Control Lyapunov functions and partial differential equations

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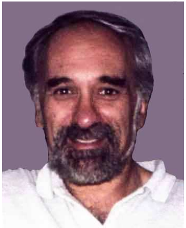

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Texts in Applied Mathematics 6

Mathematical Control Theory

Deterministic Finite Dimensional Systems

Springer-Verlag

Control Lyapunov functions and Eduardo

Control Lyapunov function is a very powerful tool for stabilization of nonlinear control system in finite dimension. Let us mention that this tool has been strongly developed by Eduardo. In particular, in his following seminal works the Lyapunov approach is a key step.

1. A Lyapunov-like characterization of asymptotic controllability (1983),
2. A "universal" construction of Artstein's theorem on nonlinear stabilization (1989),
3. Smooth stabilization implies coprime factorization (1989),
4. New characterizations of input to state stability (1996; with Yuandan Lin and Yuan Wang),
5. Asymptotic controllability implies feedback stabilization (1996; with F.H. Clarke, Yu S. Ledyaev and A.I. Subbotin),
6. A Lyapunov characterization of robust stabilization (1999; with Y. Ledyaev),

Sontag+Lyapunov gives 20,000 results with google.

Lyapunov function and PDE

Lyapunov is also a powerful tool for PDE (linear and nonlinear). However one of the problem is the LaSalle invariance principle: one needs to prove the precompactness of the trajectories, which is difficult to get for nonlinear PDE. Hence it is better to have strict Lyapunov functions. In this talk we present an example of application of strict Lyapunov function to $1-D$ hyperbolic systems.

The hyperbolic system considered

The dynamical (control) system is, with $y_t = \partial y/\partial t$ and $y_x = \partial y/\partial x$,

(1) $$y_t + A(y)y_x = 0,\ y \in \mathbb{R}^n,\ x \in [0,1],\ t \in [0,+\infty).$$

At time t, the state is the map $x \in [0,1] \mapsto y(t,x) \in \mathbb{R}^n$. We assume that

- Assumptions on A:

(2) $$A(0) = \text{diag}\ (\lambda_1, \lambda_2, \dots, \lambda_n),$$

(3) $$\lambda_i > 0,\ \forall i \in \{1, \dots, m\},\ \lambda_i < 0,\ \forall i \in \{m+1, \dots, n\},$$

(4) $$\lambda_i \neq \lambda_j,\ \forall (i,j) \in \{1, \dots, n\}^2 \text{ such that } i \neq j.$$

• Boundary conditions on y:

(1) $$\begin{pmatrix} y_+(t,0) \\ y_-(t,1) \end{pmatrix} = G \begin{pmatrix} y_+(t,1) \\ y_-(t,0) \end{pmatrix}, \; t \in [0,+\infty),$$

where

(i) $y_+ \in \mathbb{R}^m$ and $y_- \in \mathbb{R}^{n-m}$ are defined by

(2) $$y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix},$$

(ii) the map $G : \mathbb{R}^n \to \mathbb{R}^n$ vanishes at 0.

In many situations G is a feedback that can be (partially) chosen. We then have a control system and we want to stabilize the origin $\bar{y} \equiv 0$.

Notations

For $K \in \mathcal{M}_{n,m}(\mathbb{R})$,

(1) $$\|K\| := \max\{|Kx|; x \in \mathbb{R}^n,\ |x| = 1\}.$$

If $n = m$,

(2) $$\rho_1(K) := \text{Inf}\ \{\|\Delta K \Delta^{-1}\|;\ \Delta \in \mathcal{D}_{n,+}\},$$

where $\mathcal{D}_{n,+}$ denotes the set of $n \times n$ real diagonal matrices with strictly positive diagonal elements. $H^2(0,1)$ denotes the Sobolev space of $y : [0,1] \rightarrow \mathbb{R}^n$ such that y, y_x and y_{xx} are in L^2. It is equipped with the norm

(3) $$|y|_{H^2(0,1)} := \left(\int_0^1 (|y|^2 + |y_x|^2 + |y_{xx}|^2)dx\right)^{1/2}.$$

Theorem (JMC-G. Bastin-B. d'Andréa-Novel (2008))

If $\rho_1(G'(0)) < 1$, *then the equilibrium* $\bar{y} \equiv 0$ *of the quasi-linear hyperbolic system*

(1) $$y_t + A(y)y_x = 0,$$

with the above boundary conditions, is locally exponentially stable for the Sobolev H^2*-norm.*

Complements:

- $y_t + A(x)y_x + B(x)y = 0$: G. Bastin and JMC (2010), A. Diagne, G. Bastin and JMC (2010), R. Vazquez, M. Krstic and JMC (2011),
- $y_t + A(x,y)y_x + B(x,y)y = 0$: A. Diagne and A. Drici (2011), R. Vazquez, JMC, M. Krstic and G. Bastin (2011),
- Integral action: V. Dos Santos, G. Bastin, JMC and B. d'Andréa-Novel (2008), A. Drici (2010).

Estimate on the exponential decay rate

Let

(1) $$\nu \in (0, -\min\{|\lambda_1|, \dots, |\lambda_n|\} \ln(\rho_1(G'(0)))).$$

Then there exist $\varepsilon > 0$ and $C > 0$ such that, for every $y_0 \in H^2((0,1), \mathbb{R}^n)$ satisfying $|y_0|_{H^2((0,1),\mathbb{R}^n)} < \varepsilon$ (and the usual compatibility conditions at $x = 0$ and $x = L$), the classical solution y to the Cauchy problem

(2) $$y_t + A(y)y_x = 0,\ y(0,x) = y_0(x) + \text{boundary conditions}$$

is defined on $[0, +\infty)$ and satisfies

(3) $$|y(t,\cdot)|_{H^2((0,1),\mathbb{R}^n)} \leqslant Ce^{-\nu t}|y_0|_{H^2((0,1),\mathbb{R}^n)},\ \forall t \in [0, +\infty).$$

The Li Tatsien condition

(1) $$R_2(K) := \text{Max}\,\{\sum_{j=1}^{n} |K_{ij}|;\ i \in \{1, \dots, n\}\},$$

(2) $$\rho_2(K) := \text{Inf}\,\{R_2(\Delta K \Delta^{-1});\ \Delta \in \mathcal{D}_{n,+}\}.$$

Theorem (Li Tatsien, 1994)

If $\rho_2(G'(0)) < 1$, then the equilibrium $\bar{y} \equiv 0$ of the quasi-linear hyperbolic system

(3) $$y_t + A(y)y_x = 0,$$

with the above boundary conditions, is locally exponentially stable for the C^1-norm.

The Li Tatsien proof relies mainly on the use of direct estimates of the solutions and their derivatives along the characteristic curves.

C^1/H^2-exponential stability

1. Open problem: Does there exists K such that one has local exponential stability for the C^1-norm but not for the H^2-norm?
2. Open problem: Does there exists K such that one has local exponential stability for the H^2-norm but not for the C^1-norm?

Comparison of ρ_2 and ρ_1

Proposition

For every $K \in \mathcal{M}_{n,n}(\mathbb{R})$,

$$\rho_1(K) \leqslant \rho_2(K). \tag{1}$$

Example where (1) is strict: for $a > 0$, let

$$K_a := \begin{pmatrix} a & a \\ -a & a \end{pmatrix} \in \mathcal{M}_{2,2}(\mathbb{R}). \tag{2}$$

Then

$$\rho_1(K_a) = \sqrt{2}a < 2a = \rho_2(K_a). \tag{3}$$

Open problem: Does $\rho_1(K) < 1$ implies the local exponential stability for the C^1-norm?

Comparison with stability conditions for linear hyperbolic systems

Let us first point that in the linear case (i.e. when A does not depend on y and G is linear) one has the following theorem.

Theorem

Exponential stability for the C^1-norm is equivalent to the exponential stability in the H^2-norm.

For simplicity we now assume that the λ_i's are all positive: We consider the special case of linear hyperbolic systems

(1) $$y_t + \Lambda y_x = 0,\ y(t,0) = Ky(t,1),$$

where

(2) $$\Lambda := \text{diag}\ (\lambda_1, \dots, \lambda_n),\ \text{ with } \lambda_i > 0,\ \forall i \in \{1, \dots, n\}.$$

A Necessary and sufficient condition for exponential stability

Notation:

(1) $$r_i = \frac{1}{\lambda_i}, \forall i \in \{1, \ldots, n\}.$$

Theorem

$\bar{y} \equiv 0$ is exponentially stable for the system

(2) $$y_t + \Lambda y_x = 0,\ y(t,0) = Ky(t,1)$$

if and only if there exists $\delta > 0$ such that

(3)
$$\left(\mathit{det}\ (\mathit{Id}_n - (\mathit{diag}\ (e^{-r_1 z}, \ldots, e^{-r_n z}))K) = 0,\ z \in \mathbb{C}\right) \Rightarrow (\Re(z) \leqslant -\delta).$$

An example

This example is borrowed from the book Hale-Lunel (1993). Let us choose $\lambda_1 := 1$, $\lambda_2 := 2$ (hence $r_1 = 1$ and $r_2 = 1/2$) and

(1) $$K_a := \begin{pmatrix} a & a \\ a & a \end{pmatrix}, \; a \in \mathbb{R}.$$

Then $\rho_1(K) = 2|a|$. Hence $\rho_1(K_a) < 1$ is equivalent to $a \in (-1/2, 1/2)$. However exponential stability is equivalent to $a \in (-1, 1/2)$.

Robustness issues

For a positive integer n, let

(1) $$\lambda_1 := \frac{4n}{4n+1}, \ \lambda_2 = \frac{4n}{2n+1}.$$

Then

(2) $$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} \sin\left(4n\pi(t-(x/\lambda_1))\right) \\ \sin\left(4n\pi(t-(x/\lambda_2))\right) \end{pmatrix}$$

is a solution of $y_t + \Lambda y_x = 0$, $y(t,0) = K_{-1/2}y(t,1)$ which does not tends to 0 as $t \to +\infty$. Hence one does not have exponential stability. However $\lim_{n\to+\infty} \lambda_1 = 1$ and $\lim_{n\to+\infty} \lambda_2 = 2$. The exponential stability is not robust with respect to Λ: small perturbations of Λ can destroy the exponential stability.

Robust exponential stability

Notation:

(1) $$\rho_0(K) := \max\{\rho(\text{diag }(e^{\iota\theta_1}, \dots, e^{\iota\theta_n})K);\ (\theta_1, \dots, \theta_n)^{\text{tr}} \in \mathbb{R}^n\}.$$

Theorem (R. Silkowski, 1993)

If the $(r_1, \dots, r_n)$ are rationally independent, $\bar{y} \equiv 0$ is exponentially stable for the linear system $y_t + \Lambda y_x = 0$, $y(t, 0) = Ky(t, 1)$, if and only if $\rho_0(K) < 1$.

Note that $\rho_0(K)$ depends continuously on K and that "$(r_1, \dots, r_n)$ are rationally independent" is a generic condition. Therefore, if one wants to have a natural robustness property with respect to the r_i's, the condition for exponential stability is

(2) $$\rho_0(K) < 1.$$

This condition does not depend on the λ_i's!

Comparison of ρ_0 and ρ_1

Proposition (JMC-G. Bastin-B. d'Andréa-Novel, 2008)

For every $n \in \mathbb{N}$ *and for every* $K \in \mathcal{M}_{n,n}(\mathbb{R})$,

(1) $$\rho_0(K) \leqslant \rho_1(K).$$

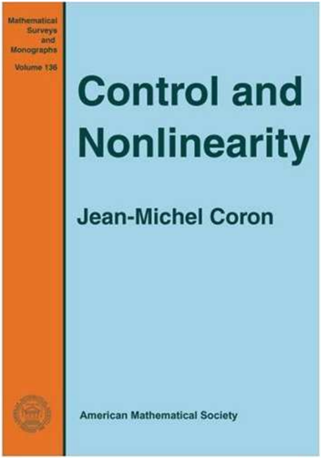
For every $n \in \{1, 2, 3, 4, 5\}$ *and for every* $K \in \mathcal{M}_{n,n}(\mathbb{R})$,

(2) $$\rho_0(K) = \rho_1(K).$$

For every $n \in \mathbb{N} \setminus \{1, 2, 3, 4, 5\}$, *there exists* $K \in \mathcal{M}_{n,n}(\mathbb{R})$ *such that* $\rho_0(K) < \rho_1(K)$.

Open problem: Is $\rho_0(G'(0)) < 1$ a sufficient condition for local exponential stability (for the H^2-norm) in the nonlinear case?

Commercial break



JMC, Control and nonlinearity, Mathematical Surveys and Monographs, 136, 2007, 427 p. Pdf file freely available from my web page.

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Mathematical Control and Related Fields

Editors in Chief: Jean-Michel Coron and Xu Zhang

American Institute of Mathematical Sciences

Proof of the exponential stability if A is constant and G is linear

Main tool: a Lyapunov approach. $A(y) = \Lambda$, $G(y) = Ky$. For simplicity, all the λ_i's are positive. A Lyapunov function candidate is

(1) $$V(y) := \int_0^1 y^{\mathrm{tr}} Q y e^{-\mu x} dx, \ Q \text{ is positive symmetric.}$$

If Q is diagonal, one gets

(2) $$\begin{aligned} \dot{V} &= -\int_0^1 (y_x^{\mathrm{tr}} \Lambda Q y + y^{\mathrm{tr}} Q \Lambda y_x) e^{-\mu x} dx \\ &= -\mu \int_0^1 y^{\mathrm{tr}} \Lambda Q y \, e^{-\mu x} dx - B, \end{aligned}$$

with

(3) $$B := [y^{\mathrm{tr}} \Lambda Q y e^{-\mu x}]_{x=0}^{x=1} = y(1)^{\mathrm{tr}} (\Lambda Q e^{-\mu} - K^{\mathrm{tr}} \Lambda Q K) y(1).$$

Let $D \in \mathcal{D}_{n,+}$ be such that $\|DKD^{-1}\| < 1$ and let $\xi := Dy(1)$. We take $Q = D^2\Lambda^{-1}$. Then

$$B = e^{-\mu}|\xi|^2 - |DKD^{-1}\xi|^2. \tag{1}$$

Therefore it suffices to take $\mu > 0$ small enough.

Remark

Introduction of μ:

- *JMC (1998) for the global asymptotic stabilization of the Euler equations.*
- *Cheng-Zhong Xu and Gauthier Sallet (2002) for symmetric linear hyperbolic systems.*

New difficulties if $A(y)$ depends on y

We try with the same V:

(1)
$$\begin{aligned}\dot{V} &= -\int_0^1 \left(y_x^{\mathsf{tr}} A(y)^{\mathsf{tr}} Q y + y^{\mathsf{tr}} Q A(y) y_x\right) e^{-\mu x} dx \\ &= -\mu \int_0^1 y^{\mathsf{tr}} A(y) Q y e^{-\mu x} dx - B + N_1 + N_2\end{aligned}$$

with

(2)
$$N_1 := \int_0^1 y^{\mathsf{tr}} (QA(y) - A(y)Q) y_x e^{-\mu x} dx,$$

(3)
$$N_2 := \int_0^1 y^{\mathsf{tr}} \left(A'(y) y_x\right)^{\mathsf{tr}} Q y e^{-\mu x} dx$$

Solution for N_1

Take Q depending on y such that $A(y)Q(y) = Q(y)A(y)$, $Q(0) = D^2F(0)^{-1}$. (This is possible since the eigenvalues of $F(0)$ are distinct.) Now

$$\dot{V} = -\mu \int_0^1 y^{\mathrm{tr}} A(y)Q(y)ye^{-\mu x}dx - B + N_2 \tag{1}$$

with

$$N_2 := \int_0^1 y^{\mathrm{tr}} \big(A'(y)y_x Q(y) + A(y)Q'(y)y_x\big)^{\mathrm{tr}} ye^{-\mu x}dx. \tag{2}$$

What to do with N_2?

Solution for N_2

New Lyapunov function:

(1) $$V(y) = V_1(y) + V_2(y) + V_3(y)$$

with

(2) $$V_1(y) = \int_0^1 y^{\text{tr}} Q(y) y \, e^{-\mu x} dx,$$

(3) $$V_2(y) = \int_0^1 y_x^{\text{tr}} R(y) y_x \, e^{-\mu x} dx,$$

(4) $$V_3(y) = \int_0^1 y_{xx}^{\text{tr}} S(y) y_{xx} \, e^{-\mu x} dx,$$

where $\mu > 0$, $Q(y)$, $R(y)$ and $S(y)$ are symmetric positive definite matrices.

Choice of Q, R and S

- Commutations:

(1) $$A(y)Q(y) - Q(y)A(y) = 0,$$

(2) $$A(y)R(y) - R(y)A(y) = 0,$$

(3) $$A(y)S(y) - S(y)A(y) = 0.$$

-

(4) $$Q(0) = D^2A(0)^{-1},\ R(0) = D^2A(0),\ S(0) = D^2A(0)^3.$$

Estimates on $\dot{V}$

Lemma

If $\mu > 0$ *is small enough, there exist positive real constants* α, β, δ *such that, for every* $y : [0,1] \to \mathbb{R}^n$ *such that* $|y|_{C^0([0,1])} + |y_x|_{C^0([0,1])} \leqslant \delta$, *we have*

$$\frac{1}{\beta}\int_0^1 (|y|^2 + |y_x|^2 + |y_{xx}|^2)dx \leqslant V(y) \leqslant \beta \int_0^1 (|y|^2 + |y_x|^2 + |y_{xx}|^2)dx,$$

$$\dot{V} \leqslant -\alpha V.$$

...

Why this miracle?

To explain simply the reason of this miracle, we assume that $n = 1$: there is no more problem of commutation of matrices. We simply take

(1) $$V_1 := \int_0^1 y^2 e^{-\mu x} dx,\ V_2 := \int_0^1 \alpha^2 e^{-\mu x} dx,\ V_2 := \int_0^1 \beta^2 e^{-\mu x} dx,$$

with $\alpha := y_x$ and $\beta := y_{xx}$. Note that, differentiating $y_t + A(y) y_x = 0$ with respect to x, one gets

(2) $$\alpha_t + A(y)\alpha_x + A'(y)\alpha^2 = 0.$$

(3) $$\begin{aligned} \dot{V}_2 &= -2\int_0^1 (A(y)\alpha_x + A'(y)\alpha^2)\alpha e^{-\mu x} dx \\ &= -\int_0^1 (\mu(A(y)\alpha^2 + A'(y)\alpha^3) e^{-\mu x} dx + \text{boundary terms}. \end{aligned}$$

Still not good: one can not bound $\int_0^1 |\alpha^3| dx$ by $(\int_0^1 \alpha^2 dx)^{3/2}$. But it sounds better since we do not have to bound a derivative of a function by the function. Encouraged, one keeps going.

Differentiating $\alpha_t + A(y)\alpha_x + A'(y)\alpha^2 = 0$ with respect to x, one gets

$$\beta_t + A(y)\beta_x + 3A'(y)\alpha\beta + A''(y)\alpha^3 = 0. \tag{1}$$

Hence

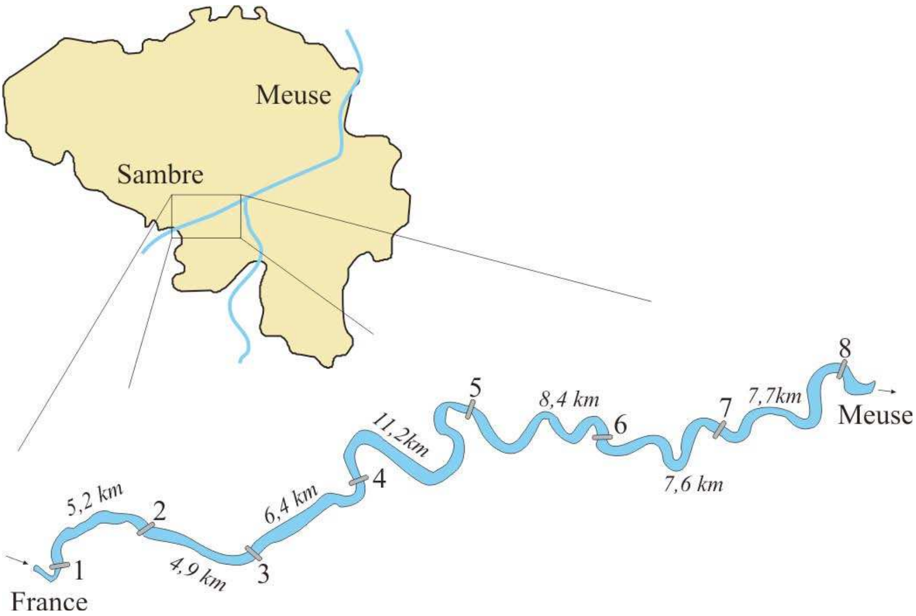
$$\begin{aligned} \dot{V}_2 &= -2\int_0^1 (A(y)\beta_x + 3A'(y)\alpha\beta + A''(y)\alpha^3)\beta e^{-\mu x} dx \\ &= -\int_0^1 (\mu A(y)\beta^2 + 5A'(y)\alpha\beta^2 + 2A''(y)\alpha^3\beta)e^{-\mu x} dx \\ &\qquad + \text{boundary terms.} \end{aligned} \tag{2}$$

It then suffices to use the Sobolev inequality

$$\max\{|\varphi(x)|;\, x \in [0,1]\} \leqslant C\left(\int_0^1 (\varphi^2 + \varphi'^2) dx\right)^{1/2}. \tag{3}$$

...

La Sambre (The same + Luc Moens)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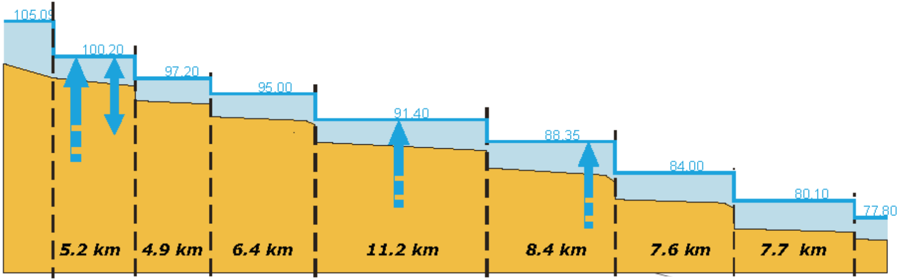
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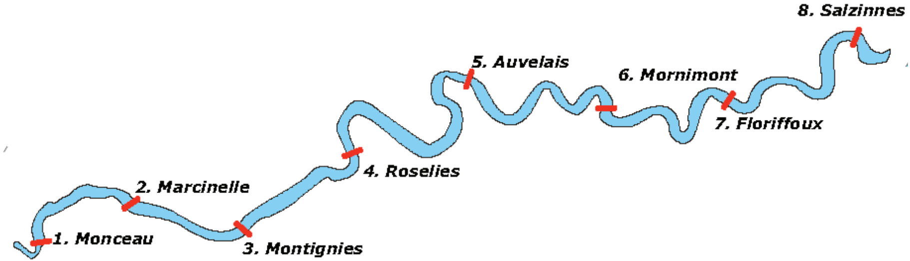
1. Monceau
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6. Mornimont
7. Floriffoux
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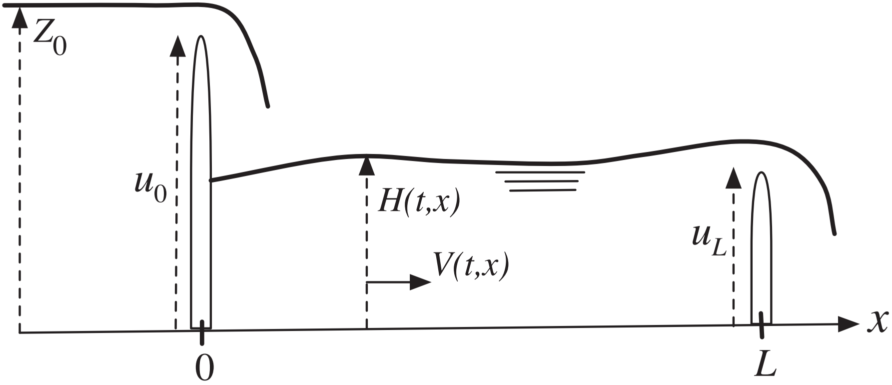

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$H(t,x)$
$V(t,x)$
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The Saint-Venant equations

The index i is for the i-th reach.
Conservation of mass:

(1) $$H_{it} + (H_i V_i)_x = 0,$$

Conservation of momentum:

(2) $$V_{it} + \left(gH_i + \frac{V_i^2}{2} \right)_x = 0.$$

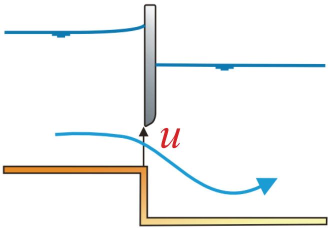
Flow rate: $Q_i = H_i V_i$.

Barré de Saint-Venant
(Adhémar-Jean-Claude)
1797-1886

Théorie du mouvement non permanent des eaux, avec applications aux crues des rivières et à l'introduction des marées dans leur lit, C. R. Acad. Sci. Paris Sér. I Math., vol. 53 (1871), pp.147–154.

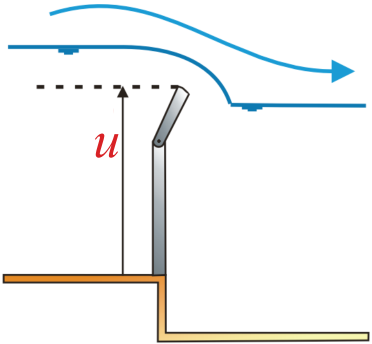
Boundary conditions

Underflow (sluice)



$$Q = K\sqrt{u(H_{up} - H_{down})}$$

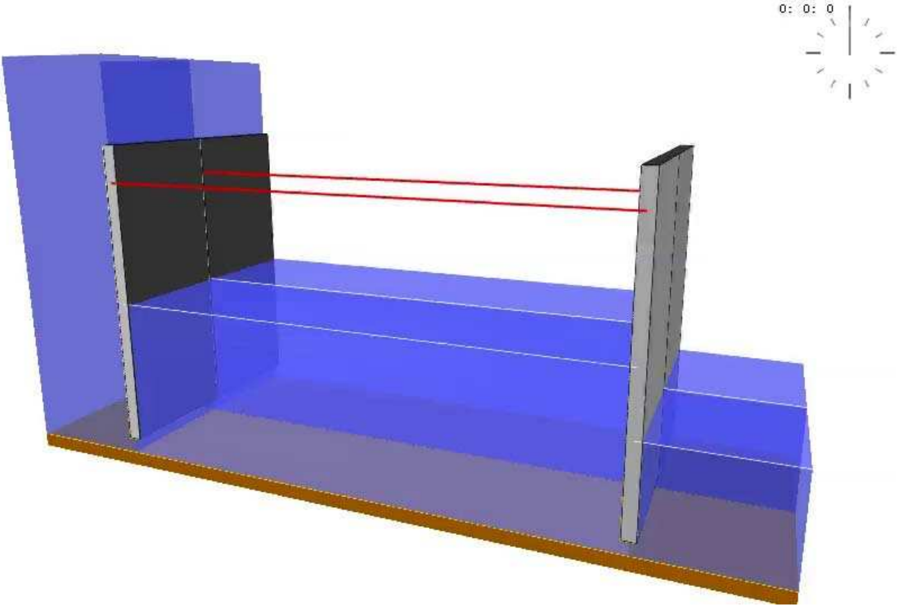
Overflow (spillway)



$$Q = K(H_{up} - u)^{3/2}$$

La Sambre: Gates

Closed loop versus open loop



Closed loop versus open loop

Work in progress: La Meuse

Balance laws

The partial differential system is now

$$(1) \qquad y_t + A(x,y)y_x + B(x,y) = 0 + \text{boundary conditions.}$$

We study only the linearized system around $y = 0$, i.e. the linear system

$$(2) \qquad y_t + \Lambda y_x + Ly = 0 + \text{linear boundary conditions.}$$

We also assume that we control $y_+(t,0)$ and $y_-(t,1)$. Hence the control system is (2) together with the boundary conditions $y_+(t,0) = u_+(t)$, $y_-(t,1) = u_-(t)$. Since the system is linear, one does not need to consider anymore V_2 and V_3. Then natural candidates for (control) Lyapunov are the *basic* functional

$$(3) \qquad V(y) := \int_0^1 y^{\text{tr}} Q(x) y dx, \text{where } Q(x)^{\text{tr}} = Q(x) \text{ and } Q(x) > 0.$$

Note that the interest of these *basic* (potential) control Lyapunov functions is that they lead to "local" control laws: the feedback laws depend only on the value of $y_-(t,0)$ and $y_+(t,1)$. (These values are usually easy to measure.)

A necessary and sufficient condition when $n = 2$ and $m = 1$

Open problem: Find a necessary and sufficient condition for the existence of a *basic* control Lyapunov. However we know the answer for $n = 2$ and $m = 1$. In this case, after a suitable change of variables the linear system takes the form:

(1) $$\begin{cases} y_{1t} + \lambda_1(x) y_{1x} + a(x) y_2 = 0, \\ y_{2t} + \lambda_2(x) y_{2x} + b(x) y_1 = 0. \end{cases}$$

with $\lambda_1(x) > 0 > \lambda_2(x)$. Let us recall that control is on both sides:

(2) $$y_1(t, 0) = u_1(t),\ y_2(t, 1) = u_2(t).$$

Theorem (G. Bastin and JMC (2010))

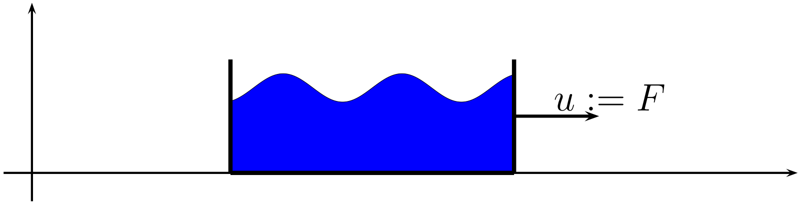
There exists a basic control Lyapunov function for (1)-(2) *if and only if the maximal solution* η *of the Cauchy problem*

(3) $$\eta' = \left| a + b\eta^2 \right|,\ \eta(0) = 0,$$

is defined on $[0, 1]$.

Complements

There are linear cases where there are no stabilizing feedback laws of the form $(y_1(0), y_2(1))^{\text{tr}} = K(y_1(1), y_2(0))^{\text{tr}}$.
A solution: Use Krstic's backstepping approach (R. Vazquez, M. Krstic and JMC (2011); R. Vazquez, JMC, M. Krstic and G. Bastin (2011)).
An open problem: Stabilization of the following $1 - D$ water tank control system around equilibria.



This system is modeled with the Saint-Venant equations. The local controllability of this control around equilibria is already known: JMC (2002).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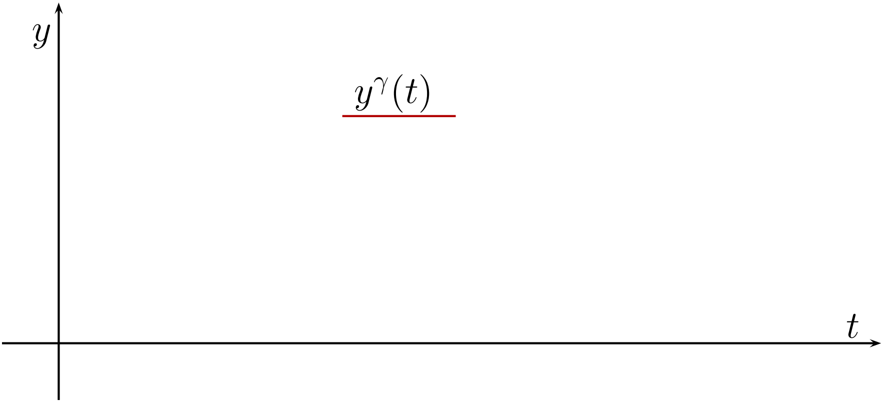
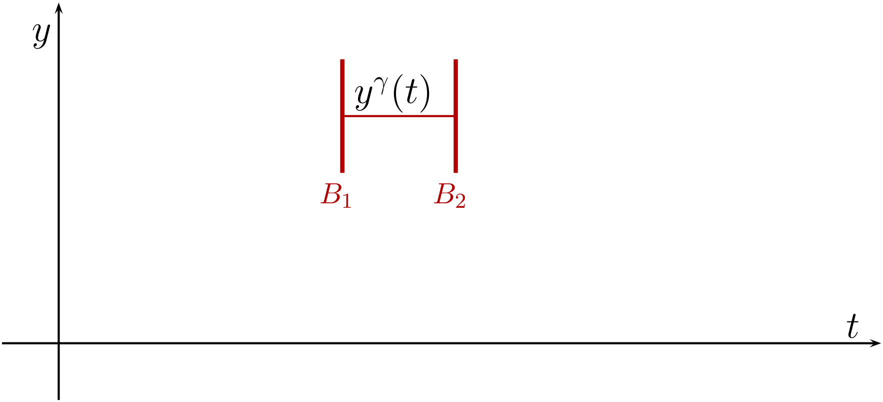
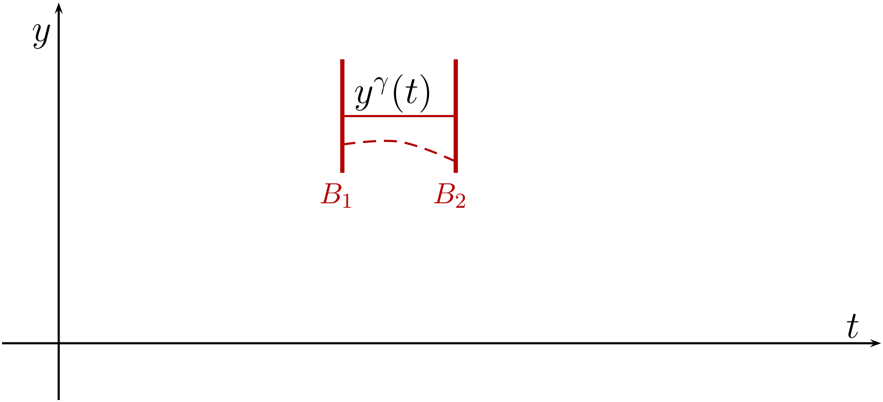
Sketch of the proof of the local controllability



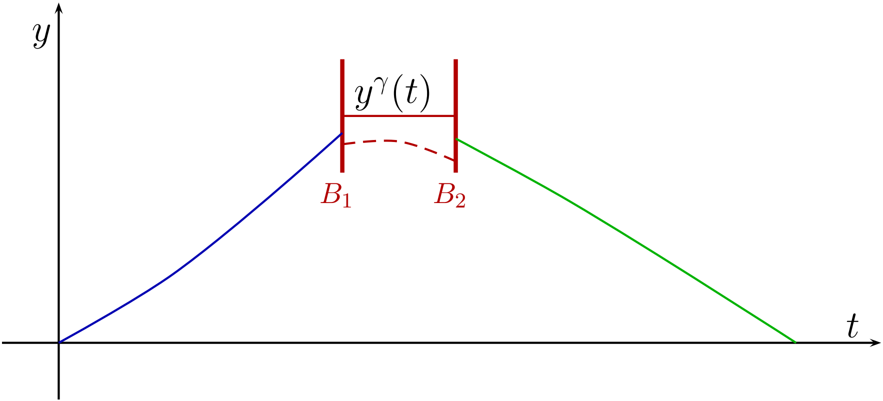
Sketch of the proof of the local controllability



Sketch of the proof of the local controllability



Sketch of the proof of the local controllability



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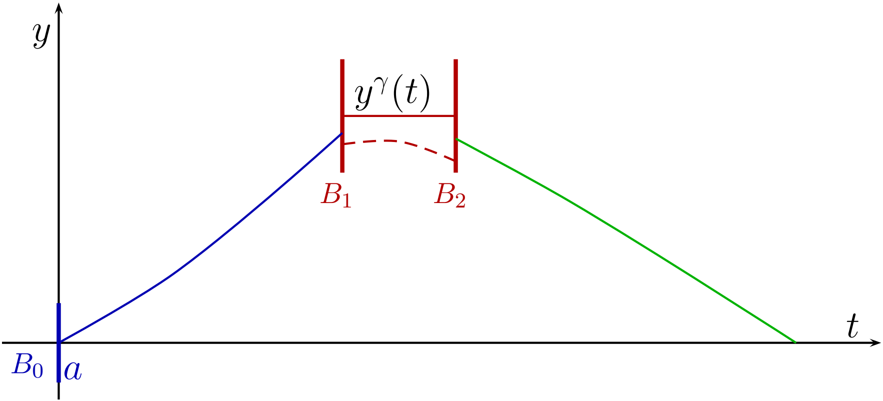

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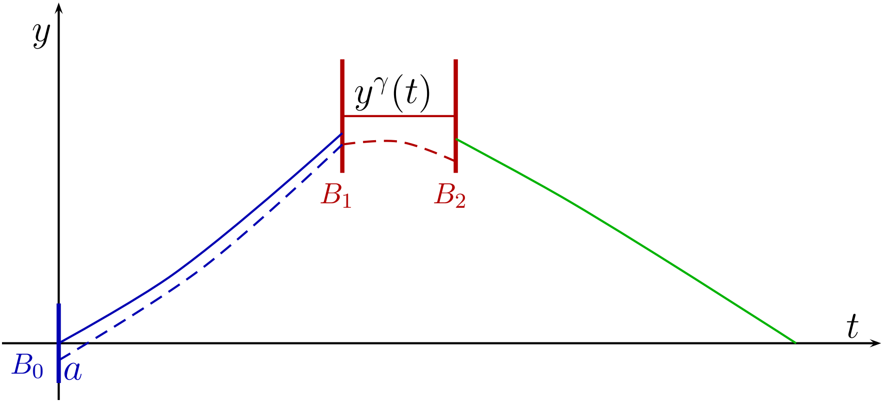

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