On the Marginal Instability of Linear Switched Systems

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Linear Switched System (continuous time) :

$$(S)\quad \dot{x}(t) = A(t)x(t) \qquad x \in \mathbb{R}^n,\ A(t) \in \mathcal{A} \subset \mathbb{R}^{n\times n}.$$

- $A(\cdot)$ = any meas. function $[0,+\infty) \to \mathcal{A}$; referred as a switching law.
- $\mathcal{A}$ compact.

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- $\mathcal{A}$ compact.

Example : $\mathcal{A} = \{A^0, A^1\}$ or $\mathcal{A} = \{\lambda A^0 + (1-\lambda)A^1 \ : \ \lambda \in [0,1]\}$

Remark : wlog $\mathcal{A}$ can be taken convex

Stability and Lyapunov exponent

Maximal Lyapunov Exponent of $\mathcal{A}$ defined as

$$\rho(\mathcal{A}) = \sup_{A(\cdot),x(0)=1} \left(\limsup_{t\to\infty} \frac{1}{t} \log \|x(t)\| \right).$$

If $\mathcal{A} = \{A\}$, $\rho(\mathcal{A}) = \max_{\lambda\in\sigma(A)} \Re(\lambda)$.

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$\rho(\mathcal{A}) < 0$

(S) Uniformly Globally Asymptotic Stable (**UGAS**)
$\to$ Uniformly Exponentially Stable (**UES**), i.e. $\exists M, \lambda > 0$ s.t. $\forall x(0) \in \mathbb{R}^n$, $t \geq 0$, $A(\cdot)$ $\|x(t)\| \leq Me^{-\lambda t}\|x(0)\|$.

$\rho(\mathcal{A}) = 0$

- (S) **marginally stable:** all traj. bounded and $\exists$ traj. not CV to 0,
- (S) **marginally unstable:** $\exists$ unbounded traj.

$\rho(\mathcal{A}) > 0$

(S) **unstable:** $\exists$ traj. going to ∞ exponentially.

Case $\rho(\mathcal{A}) = 0$ and Irreducibility

Recall that marginal instability $\Rightarrow \rho(\mathcal{A}) = 0$.

up to a translation, $\mathcal{A} \rightsquigarrow \mathcal{A} - \rho(\mathcal{A})Id_n$, reduce to "Case $\rho(\mathcal{A}) = 0$".

$\Rightarrow$ Study of the case $\rho(\mathcal{A}) = 0$ crucial to understand stability properties.

But computation of $\rho(\mathcal{A})$ is VERY HARD in general even numerically. (Maybe not?? new results from Jungers, Parillo, ...!)

Definition

$\mathcal{A}$ irreducible if $\nexists\ 0 \subsetneq V \subsetneq \mathbb{R}^n$ invariant for every $A \in \mathcal{A}$.
$\mathcal{A}$ reducible otherwise.

N.B.

$$\mathcal{A} \text{ reducible} \iff A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \ \forall A \in \mathcal{A}$$

Barabanov Norm

$\rho(\mathcal{A}) = 0$ and $\mathcal{A}$ irreducible; $\|\cdot\|$ fixed norm.

$$\forall x_0 \in \mathbb{R}^n, \qquad v(x_0) := \sup_{\mathcal{A}(\cdot)} \limsup_{t\to\infty} \|x\big(t, x_0; \mathcal{A}(\cdot)\big)\|.$$

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Theorem (N. Barabanov)

$\rho(\mathcal{A}) = 0$ and $\mathcal{A}$ irreducible. Then $v : \mathbb{R}^n \to [0, +\infty)$ is a norm s. t.:

- $v(x(t)) \leq v(x(0))$ for every switching law $A(\cdot)$ and initial cond. $x(0)$;
- $\forall x(0)$, $\exists$ traj. $x(\cdot)$ s. t. $v(x(\cdot)) \equiv v(x(0))$.

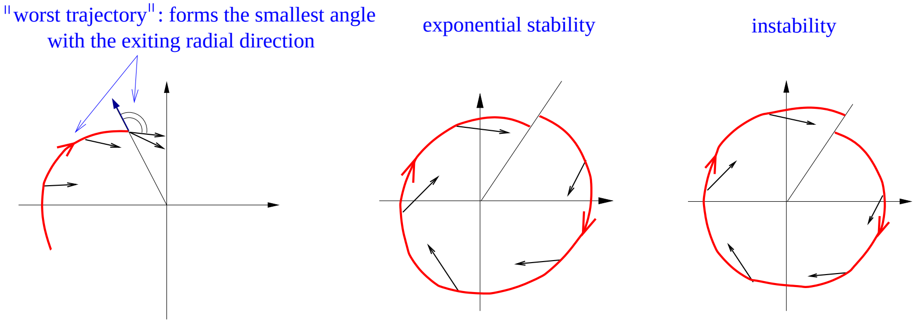
(Unit) Barabanov sphere $S_v = \{x \in \mathbb{R}^n,\ v(x) = 1\}$.

If $\rho(\mathcal{A}) = 0$ and $\mathcal{A}$ irreducible, system is marginally stable.

2D systems

$$\dot{x} = \sigma(t)A^0 x + (1-\sigma(t))A^1 x \qquad x \in \mathbb{R}^2,\ \sigma(t) \in \{0,1\}.$$

Solved (e.g. cf. U. Boscain): complete description of stability cases.
$\longrightarrow$ Method based on notion of worst trajectory WT



If $\rho(\mathcal{A}) = 0$, (often) WT corr. periodic pw. cst switching law s.t. -1 eigenvalue of $e^{t_0 A^0} e^{t_1 A^1}$.

Reducibility and Invariant flags

Maximal invariant flag for $\mathcal{A}$:

$$\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_k = \mathbb{R}^n$$

where

- E_i invariant w.r.t. each $A \in \mathcal{A}$,
- $\nexists V$ invariant w.r.t. $\mathcal{A}$ such that $E_{i-1} \subsetneq V \subsetneq E_i$.

Coordinate system adapted to the flag $\rightarrow$ $A = \begin{pmatrix} A_{11} & A_{12} & \cdots & & \\ 0 & A_{22} & A_{23} & \cdots & \\ 0 & 0 & A_{33} & A_{34} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & A_{kk} \end{pmatrix}$, $\forall A \in \mathcal{A}$

Define $\mathcal{A}_i = \{A_{ii} \ : \ A \in \mathcal{A}\}$. Then $\mathcal{A}_i$ irreducible and $\rho(\mathcal{A}) = \max_i \rho(\mathcal{A}_i)$.

Remark: maximal invariant flag for $\mathcal{A}$ not unique.

However, from Jordan-Hölder Theorem, subsystems $\mathcal{A}_i$ independent on the flag up to permutations.

Worst "polynomial" behavior - First Estimate

From now on assume $\rho(\mathcal{A}) = 0$

Let $\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{k-1} \subsetneq E_k = \mathbb{R}^n$ a maximal invariant flag. Then

$$\rho(\mathcal{A}_i) \leq 0 \quad i = 1, \ldots, k \ \text{ and } L := \# \{\mathcal{A}_i, \ \ \rho(\mathcal{A}_i) = 0\} \geq 1.$$

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Theorem (V. Protasov)

Block form + variation of constant $\rightarrow \exists C > 0$, $\forall x(0) \in \mathbb{R}^n$

$$\|x(t)\| \leq C(1 + t^{L-1})\|x(0)\|, \quad \forall t \geq 0.$$

In principle the system could be unstable with polynomial growth.

?? Relationships between subsystems to get unbounded growth ??

Towards "polynomial" blow-up - Case Study

- $\rho(\mathcal{A}) = 0$, $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ $\forall A \in \mathcal{A}$; $\|\cdot\|$ fixed norm.
- $i = 1, 2$, $\mathcal{A}_i$ irreduc., $\rho(\mathcal{A}_i) = 0$; v_i, S_i Barabanov norms and unit spheres.

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For any switching law $A(\cdot)$, let $R_i^{A(\cdot)}(\cdot, \cdot)$ corresp. resolvant.
$t(\geq t_1) \mapsto v_i(R_i(t, t_1))$ bdd by 1 and non-increasing.

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$$x_1(t) = R_1^{A(\cdot)}(t,0)x_1(0) + \int_0^t R_1^{A(\cdot)}(t,s)A_{12}(s)R_2^{A(\cdot)}(s,0)x_2(0)ds$$

$$\|x_1(t)\| \leq K\|x_1(0)\| + M\|x_2(0)\|\int_0^t v_1(R_1^{A(\cdot)}(t,s))v_2(R_2^{A(\cdot)}(s,0))ds.$$

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$\int_0^t \cdots$ bounded by t. To make it unbounded, (better have)
BOTH $v_i(R_i^{A(\cdot)}(t,s))$ must not CV to zero,

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$\int_0^t \cdots$ bounded by t. To make it unbounded, (better have)
BOTH $v_i(R_i^{A(\cdot)}(t,s))$ must not CV to zero,
(Even better) With SAME switching law $A(\cdot)$, two trajs. on S_1 and S_2.

Resonance

Definition (Resonance Chain)

$\mathcal{A}$ switched system s.t. $\rho(\mathcal{A}) = 0$. Let $(\mathcal{A}_i)$ diag. subsystems of max. inv. flag.
Resonance Chain of length $l \geq 2$: $(\mathcal{A}_{i_1}, \cdots, \mathcal{A}_{i_l})$

- $i_1 < \cdots < i_l$, $\rho(\mathcal{A}_{i_j}) = 0$, $1 \leq j \leq l$;
- $\exists$ common switching law $A(\cdot)$ s.t. $A_{i_j}(\cdot)$ gives rise to traj. on S_{i_j}, $1 \leq j \leq l$.

$$A = \begin{pmatrix} A_{11} & A_{12} & & & & & \\ 0 & A_{22} & A_{23} & & \cdots & \cdots & \\ 0 & 0 & A_{33} & A_{34} & & & \\ \vdots & & & A_{44} & A_{45} & & \\ \vdots & & \ddots & & A_{55} & A_{56} & \\ \vdots & & & & & A_{66} & A_{67} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & A_{77} \end{pmatrix}$$

$(\mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_7)$ resonance chain of length 3

(i) $\rho(\mathcal{A}_i) = 0$, $i = 4, 5, 7$

(ii) $\exists$ common switching law giving rise to trajs. on Barabanov spheres of $\mathcal{A}_i$'s

First result

Theorem (YC, P. Mason, M. Sigalotti)

Let $\mathcal{A}$ linear switched system, marginally unstable. Then,

- *$\mathcal{A}$ is reducible*
- *$\exists$ resonance chain of length $l \geq 2$.*

Simplest nontrivial case of reducible systems:

$$\mathcal{A} = conv\{A^0, A^1\}, \quad A^0 = \begin{pmatrix} A^0_{11} & A^0_{12} \\ 0 & A^0_{22} \end{pmatrix}, \quad A^1 = \begin{pmatrix} A^1_{11} & A^1_{12} \\ 0 & A^1_{22} \end{pmatrix}.$$

Assume A^0, A^1 Hurwitz and $\rho(\mathcal{A}) = 0$. Then,

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Assume A^0, A^1 Hurwitz and $\rho(\mathcal{A}) = 0$. Then,

Theorem (Pulvirenti's Conjecture)

If $n = 2, 3$ no marginal instability.

For $n = 4$ marginal instability is possible
with trajectories going to infinity polynomially as t

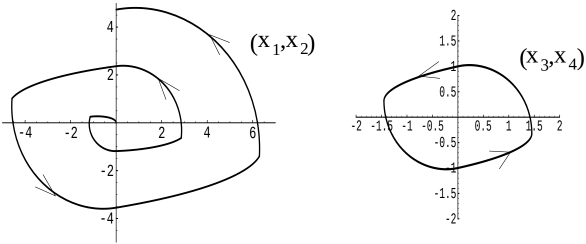
Numerical example

$$A^0 = \begin{pmatrix} A_*^0 & Id \\ 0 & A_*^0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} A_*^1 & Id \\ 0 & A_*^1 \end{pmatrix}$$

$$\text{Choose} \quad A_*^0 = \begin{pmatrix} -1 & -\alpha \\ \alpha & -1 \end{pmatrix}, \quad A_*^1 = \begin{pmatrix} -1 & -\alpha \\ 1/\alpha & -1 \end{pmatrix}.$$

For $\alpha \sim 4.5047$ one has $\rho(\mathcal{A}_*) = 0$. Worst traj. (WT): "defined" by (t_0, t_1).

$x = (x_1, x_2, x_3, x_4)$



$n = 4$: a converse result

$$A^0 = \begin{pmatrix} A^0_{11} & A^0_{12} \\ 0 & A^0_{22} \end{pmatrix}, \quad A^1 = \begin{pmatrix} A^1_{11} & A^1_{12} \\ 0 & A^1_{22} \end{pmatrix}$$

$$\mathcal{A}_1 = conv\{A^0_{11}, A^1_{11}\}$$
$$\mathcal{A}_2 = conv\{A^0_{22}, A^1_{22}\}$$

Resonance: $\rho(\mathcal{A}_1) = \rho(\mathcal{A}_2) = 0$ and SAME (t_0, t_1).

Theorem (YC, P.Mason, M. Sigalotti)

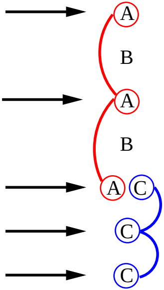
If $n = 4$ and $\mathcal{A}_1, \mathcal{A}_2$ are in resonance then, generically w.r.t. (A^0_{12}, A^1_{12}), the system is polynomially unstable as t.

Resonance Degree

Definition (Resonance Degree of a switched system $\mathcal{A}$ with $\rho(\mathcal{A}) = 0$)

Two resonance chains are connected if smallest index of one $\geq$ largest index of the other.
Chord of resonance chains $=$ collection of consecutive connected resonance chains.
Chord degree $= \Sigma$ resonance chains lengths - Nb. resonance chains (for each chord).
Resonance Degree associated to $\mathcal{A} =$ Max. chord degrees.

$$A = \begin{pmatrix} A_{11} & A_{12} & & & & & \\ 0 & A_{22} & A_{23} & & \cdots & \cdots & \\ 0 & 0 & A_{33} & A_{34} & & & \\ \vdots & & & A_{44} & A_{45} & & \\ \vdots & & \ddots & & A_{55} & A_{56} & \\ \vdots & & & & & A_{66} & A_{67} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & A_{77} \end{pmatrix}$$



letters A, B, C $=$ resonance chains. Connected chains $= (A, C), (B, C)$. Not (A, B).
resonance degree of $\mathcal{A}$ equal to $4 = 3 + 3 - 2$.

Asymptotic behavior of trajectories

Theorem (YC, P. Mason, M. Sigalotti)

L = *resonance degree of* $\mathcal{A}$. *Then* $\exists C > 0$ *s.t.* $\forall x(0) \in \mathbb{R}^n$, $\forall t \geq 0$,

$$(EST) \quad \|x(t)\| \leq C(1 + t^L)\|x(0)\|.$$

Conversely, in special cases, $\exists \hat{C} > 0$ s.t. for any $t > 0$, $\exists$ switching law and $x(0) \neq 0$ s.t.

$$\|x(t)\| \geq \hat{C} t^L \|x(0)\|,$$

i.e., (in special cases) optimality of **(EST).**

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i.e., (in special cases) optimality of **(EST).**

BUT, (in special cases) for every single traj. $x(\cdot)$,

$$\lim_{t \to \infty} \frac{\|x(t)\|}{t^L} = 0.$$

(e.g. resonance degree only reached for a chord with at least TWO connected resonance chains).

Discrete time systems

Discrete time switched systems:

$$z(k+1) = M(k)\, z(k)\,, \quad \text{where} \quad M(k) \in \mathcal{M}\,,$$

Stability characterized by JSR = Joint Spectral Radius:

$$\rho := \limsup_{k\to\infty} \Big(\max_{M(1),\dots,M(k)\in\mathcal{M}} \|M(k)\cdots M(1)\|^{1/k} \Big)$$

All the results presented above easily adapted to discrete time switched systems.

Nonnegative integer matrices cf. Blondel-Jungers-Protasov

Discrete time case with $\mathcal{M}$ made by nonnegative integer matrices already been studied in the literature (see [Jungers-book 2009]).

$\exists$ complete characterization of maximal polynomial growth of trajectories, cf. Jungers-Protasov-Blondel (2008).

BUT their methods cannot be adapted to general case considered here.

Conclusion and open problems

Main results:

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Some open questions:

- Resonance degree L "generically" best estimate for polynomial growth?
- Crucial to study dynamical system generated on Barabanov sphere:
  - if $\mathcal{A}$ irreducible, $\exists$ periodic trajectory lying on Barabanov sphere?
  - examples of "chaotic" behavior on Barabanov sphere?

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Partial result for 3×3 Hurwitz stable irred. switched systems (H_3 SI-SS.).
$\mathcal{A} = \{A^0, A^1\}$, $\rho(\mathcal{A}) = 0$, $[A^0, A^1]$ Hurwitz, $rk(A^0 - A1) = 1$.

Theorem (Barabanov)

$\exists$! periodic traj. (4 bang arcs) attracting every traj. on Barabanov sphere ($\exists$! worst trajectory and it is bang-bang).

Open problem: Complete Poincaré-Bendixon theory for H_3 SI-SS.