Workshop on Perspectives and Future Directions in Systems and Control Theory

Rutgers University

May 23-27, 2011

Prospects for a Large Signal Nonlinear Input/Output Theory

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Harvard University

Eduardo Sontag: The reason we are here!

Some of my early interactions:

Gainsville, 1975-1976 –lots of polynomials

Bordeaux Conference, 1978-- lots of energy

Bielefeld-Rome, 1981—all manner of excess
(by almost everyone)

Three questions about this talk--

a) Why this topic? b) How might it advance our field?
c) Who might be influenced by efforts of this type?

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a) Some believe that mathematical models are the main vehicle by which engineers understand problems and that it may be possible to do a better job of understanding important classes of nonlinear systems.

b) If a theory explains enough interesting examples it will generate a standard vocabulary and thus facilitate understanding and communication.

c) If successful, such a theory may help scientists and engineers gain intuition about qualitative phenomena and facilitate both analysis and design.

What do we have to build on?

1. Linear Theory

2. Volterra Series

3. Small gain theory

4. Lie theoretic methods

5. Theory of abstract dynamical systems

And something a little new

What results can we hope for?

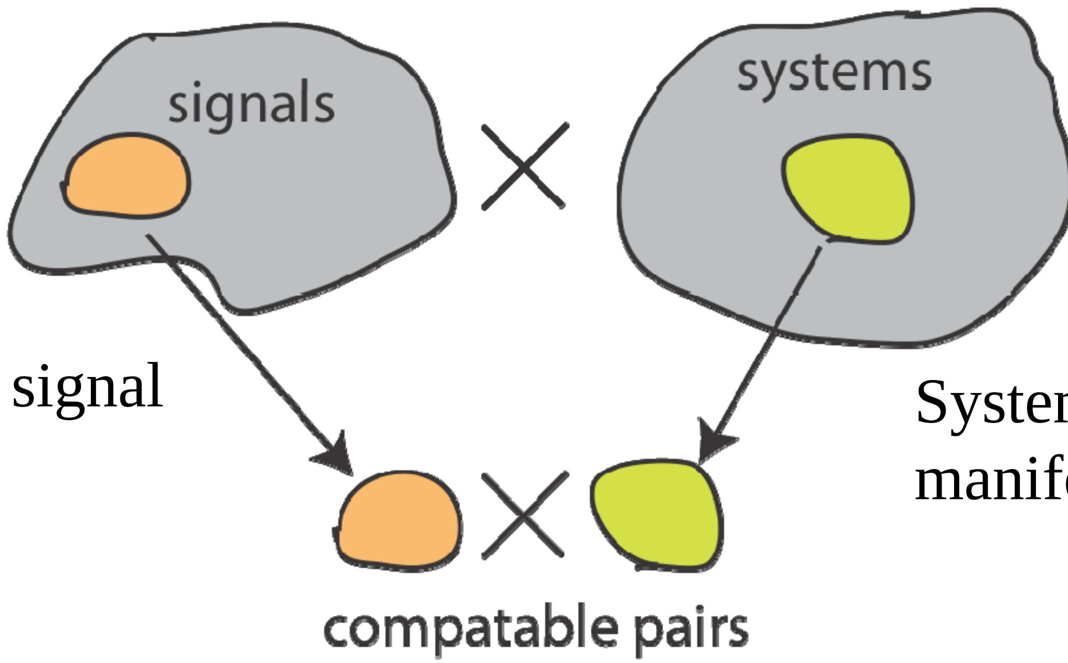
Some input/output relationships that describe, and let us reason about, strongly nonlinear phenomena such as:

a)systems with relays and/or saturating nonlinearities.

b)systems with multiple equiibria.

c)systems operating on pulses and/or nearly piecewise constant signals.

Working with suitable pairs: Don’t be too greedy!



Manifolds in signal space

Systems defined on manifolds

Manifolds are not just for differential equations and systems—they are also useful for describing sets of functions such as the subsets of Sobolev spaces characterizing pulses.

What I intend to show

By restricting the inputs to suitable sub manifolds of a Sobolev space and by making use of suitably defined closed one-forms defined on these sub manifolds, in some interesting cases it is possible to find a simple, easily understood, approximation to a highly nonlinear system. The restriction on the input space, when matched with a suitable restriction on the class of systems, results in a workable theory. For example, a pairing of this type shows promise in modeling aspects of neural systems.

Background on Volterra Series

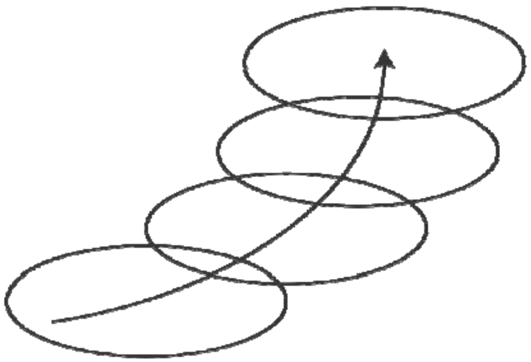
Assume that f and g are real, analytic, vector -valued functions with $f(0) = 0$. Consider
$\dot{x} = f(x) + ug(x)$; $x(0) = 0$
Then there is a locally convergent expansion
$x(t) = w_0 + \int_0^t w(t,\sigma)u(\sigma)d\sigma + \cdots$
with the convergence being on $[0,\infty)$ if $\partial f/\partial x$ has eigenvalues with negative real parts.
The omitted terms are integral operators acting on u the k^{th} being homogenous of degree k.
The kernels are exponentially decaying.

A hopeful, yet cautionary, example

Put suitable restrictions on u and consider

$$\dot{x} = -\sin 2\pi x + u$$

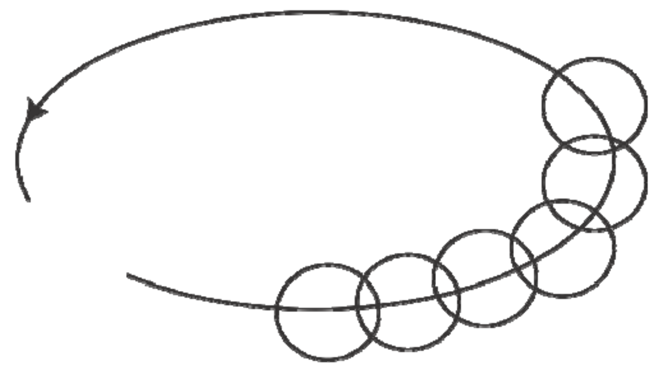
Near $(x(0), u) = (0, 0)$ we can express x as an "analytic functional" of u via a Volterra series. What stands in the way of its analytic continuation?



PB1

Limits on the domain of convergence

Any expansion near $x = n$ has kernels which decay and so for $||u||_\infty \leq a$ and some $a > 0$ all expansions represent solutions that return to their original value. Yet there are inputs in L_∞ that drive x from $x = n$ to m thus we see the need for something like a branch cut, now in L_∞

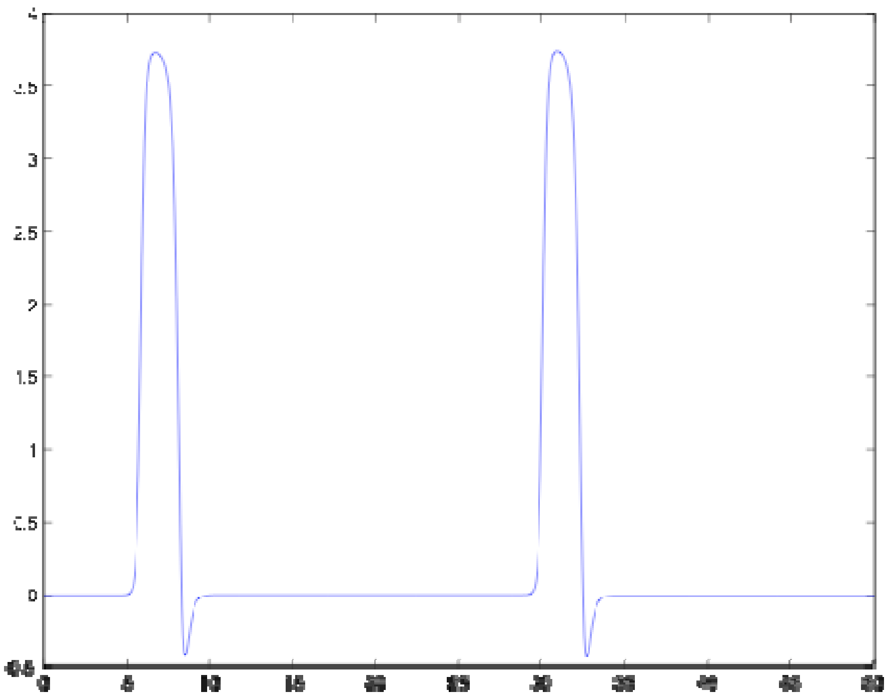


Slide 10

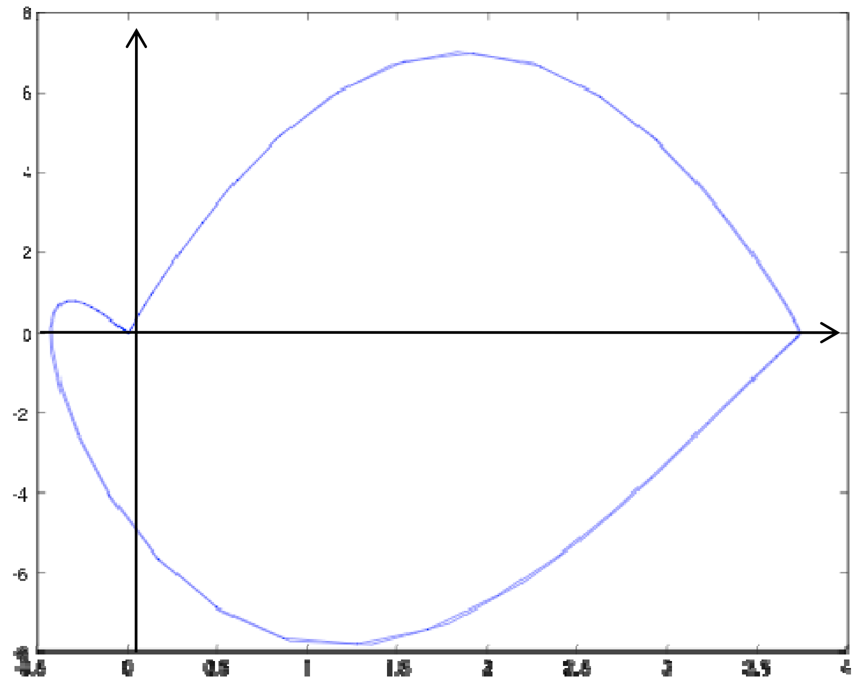
PB1 Prof. Brockett, 5/12/2011

Restricting the input space

Model for an Action Potential



u(t) verses time



du/dt verses u (phase plane plot)

Signals in the phase plane

Examples of restricting the input space

$$\dot{x} = -\sin 2\pi x + u$$

stable equilibria when $u = 0$ and $x \in \mathbb{Z}$

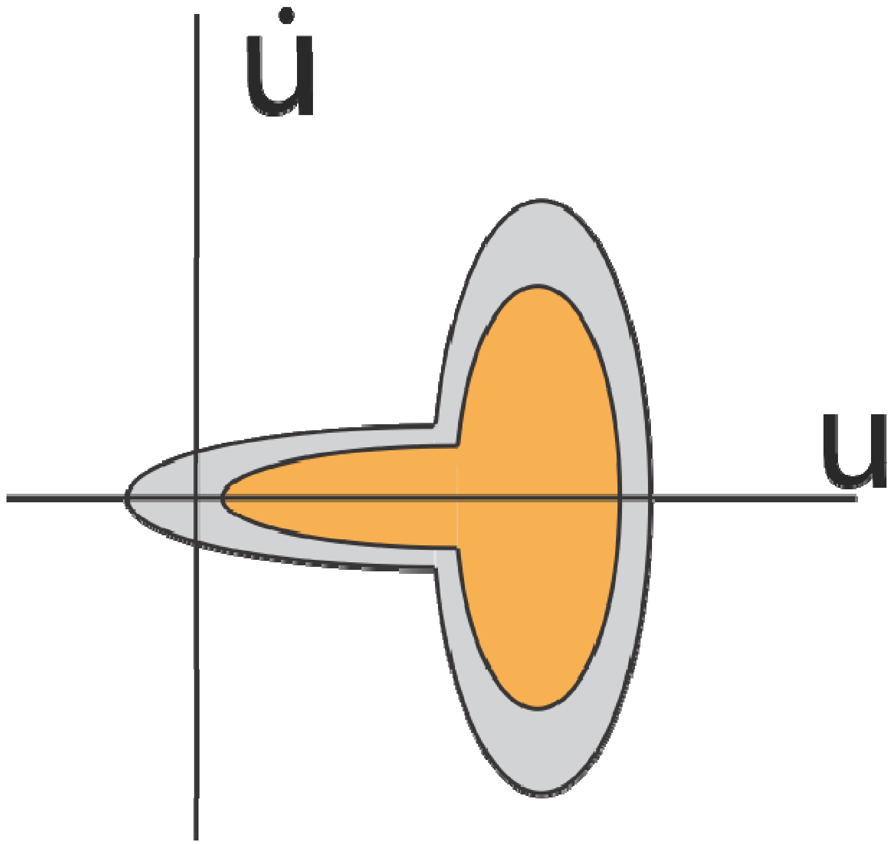
Make the definitions

$$\mathcal{A}_{kw} = \{u | |(k\dot{u})^2 + (u - .5)^2 - 1| < w\}$$

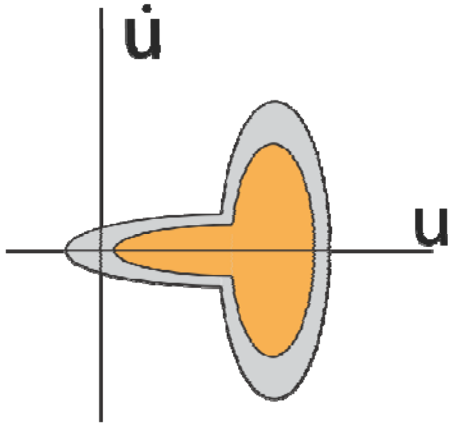
$$||u||_T = \sup_t \tfrac{1}{T} \sqrt{\int_{t-T}^{t} \dot{u}^2 \, dt}$$

$$x(t) = \int \tfrac{d}{dt} \tan^{-1} \tfrac{\dot{u}}{u} dt + x_r(t)$$

Signals in the phase plane: annular open subsets



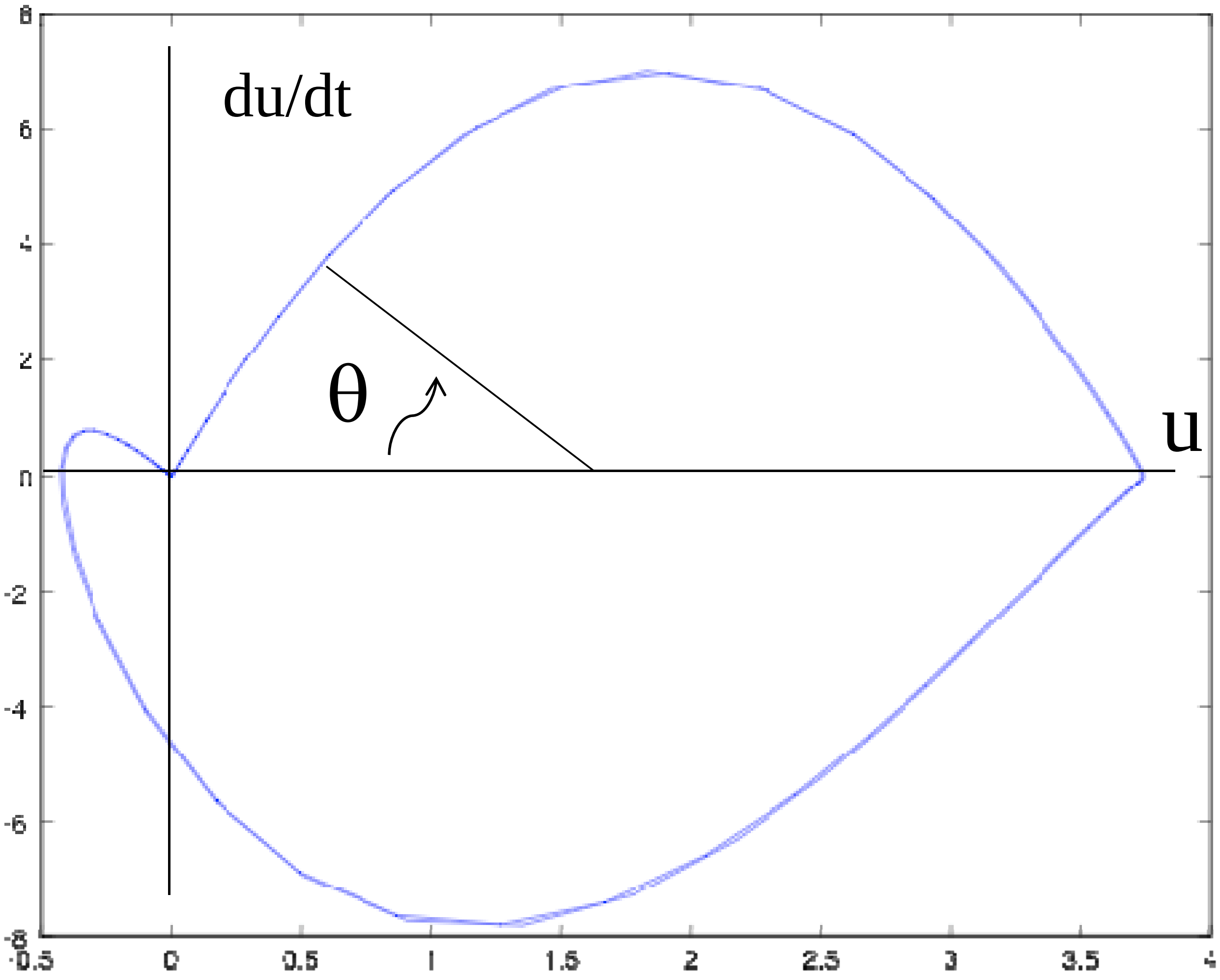
Submanifolds of Sobolev Spaces: Periodic systems



$$\dot{x} = -a\sin(2\pi x) + bu$$

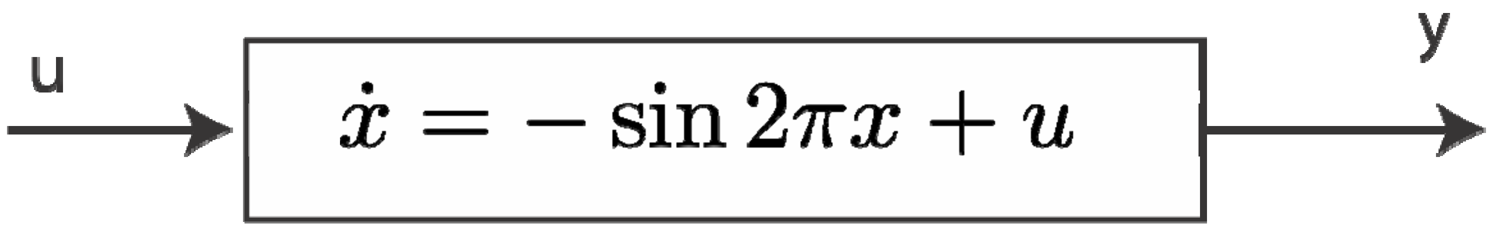
$$\begin{array}{ccc} \mathcal{U} \subset W_p^q & \longrightarrow & L_2[0,\infty) \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathcal{S} & \xrightarrow{s_{k+1} = \phi(s_k, v_k)} & \mathbb{R}^n \end{array}$$

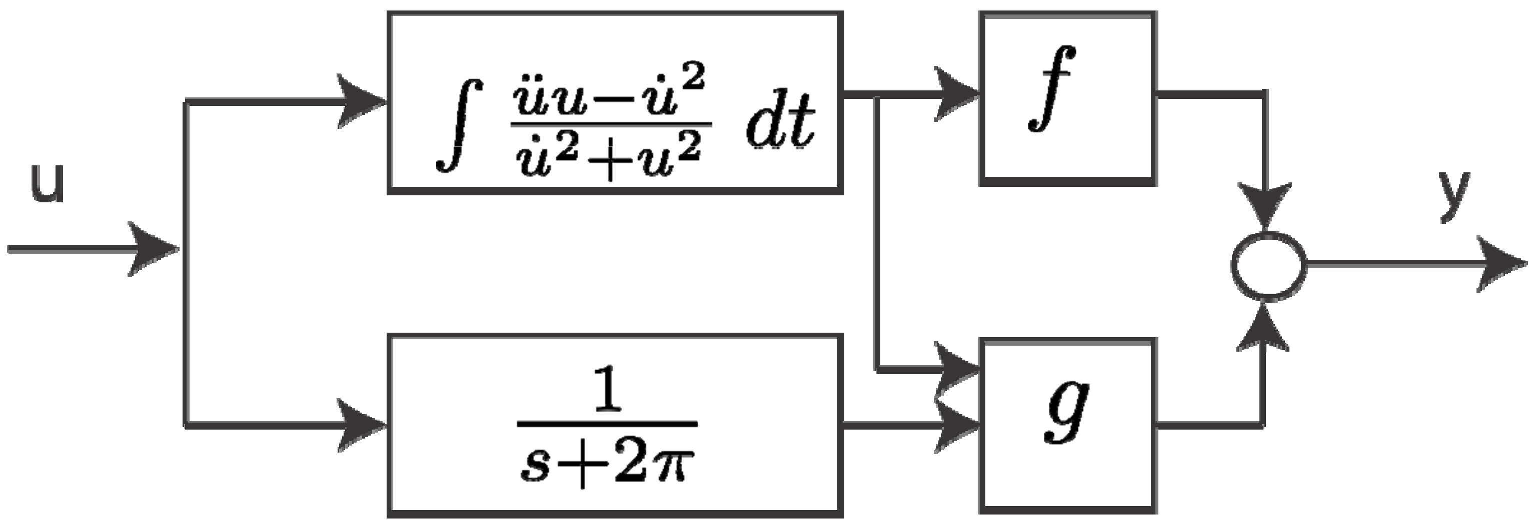
Closed but not exact forms: Angles



Phase plot for an action potential

Using a closed one-form to capture large signal effects

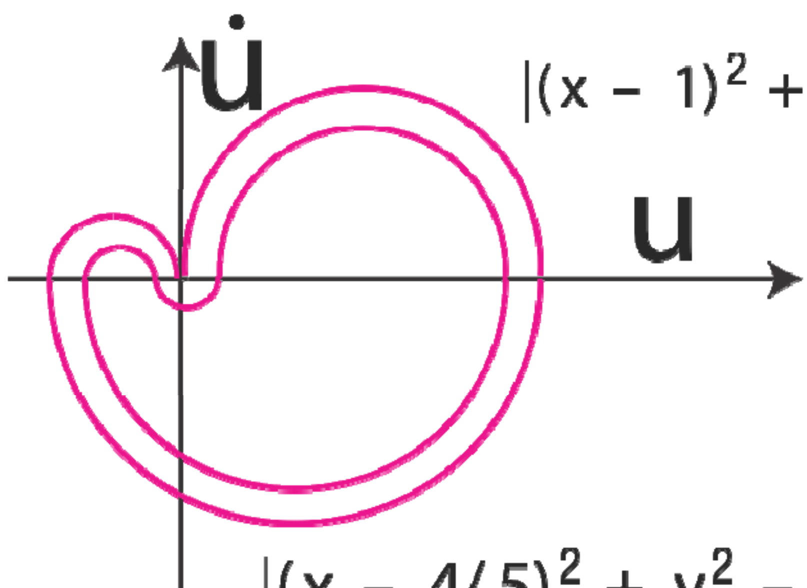




Refining the details, leaving the essentials alone

$$\dot{x} = -a\sin(2\pi x) + bu$$

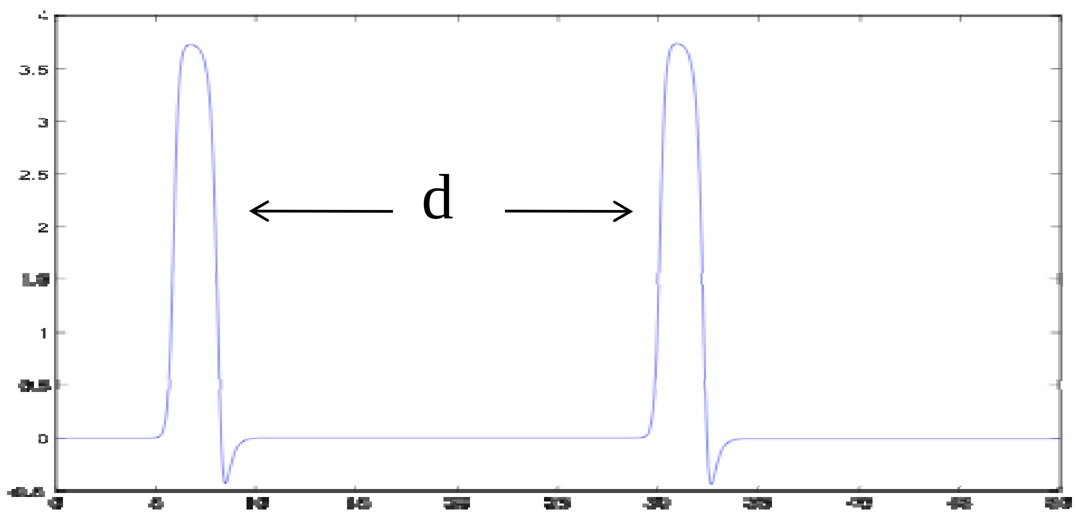
$|(x + 1/4)^2 + y^2 - 1/4| < c$



Cardioid $(x^2 + y^2 - 2ax)^2 = 4a^2(x^2 + y^2)$

An important point

Two signals that are close, in the sense that their $\dot{x}$ vs. x graphs are close, are not necessarily close in any Sobolev topology. The issue comes down to that of identifying a topology that describes a suitable neighborhood of a pulse. Does there exist a domain of holomorphy that includes inputs of the form shown, regardless of the value of d?

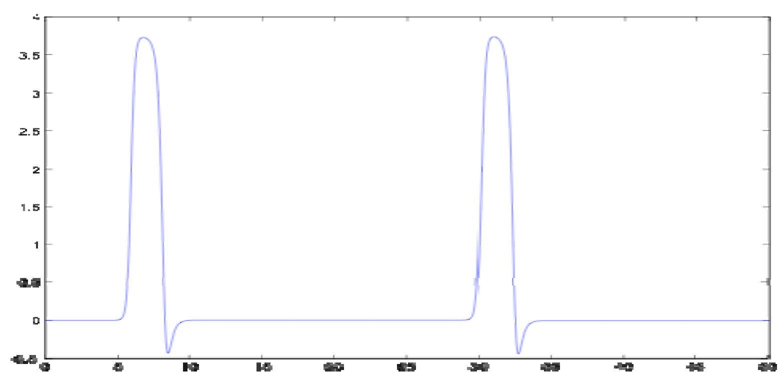


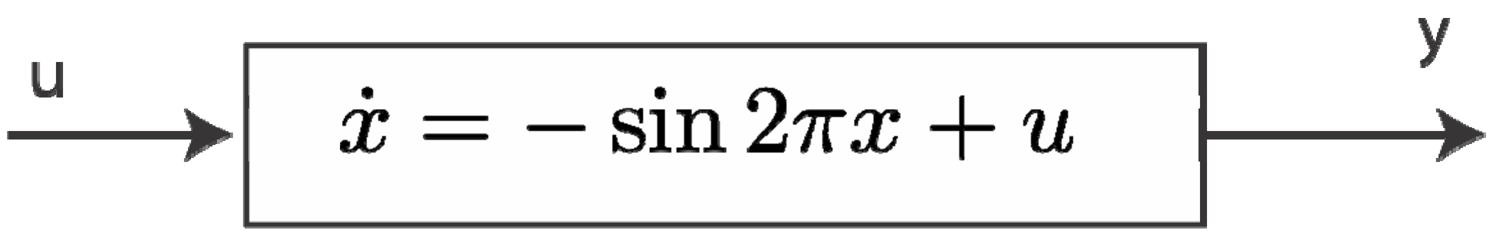
Because in between pulses the system relaxes to the equilibrium point at an exponential rate, the details of the response do depend on d. However the de Rham-like term,

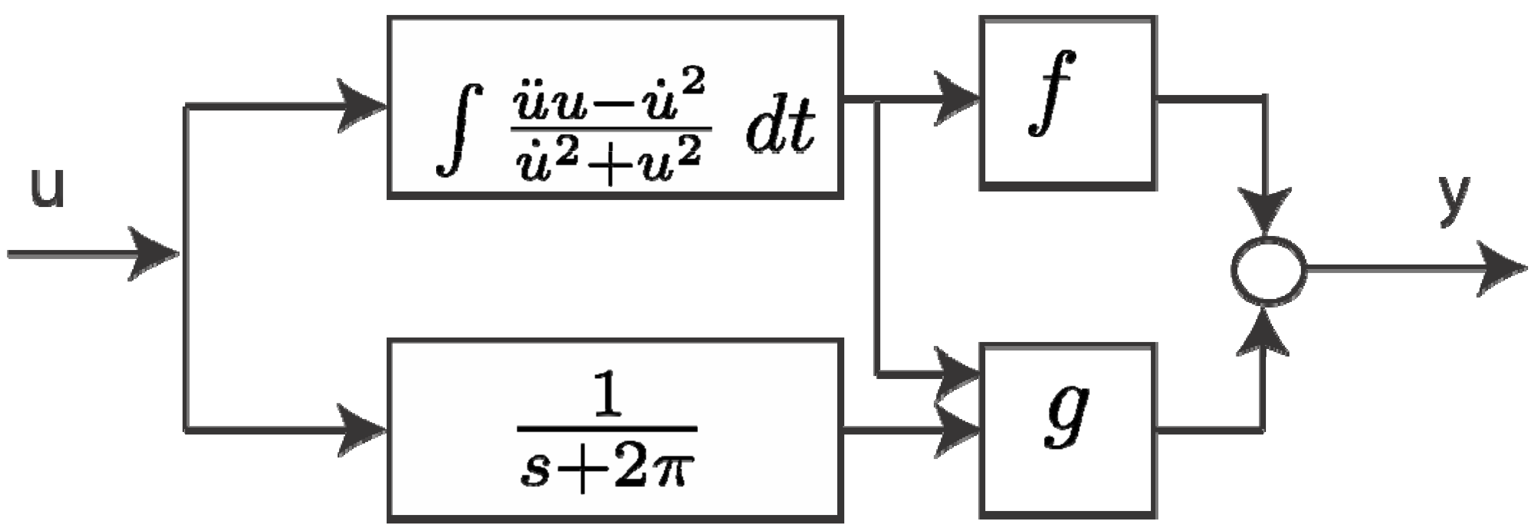
$x = \int \frac{\ddot{u}u - \dot{u}^2}{\dot{u}^2 + u^2}\, dt$

contributes little unless there is a pulse, in which case it contributes 2π and this is independent of the time between pulses. This suggests the approximation

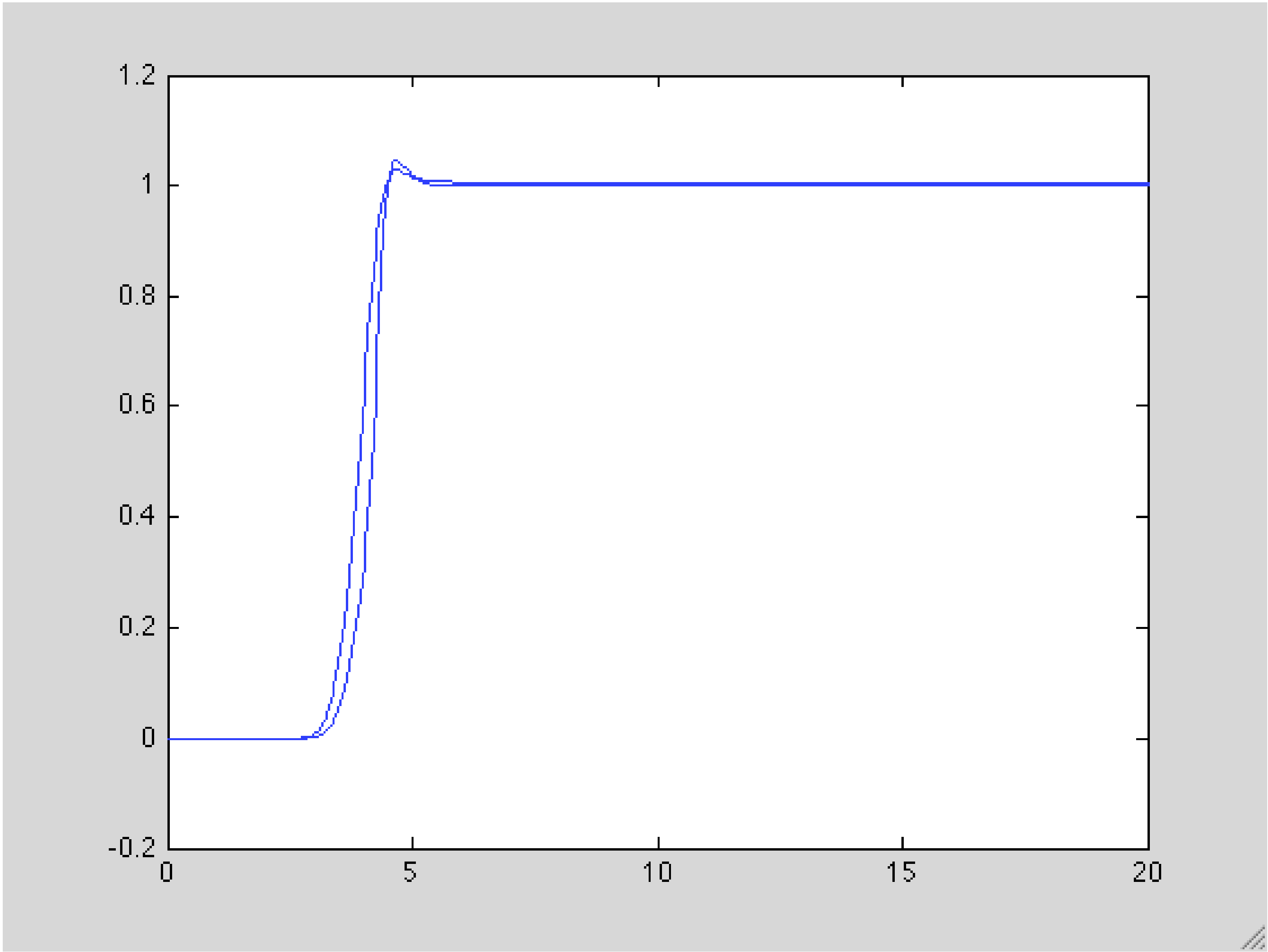
$x \approx w * (u - r) + r \;\; ; \;\; r = \int_0^t \frac{\ddot{u}u - \dot{u}^2}{\dot{u}^2 + u^2}\, dt$



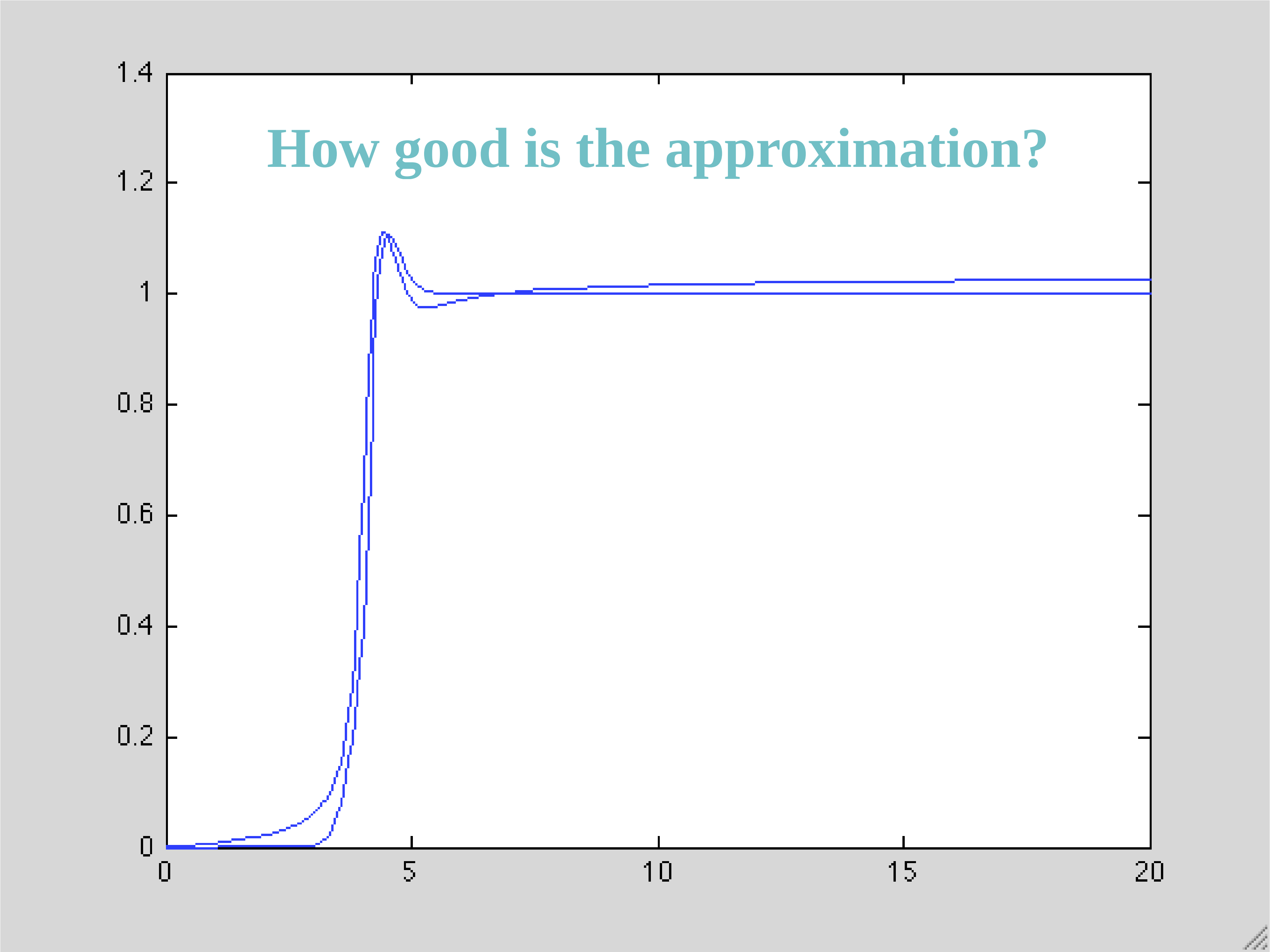

u
$\dot{x} = -\sin 2\pi x + u$
y


u
$\int \frac{\ddot{u}u - \dot{u}^2}{\dot{u}^2 + u^2}\, dt$
$\frac{1}{s+2\pi}$
f
g
y

How good is the approximation?



How good is the approximation?



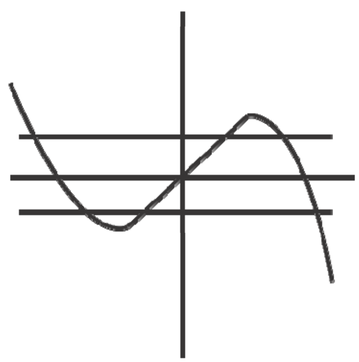
```
function xdot=derham(t,x)
f=exp(-(2*t-8)^2);
g=-4*(2*t-8)*f;
h=((16*(2*t-8)^2)-8)*f;
xdot=zeros(6,1);
xdot(1)=0;
xdot(2)=0;
xdot(3)=1;
xdot(4)=-sin(6.28*x(4)) + x(1)*f ;
xdot(5)= -(1/(2*pi))*(h*(f-.5)-g^2)/((f-.5)^2+g^2);
xdot(6)= 1*(-x(6)+x(5)+ 1.3*f);
```

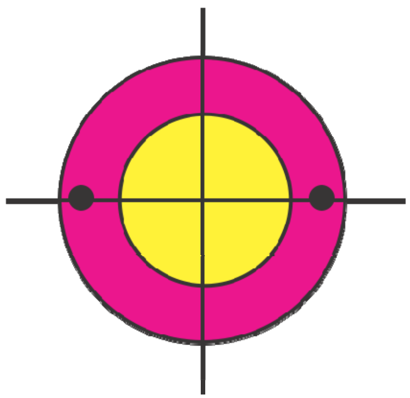
```
t0=0;tf=20;
a=1.7;
b= 1;
c= 0;

x0=[a  b  c  0  0  0];
[t,x]=ode45('derham',[t0,tf],x0);
figure
plot(t,x(:,4));
hold
%plot(t,x(:,5));
plot(t,x(:,6));
```

Operating between levels (a few remarks)

$$\dot{x} = x - \frac{1}{2}x^3 + u$$

For $u = \pm 1/2$ this equation has equilibria $x = \pm 1/2$





The Realization of Automata

$$
\begin{array}{ccc}
\mathcal{U} \subset W_p^q & \xrightarrow{\;\dot{x} = f(x,u)\;} & L_2[0,\infty) \\
\Big\downarrow \pi_1 & & \Big\downarrow \pi_2 \\
\mathcal{S} & \xrightarrow{\;s_{k+1} = \phi(s_k, v_k)\;} & \mathbb{R}^n
\end{array}
$$

$$\dot{x} = x - \frac{1}{2}x^3 + u$$

$$
\begin{array}{ccc}
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\end{array}
$$

Take home messages

1. Models that incorporate closed one-forms such as

$$\int \frac{\ddot{u}u-\dot{u}^2}{\dot{u}^2+u^2}\ dt$$

give a an additional degree of flexibility to the modeling tool kit. We have given an example to illustrate.

2. There is a significant role for models that only intend to be valid for certain subsets of a Sobolev space. These are especially interesting when the fundamental group of this subset is nontrivial.

All I've given here today is a brief outline of what a complete theory would look like.

Coming soon will be the completely dressed version, suitable for refined sensibilities.

Eduardo---

Congratulations on a distinguished career based on talent, hard work, discipline, service to the community and

Most Importantly

a genuine interest in the pursuit of knowledge.