Linear Matrix Inequalities vs Convex Sets

with admiration and friendship for **Eduardo**

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Advertisement: **Try noncommutative computation**

NCAlgebra[1]

NCSoSTools[2]

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[1] Helton, deOliveira (UCSD), Stankus (CalPoly SanLObispo), Miller

[2] Igor Klep

Ingredients of Talk: LMIs and Convexity

A **Linear Pencil** is a matrix valued function L of the form

$$L(x) := L_0 + L_1 x_1 + \cdots + L_g x_g,$$

where $L_0, L_1, L_2, \cdots, L_g$ are symmetric matrices and $x := \{x_1, \cdots, x_g\}$ are m real parameters.

A **Linear Matrix Inequality (LMI)** is one of the form:

$$L(x) \succeq 0.$$

Normalization: a **monic LMI** is one with $L_0 = I$.
The set of solutions

$$\mathcal{G} := \{(x_1, x_2, \cdots, x_g) : L_0 + L_1 x_1 + \cdots + L_g x_g \quad \text{is PosSD}\}$$

is a convex set. Solutions can be found numerically for problems of modest size. This is called

Semidefinite Programming SDP

Ingredients of Talk: Noncommutative polynomials

$x = (x_1, \cdots, x_g)$ algebraic noncommuting variables

Noncommutative polynomials: $p(x)$:

Eg. $$p(x) = x_1 x_2 + x_2 x_1$$

Evaluate p: on matrices $X = (X_1, \cdots X_g)$ a tuple of matrices.
Substitute a matrix for each variable $x_1 \to X_1$, $x_2 \to X_2$

Eg. $$p(X) = X_1 X_2 + X_2 X_1.$$

Noncommutative inequalities: p is positive means:

$p(X)$ is PSD for all X

Eduardo's secret life.

Eduardo's secret noncommutative linear life.

On Linear Systems and Noncommutative Rings

by

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ABSTRACT

This paper studies some problems appearing in the extension of the theory of linear dynamical systems to the case in which parameters are taken from noncommutative rings. Purely algebraic statements of some of the problems are also obtained.

Through systems defined by operator rings, the theory of linear systems over rings may be applied to other areas of automata and control theory; several such applications are outlined.

Introduction. The algebraic theory of linear constant systems with coefficients over an arbitrary field has gone through major developments during the past decade. A comprehensive account is given in Kalman, Falb and Arbib [9, Chap. 10]. Since a considerable number of the results remain valid for the

Examples of NC Polynomials

The Ricatti polynomial

$$r((a,b,c),x) = -xb^Tbx + a^Tx + xa + c$$

Here $a = (a,b,c)$ and $x = (x)$.

Evaluation of NC Polynomials

r is naturally evaluated on a $k+g$ tuple of (not necessarily) commuting symmetric matrices

$A = (A_1, \ldots, A_k) \in (\mathbb{R}^{n\times n})^k$ $\qquad X = (X_1, \ldots, X_g) \in (\mathbb{SR}^{n\times n})^g$

$$r((A,B,C),X) = -XB^TBX + A^TX + XA + C \in \mathbb{S}_n(\mathbb{R}).$$

Note that the form of the Riccati is independent of n.

POLYNOMIAL MATRIX INEQUALITIES

Polynomial or Rational function of matrices are PosSDef.
Example: Get Riccati expressions like

$$AX + XA^T - XBB^TX + CC^T \succ 0$$

OR Linear Matrix Inequalities (LMI) like

$$\begin{pmatrix} AX + XA^T + C^TC & XB \\ B^TX & I \end{pmatrix} \succ 0$$

which is equivalent to the Riccati inequality.

NC Polynomials in a and x

• Let $\mathbb{R}\langle a, x\rangle$ denote the algebra of polynomials in the $k+g$ non-commuting variables,

$$a = (a_1, \ldots, a_k), \qquad x = (x_1, \ldots, x_g).$$

• There is the involution ^{T} satisfying, $(fg)^T = g^T f^T$, which reverses the order of words.

• The variables x are assumed formally symmetric, $x = x^T$.

Can manipulate with computer algebra eg. NCAlgebra

Outline

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Linear Systems give NC Polynomial Inequalities

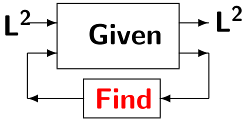
We Need Theory of NC Real Algebraic Geometry

Dimension Free Convexity vs NC LMIs

Change of Variables to achieve NC Convexity

nc maps

Linear Systems Problems $\rightarrow$ Matrix Inequalities

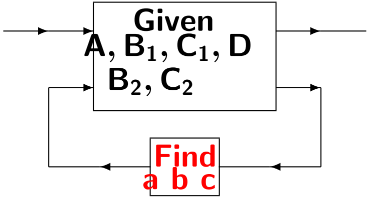


Many such problems Eg. H^∞ control

The problem is Dimension free: since it is given only by signal flow diagrams and L^2 signals.

Dim Free System Probs is Equivalent to Noncommutative Polynomial Inequalities

GET ALGEBRA



$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

DYNAMICS of "closed loop" system: BLOCK matrices

$$\mathcal{A} \quad \mathcal{B} \quad \mathcal{C} \quad \mathcal{D}$$

ENERGY DISSIPATION:

$$H := \mathcal{A}^T E + E\mathcal{A} + E\mathcal{B}\mathcal{B}^T E + \mathcal{C}^T\mathcal{C} \preceq 0$$

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \qquad E_{12} = E_{21}{}^T$$

$$H = \begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & E_{yy} \end{pmatrix} \qquad H_{xy} = H_{yx}^T$$

H^∞ Control

ALGEBRA PROBLEM:

Given the polynomials:

$$H_{xx} = E_{11}\, A + A^T\, E_{11} + C_1^T\, C_1 + {E_{12}}^T\, b\, C_2 + C_2^T\, b^T\, {E_{12}}^T + E_{11}\, B_1\, b^T\, {E_{12}}^T + E_{11}\, B_1\, B_1^T\, E_{11} + E_{12}\, b\, b^T\, {E_{12}}^T + E_{12}\, b\, B_1^T\, E_{11}$$

$$H_{xz} = E_{21}\, A + \frac{a^T\,(E_{21} + {E_{12}}^T)}{2} + c^T\, C_1 + E_{22}\, b\, C_2 + c^T\, B_2^T\, {E_{11}}^T + \frac{E_{21}\, B_1\, b^T\,(E_{21} + {E_{12}}^T)}{2} + E_{21}\, B_1\, B_1^T\, {E_{11}}^T + \frac{E_{22}\, b\, b^T\,(E_{21} + {E_{12}}^T)}{2} + E_{22}\, b\, B_1^T\, {E_{11}}^T$$

$$H_{zx} = A^T\, {E_{21}}^T + C_1^T\, c + \frac{(E_{12} + {E_{21}}^T)\, a}{2} + E_{11}\, B_2\, c + C_2^T\, b^T\, {E_{22}}^T + E_{11}\, B_1\, b^T\, {E_{22}}^T + E_{11}\, B_1\, B_1^T\, {E_{21}}^T + \frac{(E_{12} + {E_{21}}^T)\, b\, b^T\, {E_{22}}^T}{2} + \frac{(E_{12} + {E_{21}}^T)\, b\, B_1^T\, {E_{21}}^T}{2}$$

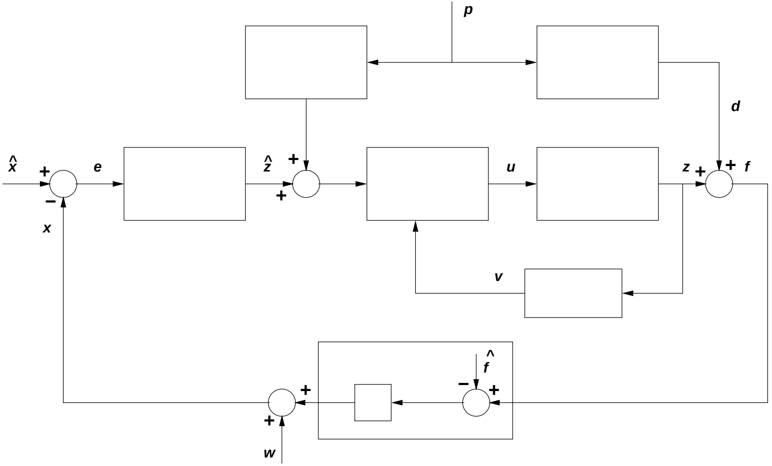
$$H_{zz} = E_{22}\, a + a^T\, {E_{22}}^T + c^T\, c + E_{21}\, B_2\, c + c^T\, B_2^T\, {E_{21}}^T + E_{21}\, B_1\, b^T\, {E_{22}}^T + E_{21}\, B_1\, B_1^T\, {E_{21}}^T + E_{22}\, b\, b^T\, {E_{22}}^T + E_{22}\, b\, B_1^T\, {E_{21}}^T$$

(PROB) A, B_1, B_2, C_1, C_2 **are knowns.**

Solve the inequality $\begin{pmatrix} H_{xx} & H_{xz} \\ H_{zx} & H_{zz} \end{pmatrix} \preceq 0$ **for unknowns**

a, b, c **and for** E_{11}, E_{12}, E_{21} **and** E_{22}

More complicated systems give fancier nc polynomials



Engineering problems defined entirely by signal flow diagrams and L^2 performance specs

are equivalent to

Polynomial Matrix Inequalities

How and why is a long story but the correspondence between linear systems and noncommutative algebra is on the next slides:

Linear Systems and Algebra Synopsis

A Signal Flow Diagram with L^2 based performance, eg H^∞ gives precisely a nc polynomial

$$p(a, x) := \begin{pmatrix} p_{11}(a, x) & \cdots & p_{1k}(a, x) \\ \vdots & \ddots & \vdots \\ p_{k1}(a, x) & \cdots & p_{kk}(a, x) \end{pmatrix}$$

Such linear systems problems become exactly:

> **Given matrices A.**
> **Find matrices X so that P(A, X) is PosSemiDef.**

WHY? Turn the crank using quadratic storage functions.

BAD Typically p is a mess, until a hundred people work on it and maybe convert it to CONVEX Matrix Inequalities.

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Convexity vs LMIs

QUESTIONS (Vague) :

WHICH DIM FREE PROBLEMS "ARE" LMI PROBLEMS.
Clearly, such a problem must be convex and "semialgebraic".
Which convex nc problems are NC LMIS?

WHICH PROBLEMS ARE TREATABLE WITH LMI's?
This requires some kind of change of variables theory.

The first is the main topic of this talk

Partial Convexity of NC Polynomials

The polynomial $p(a, x)$ is convex in x for all A if for each X, Y and $0 \le \alpha \le 1$,

$$p(A, \alpha X + (1-\alpha)Y) \preceq \alpha p(A, X) + (1-\alpha)p(A, Y).$$

The Riccati $r(a, x) = c + a^T x + xa - xb^T bx$ is concave, meaning $-r$ is convex in x (everywhere).

Can localize A to an nc semialgebraic set.

Structure of Partially Convex Polys

THM (Hay-Helton-Lim- McCullough)

SUPPOSE $p \in \mathbb{R}\langle a, x\rangle$ is convex in x THEN

$$p(a, x) = L(a, x) + \tilde{L}(a, x)^T Z(a) \tilde{L}(a, x),$$

where,

- $L(a, x)$ has degree at most one in x;
- $Z(a)$ is a symmetric matrix-valued NC polynomial;
- $Z(A) \succeq 0$ for all A;
- $\tilde{L}(a, x)$ is linear in x. $\tilde{L}(a, x)$ is a (column) vector of

. NC polynomials of the form $x_j m(a)$.

This also works fine if p is a matrix of nc polynomials.

This also works fine if A only belongs to an open nc semi-algebraic set (will not be defined here) .

Structure of Partially Convex Polys

COR SUPPOSE $p \in \mathbb{R}\langle a, x \rangle$ is convex in x

THEN there is a linear pencil $\Lambda(a, x)$ such that the set of all solutions to $\{X : p(A, X) \succeq 0\}$ equals $\{X : \Lambda(A, X) \succeq 0\}$.

Proof: p is a Schur Complement of some Λ by the previous theorem.

The (SAD) MORAL OF THE STORY

A CONVEX problem specified entirely by a signal flow diagram and L^2 performance of signals is equivalent to some LMI.

Context: Related Areas

Convex Algebraic Geometry **(mostly commutative)**
NSF FRG: Helton -Nie- Parrilo- Strumfels- Thomas

One aspect: Convexity vs LMIs.

Now there is a roadmap with some theorems and conjectures.

Three branches:

1. Which convex semialgebraic sets in $\mathbb{R}^g$ have an LMI rep?
(Line test) Is it necessary and sufficient? **Ans: Yes if $g \leq 2$.**

2.Which convex semialgebraic sets in $\mathbb{R}^g$ lift to a set with an LMI representation? **Ans: Most do.**

3. Which noncommutative semialgebraic convex sets have an LMI rep? **Ans: All do. (like what you have seen.)**

NC Real Algebraic Geometry **(since 2000)**

We have a good body of results in these areas.
Eg. Positivestellensatz

Algebraic Certificates of Positivity

Positivestellensatz (H-Klep-McCullough): Certificates equivalent to

$$\mathbf{p(X) \text{ is PSD where } L(X) \text{ is PSD}}$$

is the same as

$$p = \mathrm{SoS} + \sum_{j}^{\text{finite}} f_j^T \, L \, f_j$$

whenever L is a monic linear pencil. Here

$$\text{degree } \mathrm{SoS} \;=\; \deg f_j^T f_j \;=\; \deg p$$

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Changing variables to achieve NC convexity

Our Main current campaign

NC (free) analytic maps

Change of variables

We use NC (free) analytic maps

(1) **Analytic nc polynomials have no $\mathbf{x}^*$.**

(2) $\mathbf{f}(\mathbf{x}_1) = \mathbf{x}_1^*$ **is not an nc analytic map.**

$$\mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_{\tilde{\mathbf{g}}} \end{pmatrix}$$

Given p nc polynomial. Does it have the form

$$\mathbf{p(x) = c(f(x))}$$

for c a convex polynomial, with f nc bianalytic?
If yes

$$p = \sum_{j}^{k} F_j^T F_j + \sum_{j=k+1}^{g} H_j H_j^T$$

with F_j, H_j analytic. Note g terms where $x = (x_1, \cdots, x_g)$.

A Baby Step

THM (H-Klep- McCullough-Slinglend, JFA 2009) If yes, then p is a (unique) sum of g nc squares. One can compute this explicitly.

For bigger steps see two more papers on ArXiv.

Analogy:
Finding SoS is a highly nonconvex problem but converts to LMIs. (Parrilo, Lasserre)

Is it concievable that nc change of variables will also convert to an LMI?

I guess it is a long shot

but

Long Shots, as does Eduardo, are what make the world fun.

CONGRATS

and

THANKS

to

THE AMAZING EDUARDO

for 37 (a prime) years of inspiring friendship