Model reduction of large-scale systems

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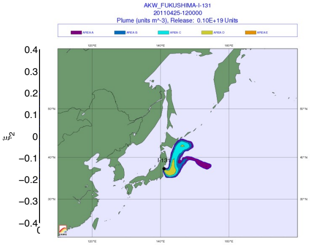
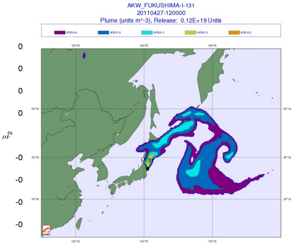
Motivating example I: pollution propagation (Fukushima)

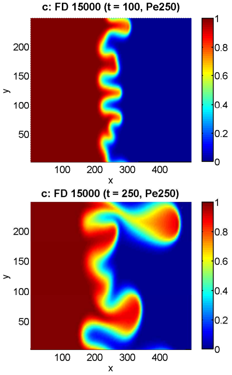




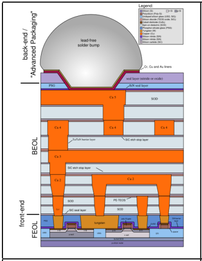
Stokes equation and advection diffusion equation

Motivating example II: Viscous fingering in porous media (EOR)

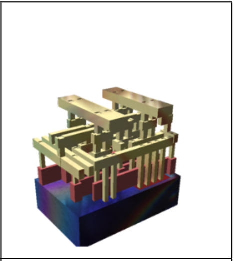
Enhanced oil recovery from underground reservoirs



Darcy's law and advection diffusion equation (twice)



CMOS Chip

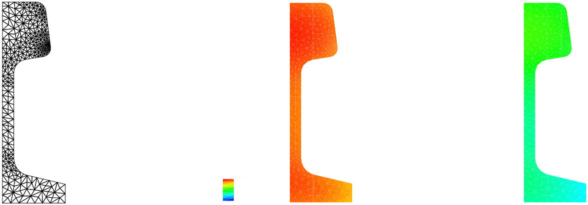


3D Silicon Chip

nanometer details	10^8 components
several GHz speed	several km interconnect
$\approx$ 10 layers	

Interconnect analysis: signal distortions & delays $\Rightarrow$ **Maxwell's equations**

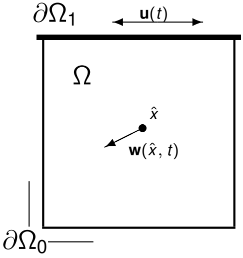
Motivating example IV: A steel cooling model



Advection diffusion equation

Motivating example V: Driven Cavity Flow

A cavity is filled with viscoelastic material and is excited through shearing forces $\mathbf{u}(t)$ of the lid. We are interested in the displacement of the material, $\mathbf{w}(\hat{x}, t)$, at the center.



$\Rightarrow$ **wave equation with hereditary damping**

Outline

1. **Introduction**
2. **Approximation: SVD based methods**
   - POD
   - Balanced reduction
3. **Approximation: Krylov-based or interpolatory methods**
   - Choice of interpolation points: Passivity preserving reduction
   - Choice of interpolation points: Optimal $\mathcal{H}_2$ reduction
   - Reduction of models in generalized form
4. **Conclusions and References**

Outline

The overall problem

Physical system

$\Downarrow$

PDEs

$\Downarrow$

ODEs

$\Downarrow$

Model reduction
(Reduced number of ODEs) $\Longleftarrow$ **DATA**

$\Downarrow$

Simulation, Design, Control

Model reduction via projection

Given is $\begin{array}{l} \mathbf{f}(\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}(t)) = \mathbf{0} \\ \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \end{array}$ or $\begin{array}{l} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{array}$.

Common framework for (most) model reduction methods:

Petrov-Galerkin projective approximation.

Choose k-dimensional subspaces, $\mathcal{V}_k = \text{Range}(\mathbf{V}_k)$, $\mathcal{W}_k = \text{Range}(\mathbf{W}_k) \subset \mathbb{C}^n$.

Find $\mathbf{v}(t) = \mathbf{V}_k\mathbf{x}_k(t) \in \mathcal{V}_k$, $\mathbf{x}_k \in \mathbb{C}^r$, such that

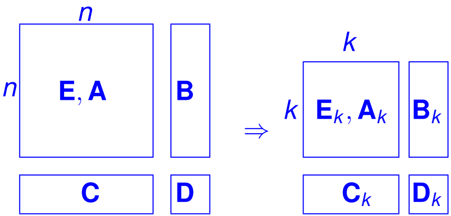
$$\mathbf{E}\dot{\mathbf{v}}(t) - \mathbf{A}\mathbf{v}(t) - \mathbf{B}\,\mathbf{u}(t) \;\perp\; \mathcal{W}_r \quad \Rightarrow$$

$$\mathbf{W}_k^* \left(\mathbf{E}\mathbf{V}_k\dot{\mathbf{x}}_k(t) - \mathbf{A}\mathbf{V}_k\mathbf{x}_k(t) - \mathbf{B}\,\mathbf{u}(t)\right) = \mathbf{0}, \quad \mathbf{y}_k(t) = \mathbf{C}\mathbf{V}_k\mathbf{x}_k(t) + \mathbf{D}\mathbf{u}(t),$$

Reduced order system

$$\mathbf{E}_k = \mathbf{W}_k^*\mathbf{E}\mathbf{V}_k, \quad \mathbf{A}_k = \mathbf{W}_k^*\mathbf{A}\mathbf{V}_k, \quad \mathbf{B}_k = \mathbf{W}_k^*\mathbf{B}, \quad \mathbf{C}_k = \mathbf{C}\mathbf{V}_k.$$

The quality of the reduced system depends on the choice of $\mathcal{V}_r$ and $\mathcal{W}_r$.



Norms:

- $\mathcal{H}_\infty$-norm: worst output error $\|\mathbf{y}(t) - \hat{\mathbf{y}}(t)\|$ for $\|\mathbf{u}(t)\| = 1$.
- $\mathcal{H}_2$-norm: $\|\mathbf{h}(t) - \hat{\mathbf{h}}(t)\|$

Consider a system described by **implicit nonlinear equations (DAEs)**:

$$\mathbf{f}\left(\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}(t)\right) = \mathbf{0}, \quad \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)),$$

with: $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{y}(t) \in \mathbb{R}^p$.
Approximate by means of a Petrov-Galerkin projection $\Pi = \mathbf{V}_k \mathbf{W}_k^*$:

$$\mathbf{W}_k^* \mathbf{f}\left(\mathbf{V}_k \dot{\mathbf{x}}_k(t), \mathbf{V}_k \mathbf{x}_k(t),\ \mathbf{u}(t)\right) = \mathbf{0}, \quad \mathbf{y}_k(t) = \mathbf{h}(\mathbf{V}_k \mathbf{x}_k(t),\ \mathbf{u}(t))$$

where $\mathbf{x}_k \in \mathbb{R}^k$, $k \ll n$. The approximation is ”good” if $\mathbf{x} - \Pi \mathbf{x}$ is ”small”.

Issues and requirements

Issues with large-scale systems

1. Storage – Computational speed – Accuracy
2. System theoretic properties

Requirements for model reduction

1. Approximation error small
2. Structure preservation (e.g. stability/passivity)
3. Procedure computationally efficient and automatic
4. In addition: many ports, parameters, nonlinearities, ...

Outline

Approximation methods: Overview

Krylov

- Realization
- Interpolation
- Lanczos
- Arnoldi

SVD

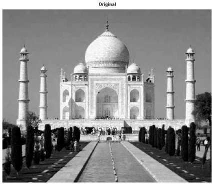
Nonlinear systems	**Linear systems**
• POD methods • Empirical Gramians	• Balanced truncation • Hankel approximation

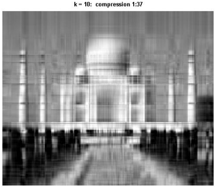
SVD

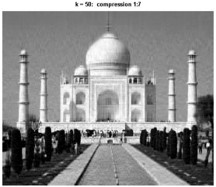
Prototype approximation problem: **SVD** (Singular Value Decomposition):

$$\mathbf{A} = \ \mathbf{U}\,\boldsymbol{\Sigma}\,\mathbf{V}^*$$

Singular values $\boldsymbol{\Sigma}$ provide trade-off between accuracy and complexity.







original 599×726	$k = 10$	$k = 50$

POD

POD (Proper Orthogonal Decomposition):
Consider: $\mathbf{f}(\dot{\mathbf{x}}(t), \mathbf{x}(t), \mathbf{u}(t)) = 0$, $\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t))$.

Snapshots of the state: $\mathcal{X} = [\mathbf{x}(t_1) \;\; \mathbf{x}(t_2) \;\; \cdots \;\; \mathbf{x}(t_N)] \in \mathbb{R}^{n \times N}$.

SVD: $\mathcal{X} = \mathbf{U}\Sigma\mathbf{V}^* \approx \mathbf{U}_k\Sigma_k\mathbf{V}_k^*, \;\; k \ll n$. Approximation of the state:

$$\mathbf{x}_k(t) = \mathbf{U}_k^*\mathbf{x}(t) \;\; \Rightarrow \;\; \mathbf{x}(t) \approx \mathbf{U}_k\mathbf{x}_k(t), \;\; \mathbf{x}_k(t) \in \mathbb{R}^k$$

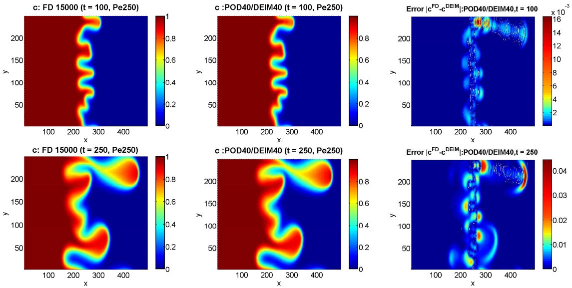
Project state and output equations. Reduced order system:

$$\boxed{\mathbf{U}_k^*\mathbf{f}(\mathbf{U}_k\dot{\mathbf{x}}_k(t), \mathbf{U}_k\mathbf{x}_k(t), \mathbf{u}(t)) = 0, \;\; \mathbf{y}_k(t) = \mathbf{h}(\mathbf{U}_k\mathbf{x}_k(t), \mathbf{u}(t))}$$

$\Rightarrow$ $\mathbf{x}_k(t)$ eolves in a **low-dimensional** space.

Issues with POD: (a) Choice of snapshots, (b) singular values not I/O invariants, (c) computation of $\mathbf{U}_\mathbf{k}^*\mathbf{f}$ costly.

Viscous fingering in porous media



$$\nabla \cdot \mathbf{u} = 0 \qquad \text{(incompressibility)}$$

$$\nabla \pi = -\mu \, \mathbf{u} \qquad \text{(Darcy's law)}$$

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = \alpha \, \nabla^2 c + \beta \, f(c) \qquad \text{(convection, diffusion for } c\text{)}$$

$$\frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla \Theta = \gamma \, \nabla^2 \Theta + \delta \, f(c) \qquad \text{(convection, diffusion for } \Theta\text{)}$$

$\mathbf{u}$: velocity, π: pressure, c: concentration, Θ: temperature, $\alpha, \beta, \gamma, \delta$ constants, $\mu(c, \Theta)$: viscosity of injected fluid, $f(c)$ nonlinear function of c.

SVD methods: balanced truncation

Given linear system $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$, $\det \mathbf{E} \neq \mathbf{0}$, $(\mathbf{A}, \mathbf{E})$ stable, use **state** and **output**. This implies the computation of the gramians which satisfy the generalized **Lyapunov equations**:

$$\mathbf{APE}^* + \mathbf{EPA}^* + \mathbf{BB}^* = \mathbf{0}, \quad \mathbf{P} > \mathbf{0}, \;\; \mathbf{A}^*\mathbf{QE} + \mathbf{E}^*\mathbf{QA} + \mathbf{C}^*\mathbf{C} = \mathbf{0}, \quad \mathbf{Q} > \mathbf{0} \quad \Rightarrow$$

$\boxed{\sigma_i = \sqrt{\lambda_i(\mathbf{PE}^*\mathbf{QE})}}$: **Hankel singular values**: provide trade-off between accuracy and complexity.

Properties

1. Stability is preserved
2. Global error bound: $\sigma_{k+1} \leq \| \mathbf{H}(s) - \hat{\mathbf{H}}(s) \|_\infty \leq 2(\sigma_{k+1} + \cdots + \sigma_n)$

Iterative solution of Lyapunov equations

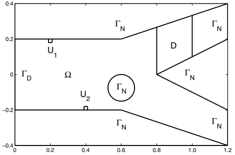
Drawbacks

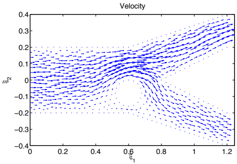
1. Dense computations, matrix factorizations and inversions $\Rightarrow$ may be ill-conditioned; number of operations $\mathcal{O}(n^3)$
2. Bottleneck: solution of Lyapunov equations: $\mathbf{APE}^* + \mathbf{EPA}^* + \mathbf{BB}^* = \mathbf{0}$.
   For large $\mathbf{A}$ such equations cannot be solved exactly.
   Instead, since $\mathbf{P} > 0 \Rightarrow$ **square root $\mathbf{L}$ exists**: $\mathbf{P} = \mathbf{LL}^*$.
   Hence compute **approximations** $\mathbf{V}$ to $\mathbf{L}$: $\hat{\mathbf{P}} = \mathbf{VV}^*$: $\mathrm{rank}\,\mathbf{V} = k \ll n$:

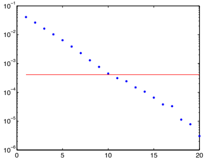
$$P = \mathbf{L} \; \mathbf{L}^* \approx \mathbf{V} \; \mathbf{V}^* = \hat{\mathbf{P}}$$

Iterative solution: ADI, modified Smith (guaranteed convergence).

Example







Semidiscretized advection diffusion equation (concentration of pollutant c):

$$\frac{\partial}{\partial t}c(\xi, t) - \nabla(\kappa \nabla c(\xi, t)) + \mathbf{v}(\xi) \cdot \nabla c(\xi, t) = u(\xi, t)$$

advection $\mathbf{v}$: solution of steady state Stokes equation; diffusivity $\kappa = 0.005$.

Original	Reduced
$m = 16$	$m = 16$
$n = 2673$	$k = 10$
$p = 283$	$p = 283$

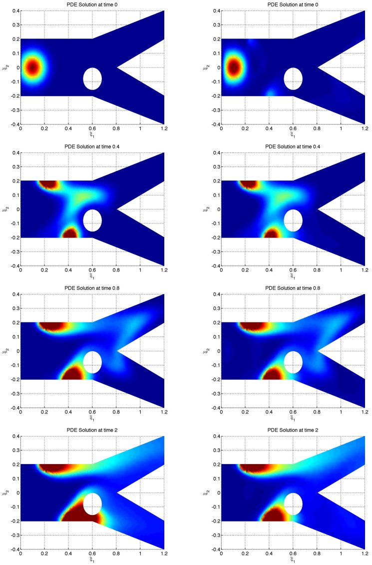
Example

$t = 0.0$

$t = 0.4$

$t = 0.8$

$t = 2.0$



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4 Conclusions and References

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SVD

Nonlinear systems	Linear systems
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Krylov methods: Approximation by moment matching

Given $\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$, expand transfer function around s_0:

$\mathbf{H}(s) = \eta_0 + \eta_1(s - s_0) + \eta_2(s - s_0)^2 + \eta_3(s - s_0)^3 + \cdots,\ \ \eta_j$: **moments at** s_0

Find $\mathbf{E}_k\dot{\mathbf{x}}_k(t) = \mathbf{A}_k\mathbf{x}_k(t) + \mathbf{B}_k\mathbf{u}(t),\ \mathbf{y}_k(t) = \mathbf{C}_k\mathbf{x}_k(t) + \mathbf{D}_k\mathbf{u}(t)$, with

$$\mathbf{H}_k(s) = \theta_0 + \theta_1(s - s_0) + \theta_2(s - s_0)^2 + \theta_3(s - s_0)^3 + \cdots$$

such that for appropriate s_0 and ℓ: $\boxed{\eta_j = \theta_j,\ \ j = 1, 2, \cdots, \ell}$ $\Rightarrow$

Approximation by interpolation

The general interpolation framework

- **Goal**: produce $\mathbf{H}_k(s)$, that approximates a large order $\mathbf{H}(s)$, by means of **interpolation** at a set of points σ_i: $\mathbf{H}_k(\sigma_i) = \mathbf{H}(\sigma_i)$, $i = 1, \cdots, k$.
- For MIMO systems interpolation conditions are imposed in specified directions: **tangential interpolation**.

Problem: Find reduced model satisfying:

$$\ell_i^* \mathbf{H}_k(\mu_i) = \ell_i^* \mathbf{H}(\mu_i), \; \mathbf{H}_k(\lambda_j)\mathbf{r}_j = \mathbf{H}(\lambda_j)\mathbf{r}_j, \; i, j = 1, \cdots, k.$$

Interpolatory projections

$$\mathbf{V}_k = \left[(\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{r}_1, \; \cdots, \; (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\mathbf{r}_k \right], \quad \mathbf{W}_k^* = \begin{bmatrix} \ell_1^* \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \ell_k^* \mathbf{C}(\mu_k \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}.$$

• **Consequence**: Krylov methods match moments **without** computing them.

Q: **How to choose the interpolation points and tangential directions?**

Choice of interpolation points: Passivity preserving model reduction

Recall: $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is passive $\Leftrightarrow$ $\mathbf{H}(s)$ is positive real.

$\Rightarrow$ implies **spectral factorization $\mathbf{H}(s) + \mathbf{H}^*(-s) = \Phi(s)\Phi^*(-s)$**. The *spectral zeros* are λ such that: $\Phi(\lambda)$, loses rank. Hence $\exists$ right spectral zero direction, $\mathbf{r}$, such that $(\mathbf{H}(\lambda) + \mathbf{H}^*(-\lambda))\mathbf{r} = \mathbf{0}$

- **Method**: Interpolatory reduction
- **Solution**: interpolation points = spectral zeros

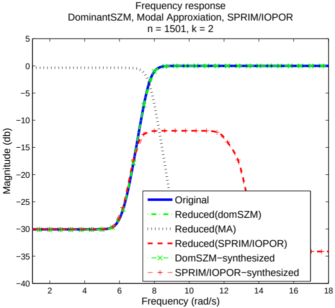
Passivity preserving tangential interpolation

Given $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$, stable and passive, let $\lambda_1, \cdots, \lambda_k$ be stable spectral zeros with corresponding right directions $\mathbf{r}_1, \cdots, \mathbf{r}_k$.

If a reduced order system $\mathbf{H}_k(s)$ is obtained by interpolatory projection with right data λ_i, $\mathbf{r}_i$, and left data $\mu_i = -\overline{\lambda_i}$, $\mathbf{r}_i^*$ for $i = 1, \cdots, k$, then $\mathbf{H}_k(s)$ is stable and passive.

Example of passivity presrving reduction

An RLC transmission line is reduced with dominant SZM, SPRIM, modal approximation (MA). Dominant SZM gives the best approximation.



System	Dim.	R	C	L	VCCs	States	Sim. time
Original	1501	1001	500	500	500	1500	0.50 *s*
Dominant SZM	2	3	2	0	-	4	0.01 *s*
SPRIM/IOPOR	2	6	3	1	-	4	0.01 *s*

Choice of interpolation points: Optimal $\mathcal{H}_2$ model reduction

Recall: the $\mathcal{H}_2$ norm of a stable system $\boldsymbol{\Sigma}$ is:

$$\|\boldsymbol{\Sigma}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{trace } [\mathbf{H}(i\omega)\mathbf{H}^*(-i\omega)] \; d\omega \right)^{1/2}$$

where $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, is the system transfer function.

Goal: construct a *Krylov projector* such that $\boldsymbol{\Sigma}_k = \underset{\deg(\hat{\boldsymbol{\Sigma}})=k}{\arg\min} \left\| \boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}} \right\|_{\mathcal{H}_2}$.

The optimization problem is **nonconvex**. We propose finding reduced order models that satisfy first-order necessary optimality conditions.

First-order necessary optimality conditions

Let $\mathbf{H}_k$ solve the optimal $\mathcal{H}_2$ problem and let $\hat{\lambda}_i$ denote its poles. Assuming for simplicity that $m = p = 1$, the following **interpolation conditions** hold:

$$\mathbf{H}(-\hat{\lambda}_i^*) = \mathbf{H}_k(-\hat{\lambda}_i^*) \;\text{ and }\; \frac{d}{ds}\mathbf{H}(s)\Big|_{s=-\hat{\lambda}_i^*} = \frac{d}{ds}\mathbf{H}_k(s)\Big|_{s=-\hat{\lambda}_i^*}$$

Thus the optimal reduced system $\mathbf{H}_k$ matches the first two moments of the original system at the **mirror image of its poles**.

1. Make an initial selection of σ_i, for $i = 1, \cdots, k$
2. $\mathbf{W} = [(\sigma_1\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*, \cdots, (\sigma_k\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*]$
3. $\mathbf{V} = [(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, \cdots, (\sigma_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}]$
4. while (not converged)
   - $\mathbf{E}_k = \mathbf{W}^*\mathbf{E}\mathbf{V}$, $\mathbf{A}_k = \mathbf{W}^*\mathbf{A}\mathbf{V}$,
   - $\sigma_i \longleftarrow -\lambda_i(\mathbf{A}_k, \mathbf{E}_k)$ + Newton correction, $i = 1, \cdots, k$
   - $\mathbf{W} = [(\sigma_1\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*, \cdots, (\sigma_k\mathbf{E}^* - \mathbf{A}^*)^{-1}\mathbf{C}^*]$
   - $\mathbf{V} = [(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, \cdots, (\sigma_k\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}]$
5. $\mathbf{E}_k = \mathbf{W}^*\mathbf{E}\mathbf{V}$, $\mathbf{A}_k = \mathbf{W}^*\mathbf{A}\mathbf{V}$, $\mathbf{B}_k = \mathbf{W}^*\mathbf{B}$, $\mathbf{C}_k = \mathbf{C}\mathbf{V}$

A numerical algorithm for optimal $\mathcal{H}_2$ model reduction

- Global minimizers are difficult to obtain with certainty; current approaches favor seeking reduced order models that satisfy a local (first-order) necessary condition for optimality.

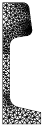
- The main computational cost of this algorithm involves solving $2k$ linear systems to generate **V** and **W**. Computing the eigenvectors **Y** and **X**, and the eigenvalues of the reduced pencil $\lambda \mathbf{E}_k - \mathbf{A}_k$ is cheap since k is small.

- The resulting algorithm (IRKA) has been successfully applied to finding $\mathcal{H}_2$-optimal reduced models for systems of order $n > 160,000$.

- Cooling process in a rolling mill.

Boundary control of 2D heat equation:
finite element discretization $\Rightarrow$ $n = 79,841$:

$\mathbf{A},\ \mathbf{E} \in \mathbb{R}^{79841 \times 79841},\ \mathbf{B} \in \mathbb{R}^{79841 \times 7},\ \mathbf{C} \in \mathbb{R}^{6 \times 79841}$.

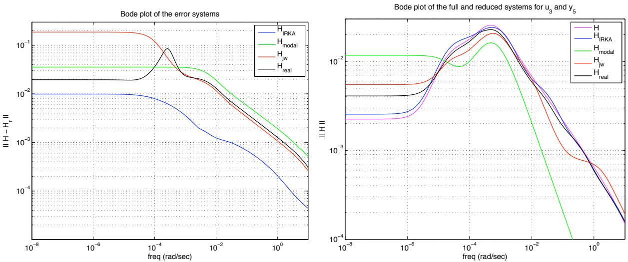


Numerical results

IRKA is compared with:

1. Modal Approximation $\mathbf{H}_{\rm modal}$: choose 20 dominant modes of $\mathbf{H}(s)$.
2. $\mathbf{H}_{\jmath\omega}$: interpolation points $j\omega$ where $\|\mathbf{H}(\jmath\omega)\|$ is dominant.
3. $\mathbf{H}_{\rm real}$: 20 interpolation points in the mirror images of the poles of $\mathbf{H}(s)$.

	$\mathbf{H}_{\rm IRKA}$	$\mathbf{H}_{\rm modal}$	$\mathbf{H}_{\jmath\omega}$	$\mathbf{H}_{\rm real}$
Relative $\mathcal{H}_\infty$ error	0.030	0.103	0.542	0.247



Models in generalized form

Forced vibration of an (isotropic) incompressible viscoelastic solid:

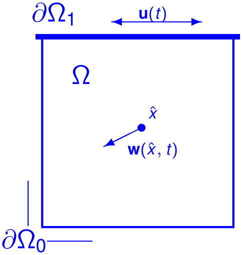
$$\partial_{tt}\mathbf{w}(x,t) - \eta\,\Delta\mathbf{w}(x,t) - \int_0^t \rho(t-\tau)\,\Delta\mathbf{w}(x,\tau)\,d\tau + \nabla\boldsymbol{\pi}(x,t) = \mathbf{b}(x)\cdot\mathbf{u}(t),$$
$$\nabla\cdot\mathbf{w}(x,t) = 0, \quad \text{and} \quad \mathbf{y}(t) = [\mathbf{w}(\hat{x}_1,t),\cdots\mathbf{w}(\hat{x}_p,t)]^*,$$

$\mathbf{w}(x,t)$: displacemet, $\pi(x,t)$: pressure;

$\nabla\cdot\mathbf{w} = 0$ incompressibility constraint;

$\rho(\tau) \geq 0$ is a known “relaxation function”;

$\mathbf{b}(x)\cdot\mathbf{u}(t) = \sum_{i=1}^{m} b_i(x)\,u_i(t)$.



Models in generalized form

Semidiscretization with respect to space gives:

$$\mathbf{M}\,\ddot{\mathbf{x}}(t) - \eta\,\mathbf{K}\,\mathbf{x}(t) - \int_0^t \rho(t-\tau)\,\mathbf{K}\,\mathbf{x}(\tau)\,d\tau + \mathbf{D}\,\mathbf{p}(t) = \mathbf{B}\,\mathbf{u}(t),$$
$$\mathbf{D}^*\,\mathbf{x}(t) = \mathbf{0}, \quad \text{and} \quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t).$$

$\mathbf{x} \in \mathbb{R}^{n_1}$: discretization of $\mathbf{w}$;

$\mathbf{p} \in \mathbb{R}^{n_2}$: discretization of pressure $\boldsymbol{\pi}$.

$\mathbf{M}, \mathbf{K} > \mathbf{0}$.

$$\Rightarrow \quad \mathbf{y}(s) = \underbrace{[\mathbf{C} \quad \mathbf{0}]}_{\mathcal{C}} \underbrace{\begin{bmatrix} s^2\mathbf{M} + (\rho(s)+\eta)\,\mathbf{K} & \mathbf{D} \\ \mathbf{D}^* & \mathbf{0} \end{bmatrix}^{-1}}_{\mathcal{K}} \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}}_{\mathcal{B}} \mathbf{u}(s) \;=\; \mathbf{H}(s)\,\mathbf{u}(s),$$

where $\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s)$. The system is described by

DAEs with hereditary damping.

Reduction of models in generalized form

We seek a **structure preserving** reduced model:

$$\mathbf{M}_r\,\ddot{\mathbf{x}}_r(t)\,-\,\eta\,\mathbf{K}_r\,\mathbf{x}_r(t)\,-\,\int_0^t \rho(t-\tau)\,\mathbf{K}_r\,\mathbf{x}_r(\tau)\,d\tau+\mathbf{D}_r\,\mathbf{p}_r=\mathbf{B}_r\,\mathbf{u}(t)$$

$$\mathbf{D}_r^*\,\mathbf{x}_r(t)=\mathbf{0}\ \text{ and }\ \mathbf{y}_r(t)=\mathbf{C}_r\,\mathbf{x}_r(t).$$

We construct $\mathbf{U}$, $\mathbf{Z}$, $\mathbf{x}(t)\approx\mathbf{U}\mathbf{x}_r(t)$, $\mathbf{p}(t)\approx\mathbf{Z}\mathbf{p}_r(t)$:

$$\mathbf{M}_r=\mathbf{U}^*\mathbf{M}\mathbf{U},\ \mathbf{K}_r=\mathbf{U}^*\mathbf{K}\mathbf{U},\ \mathbf{D}_r=\mathbf{U}^*\mathbf{D}\mathbf{Z},\ \mathbf{B}_r=\mathbf{U}^*\mathbf{B},\ \mathbf{C}_r=\mathbf{C}\mathbf{U}.$$

Thus: no mixing of $\mathbf{w}_r$ and $\mathbf{p}_r$; symmetry and definiteness are preserved.

Reduced model: choose $\mathbf{U}$ and $\mathbf{Z}$ so that the reduced model $\mathbf{H}_r(s)=\mathcal{C}_r(s)\mathcal{K}_r(s)^{-1}\mathcal{B}_r(s)$ interpolates $\mathbf{H}(s)$ at given frequency points.

$$\begin{bmatrix}\mathbf{U}\\ \mathbf{Z}\end{bmatrix}=\left[\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1)\mathbf{b}_1,\ \cdots,\ \mathcal{K}(\sigma_r)^{-1}\mathcal{B}(\sigma_r)\mathbf{b}_r\right]$$

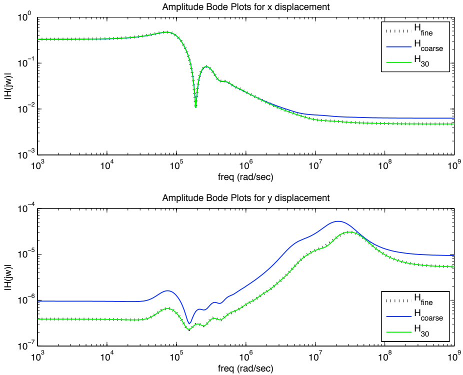
Then, tangential interpolation holds: $\mathbf{H}(\sigma_i)\mathbf{b}_i=\mathbf{H}_r(\sigma_i)\mathbf{b}_i$, $\mathbf{b}_i^*\mathbf{H}(\sigma_i)=\mathbf{b}_i^*\mathbf{H}_r(\sigma_i)$.

Example • Cavity filled with polymer *BUTYL B252* $\Rightarrow \rho(s) = s^{\alpha}$, $\alpha = 0.519$.

• $\mathbf{H}_{\text{fine}}$, using Taylor-Hood FEM discretization with 51,842 displacement and 6,651 pressure degrees of freedom (mesh size $h = \frac{1}{80}$);

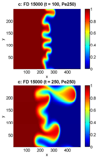
• $\mathbf{H}_{\text{coarse}}$, for a coarse mesh discretization with 29,282 displacement degrees of freedom and 3721 pressure degrees of freedom (mesh size $h = \frac{1}{60}$);

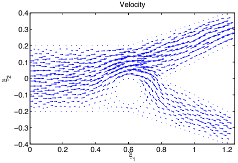
• $\mathbf{H}_{30}$, interpolatory reduced order model with 30 displacement and 30 pressure degrees of freedom. Interpolation points: chosen on the imaginary axis between 10^4 and 10^9.

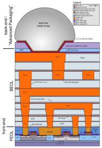


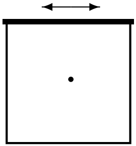
Outline

Summary









Equations

- Darcy's
- Stokes
- Advection diffusion
- Maxwell's
- Wave

Methods

- POD (SVD)
- Approximate balanced truncation (SVD)
- Passivity preserving (interpolatory)
- Optimal $\mathcal{H}_2$ (interpolatory)
- Generalized interpolatory approach

Conclusions: SVD-based reduction methods

- **POD**: method of choice for NL model reduction
  - Chaturantabut, Sorensen, Nonlinear model reduction via discrete empirical interpolation, SIAM J. Sci. Comp., 32: 2737-2764 (2010).
- **Balanced truncation**:

  has apriori computable error bound

  Applicable to small systems

  Bottleneck: solution of the Lyapunov equations
  - Reis, Heinkenschloß, Antoulas, Automatica, 47: 559-564 (2011).

Conclusions: Krylov-based or interpolatory reduction methods

- Passivity preserving model reduction
  - Antoulas, Sorensen: Systems and Control Letters (2005)
  - Ionutiu, Rommes, Antoulas: Passivity-Preserving Model Reduction Using Dominant Spectral-Zero Interpolation, IEEE Trans. CAD Integrated Circ. Syst., 27: 2250 - 2263 (2008).
- Optimal $\mathcal{H}_2$ model reduction
  - Gugercin, Antoulas, Beattie: SIAM J. Matrix Anal. Appl. (2008)
  - Kellems, Roos, Xiao, Cox: Low-dimensional, morphologically accurate models of subthreshold membrane potential, J. Comput. Neuroscience, 27:161-176 (2009).
- Interpolatory model reduction
  - A.C. Antoulas, C.A. Beattie, and S. Gugercin, *Interpolatory model reduction of large-scale systems*, in Efficient modeling and control of large-scale systems, Springer Verlag, pages 3-58 (2010).
  - C.A. Beattie and S. Gugercin, *Interpolatory projection methods for structure preserving model reduction*, Syst. Cont. Lett., 58 (2009).

(Some) Challenges in model reduction

- Model reduction from data: Loewner approach
  - Mayo, Antoulas, *A framework for the solution of the generalized realization problem*, LAA, 425: 634-662 (2007).
  - Lefteriu, Antoulas: A New Approach to Modeling Multiport Systems from Frequency-Domain Data, IEEE Trans. CAD, 29: 14-27 (2010).
- Systems depending on parameters
  - Antoulas, Ionita, Lefteriu, On two-variable interpolation, LAA (2011).
- Sparsity preservation
  - Ionutiu, Model order reduction for multi-terminal systems with application to circuit simulation, PhD Thesis 2011.
- Non-linear systems (besides POD: Astolfi, Krener, Scherpen)
- Domain decomposition - many inputs/outputs
- MEMS and multi-physics problems (micro-fluidic bio-chips)
- ...

Collaborators

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- Cosmin Ionita
- Roxana Ionutiu
- Sanda Lefteriu
- Andrew Mayo
- Timo Reis
- Joost Rommes
- Dan Sorensen

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ΘΑΝΟΣ 23/5/2011