \dot{X}_2(t) &= U(t)\end{aligned}$$

$$\begin{aligned}U(t) &= -c_2\left(X_2(t) + c_1 P_1(t)\right) - c_1 \frac{X_2(t)}{1 - a\sin\left(2P_1(t)\right) X_2(t)} \\ P_1(\theta) &= X_1(t) + \int_{t - a\sin^2 X_1(t)}^{\theta} \frac{X_2(s)\, ds}{1 - a\sin\left(2P_1(s)\right) X_2(s)}\end{aligned}$$



$X_1(t)$
$X_2(t)$
t
$U(t)$
$\sigma(t)$
$\phi(t) = t - 0.3\sin^2(X_1(t))$

Feliz cumpleaños!