Curvatures of control systems

and feedback focusing

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Perspectives and Future Directions
in Control and System Theory

Meeting Eduardo:

- 1978 - Bordeaux Conference
- 1980 - Princeton Conference
- 1986-87 - 3 semesters at Rutgers

Plan

Part I: Curvature of second order dynamical systems

- Second order dynamical systems DS
- Curvature of DS
- Position focusing

Part II Feedback focusing

- Second order control systems CS
- Feedback focusing problem
- Curvature and feedback focusing

Part III

- Curvature of control systems

Most results are based on a joint work with W. Kryński

Second order dynamical system

By second order dynamical system DS we mean

$$\ddot{x} = F(t, x, \dot{x}), \quad x \in \mathbb{R}^n.$$

It will be written as:

$$\dot{x} = y, \quad \dot{y} = F(t, x, y), \qquad (DS)$$

where $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, $y = (y^1, \ldots, y^n) \in \mathbb{R}^n$,
$F = (F^1, \ldots, F^n)$.

In mechanics:

x - generalized position,

y - generalized velocity,

$F(t, x, y)$ - "force".

Curvature of second order dynam. system

Def Curvature of DS is the $n \times n$ matrix $K = K(t, x, y)$,

$$K = -F_x - \frac{1}{4}F_y^2 + \frac{1}{2}FF_{yy} + \frac{1}{2}yF_{xy} + \frac{1}{2}F_{ty},$$

where $F_x = (F^i_{x^j})$ - matrix of partial deriv. with respect to x^j,
$F_y = (F^i_{y^j})$ - matrix of partial deriv. with respect to y^j,
$F_{ty} = (F^i_{ty^j})$ - matrix of second partial deriv. w. r. to t and y^j,
$F_{yy} = (F^i_{y^j y^k})$ - n matrices of partial der. with respect to y^j, y^k
$F_{xy} = (F^i_{x^j y^k})$ - n matrices of partial der. with respect to x^j, y^k
$K = (K^i_j)$, where, with the summation convention,

$$K^i_j = -F^i_{x^j} - \frac{1}{4}F^i_{y^k}F^k_{y^j} + \frac{1}{2}F^k F^i_{y^k y^j} + \frac{1}{2}y^k F^i_{x^k y^j} + \frac{1}{2}F^i_{ty^j},$$

Example: potential force

If the "force" $F = F(t, x)$ depends on position and time, only, then

$$K = -F_x(x) = \left(\frac{\partial F^i}{\partial x^j}\right).$$

This happens, for example, if the force is potential,

$$F = F(x) = -\phi_x(x),$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ - a potential. Then the curvature is given by the Hessian of the potential,

$$K = \phi_{xx} = \left(\phi_{x^i x^j}\right),$$

and is a symmetric matrix.

Transformation rule for curvature

With a change of coordinates

$$x = x(\widetilde{x})$$

the curvature matrix $K = (K^i_j)$ transforms according to

$$\widetilde{K} = A^{-1} K A,$$

where A is the Jacobi matrix,

$$A = \left(\frac{\partial x^j}{\partial \widetilde{x}^i} \right).$$

Position focusing

Def We will say that a second order dynamical system

$$\dot{x} = y, \quad \dot{y} = F(t, x, y), \qquad (DS)$$

has a position focusing property along a trajectory

$$\gamma : [0, T] \ni t \mapsto (x(t), y(t))$$

if there is $t^* \in (0, T]$ such that all trajectories starting from the initial position $x_0 = x(0)$, with arbitrary initial velocities $y(0)$, end at the same position $x(t^*)$, with some velocities $y(t^*)$. Thus, for all of them:

$$x(0) \quad \text{are the same and} \quad x(t^*) \quad \text{are the same.}$$

Position focusing II

Question 1 When a second order dynamical system

$$\dot{x} = y, \quad \dot{y} = F(t, x, y), \tag{DS}$$

has the position focusing property (along a given trajectory or along any trajectory)?

THM (Sufficient condition) Given a trajectory γ of DS, if the curvature matrix K is constant along γ on $[0, T]$ and equal to a positive multiple of identity,

$$K(t, x(t), y(t)) \ = \ \kappa\, I, \quad \kappa > \frac{\pi^2}{T^2},$$

then DS has the focusing property along γ and the trajectories starting from $x(0) = x_0$ focus at $x(t^*)$, where

$$t^* = \frac{\pi}{\sqrt{\kappa}}.$$

Part II: Second order control systems

Main message:

We may modify the curvature of a system by using feedback control so that the system has desired properties (e.g. feedback focusing).

Second order control system

Consider a control system

$$\dot{x} = y, \quad \dot{y} = F(t, x, y) + G(t, x, y)u, \tag{CS}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $u \in \mathbb{R}^n$ or in open subsets in $\mathbb{R}^n$.
G is $n \times n$ matrix, assumed invertible.
We call such systems fully actuated.

Question 2 Does there exist a feedback control

$$u = u(t, x, y)$$

such that the resulting dynamical system

$$\dot{x} = y, \quad \dot{y} = F(t, x, y) + G(t, x, y)u(t, x, y), \tag{DS}$$

has the position focusing property?
If so, we say that CS has the feedback focusing property.

Feedback focusing

THM Any fully actuated second order control system has the feedback focusing property, along any trajectory.

Moreover, given any trajectory γ of the dynamical system

$$\dot{x} = y, \quad \dot{y} = F(t, x, y),$$

the focusing feedback control $u(t, x, y)$ along γ vanishes on γ, i.e. can be taken so that

$$u(t, x(t), y(t)) = 0, \quad \text{for} \ \ (x(t), y(t)) = \gamma(t).$$

Constructing focusing feedback

Suppose, we have a trajectory $\gamma : [0,T] \to \mathbb{R}^n \times \mathbb{R}^n$ of DS.
We want to find feedback control $u = u(x,y)$ such that the resulting DS has the feedback focusing property along γ, at a given time $t^* \in (0,T]$.
Take a system given by geodesic equations on the sphere S^n of radius $r = \frac{t^*}{\pi}$:

$$\dot{x} = y, \quad \dot{y} = F_S(x,y). \tag{S}$$

Take a geodesic γ_S of (S) (half of a great circle) and map it onto $\gamma|_{[0,t^*]}$, together with its nbhd. Denote the system transformed by this mapping, by

$$\dot{x} = y, \quad \dot{y} = \widetilde{F}_S(x,y) \tag{$\widetilde{S}$}$$

(the transformed variables are still denoted x and y).

Constructing focusing feedback II

Feedback control $u = u(x, y)$ applied in CS produces the dynamical system:

$$\dot{x} = y, \quad \dot{y} = F(t, x, y) + G(t, x, y)u(t, x, y).$$

Thus we should solve for u the equation

$$F(t, x, y) + G(t, x, y)u(t, x, y) = \widetilde{F}_S(x, y).$$

The solution is

$$u(t, x, y) = G^{-1}(t, x, y)(\widetilde{F}_S(x, y) - F(t, x, y)).$$

By construction, this feedback control is focusing.

Non-exact feedback focusing

What happens if, in the above construction, the feedback control

$$u = u(t, x, y)$$

is only approximately equal to the feedback focusing one?

Claim. Then feedback focusing on the interval $[0, t^*]$ is lost but a weaker property holds:
There are n conjugate times, counted with multiplicity, close to the focusing time t^*.

This is proved by using our recent common results with Wojciech Kryński on estimates from below and from above for conjugate times of general systems. Classical methods do not work because the system and its curvature are not symmetric.

Part III:

Curvature(s) of general control systems

Control-affine systems

The above results can be generalized to control-affine systems

$$\Sigma : \quad \dot{x} = X(x) + \sum_{j=1}^{n} u_j Y_j(x),$$

$$x \in \mathbb{R}^{2n} \text{ or } x \in M^{2n},$$

satisfying 1-regularity condition:
The vector fields

$Y_1, \ldots, Y_n,\ [X, Y_1], \ldots, [X, Y_n]$ are pointwise linearly independent.

We denote $M = \mathbb{R}^{2n}$, or $M = M^{2n}$.

Curvature of control system

Arrange control vector fields and Lie brackets into row vectors:

$$Y = (Y_1, \ldots, Y_n),$$

$$[X, Y] = ([X, Y_1], \ldots, [X, Y_n]),$$

$$[X, [X, Y]] = ([X, [X, Y_1]], \ldots, [X, [X, Y_n]]).$$

1-regularity implies that there are $n \times n$ matrices H_0, H_1 such that

$$[X, [X, Y]] = Y H_0 + [X, Y] H_1.$$

Def The curvature matrix of a control system Σ is

$$K = -H_0 + \frac{1}{2} X(H_1) - \frac{1}{4} H_1^2.$$

Conjugate points I

Let

$$D(x) = \text{span}\,\{Y_1(x), \ldots, Y_n(x)\}$$

be the distribution spanned by control vector fields.

Let $c : [0, T] \to M$ - trajectory of X, $c(0) = x_0$. The following definition and results generalize classical theorems of Cartan-Hadamard and Bonnet-Myers. Specified to Hamiltonian setting give results of Agrachev et. al.

Def (Agrachev?) A point $0 < \tau \leq T$ is called *conjugate time* if

$$\exp(\tau X)_* D(x_0) \cap D(c(\tau)) \neq \{0\}.$$

Normal basis of D

We will use special bases of control distribution

$$D(x) = \text{span}\,\{Y_1(x), \ldots, Y_n(x)\}.$$

Def A basis $\widehat{Y}_1, \ldots, \widehat{Y}_n$ of D is normal if $\exists$ functions K_i^j s.t.

$$[X, [X, \widehat{Y}_i]] \;=\; \sum_j K_i^j \, \widehat{Y}_j.$$

Fact 1 Normal bases exist.

Fact 2 The matrix $K = (K_i^j)$ coincides with the curvature matrix of system Σ.

Conjugate points II

Let $K(t) = (K^i_j(t))$ - curvature of Σ along $c(t)$, $t \in [0, T]$, in a normal basis $\widehat{Y}_1, \ldots, \widehat{Y}_n$.

THM (a) If the largest eigenvalue $\lambda^K(t)$ of the symmetric part of $K(t)$ satisfies

$$\lambda^K(t) \leq \lambda, \quad t \in [0, T],$$

for a constant $\lambda \leq 0$, then there are no conjugate times in $(0, T]$. If $\lambda > 0$ then there are no conjugate times in $(0, T]$, if $T < \frac{\pi}{\sqrt{\lambda}}$.

(b) If $K(t)$ is symmetric along c, tr $K_0 > \kappa > 0$ and $T \geq \pi\sqrt{\frac{m}{\kappa}}$, then there is a conjugate time in $(0, T]$.

Remarks

- A priori, there is no metric in the statement of our result.

The curvature matrix K is symmetric:

- in the Riemannian case,
- in the case of X being a Hamiltonian vector field and $\mathcal{V}$ being a Lagrangian distribution on a symplectic manifold (M, σ) (Agrachev, Gamkrelidze, Chtcherbakova, Zelenko).

Our definition of curvature is much simpler in this case.

Conjugate points can also be defined for control systems satisfying a r-regularity condition. In this case one can prove, for $\mathcal{V}$ of rank one, that there are no conjugate points if all curvatures $K_0, \ldots, K_{r-1}$ are nonpositive scalars (BJ 2009).

r-regular control systems

Consider a scalar-control system

$$\Sigma: \quad \dot{x} = X(x) + uY(x).$$

Def Σ is r-regular at x, $r \geq 1$, if $X(x) \neq 0$, the vector fields

$$Y, \mathrm{ad}_X Y, \ldots \mathrm{ad}_X^r Y$$

are linearly independent at x and there are functions $h_0, \ldots, h_r$ such that, in a nbhd of x,

$$\mathrm{ad}_X^{r+1} = h_0 Y + h_1 \mathrm{ad}_X Y + \cdots + h_r \mathrm{ad}_X^r Y.$$

Fact. In a nbhd of a r-regular point there is a non-vanishing function g such that the vector field $\hat{Y} = gY$ satisfies

$$\mathrm{ad}_X^{r+1} \hat{Y} = -k_0 \hat{Y} + k_1 \mathrm{ad}_X \hat{Y} + \cdots + (-1)^r k_{r-1} \mathrm{ad}_X^{r-1} \hat{Y}.$$

Def The functions $k_0, \ldots k_{r-1}$ are called curvatures of Σ.

Absence of conjugate points

Claim If Σ is r-regular and the curvatures $k_0, \ldots, k_{r-1}$ are non-positive, then Σ has no conjugate points.

Precisely, let $c : [0, T] \to M$ be a trajectory of X.

Def A point $0 < \tau \leq T$ is conjugate time for r-regular Σ, if

$$\exp(\tau X)_* D(x_0) \cap (D + \mathrm{ad}_X D + \cdots + \mathrm{ad}_X^{r-1} D)(c(\tau)) \neq \{0\}.$$

THM (BJ 2009) If the system Σ is r-regular along c and all curvatures $k_0, \ldots, k_{r-1}$ are nonpositive along c, then Σ has no conjugate times.