The Foldings of a Square to Convex Polyhedra

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Abstract

The combinatorial, geometric, and topological structure of the set of all convex polyhedra foldable from a square is detailed. It is proved that five combinatorially distinct nondegenerate polyhedra, and four different flat polyhedra, are realizable. All the polyhedra are continuously deformable into each other, with the space of polyhedra forming four connected rings.

1 Introduction

If the perimeter of a polygon is glued to itself in a length-preserving manner in such a way that the resulting complex is homeomorphic to a sphere, then a theorem of Alexandrov establishes that as long as no more than 2π face angle is glued together at any point, the gluing corresponds to a unique convex polyhedron (where "polyhedron" here includes doubly-covered flat polygons). Exploration of the possible foldings of a polygon to convex polyhedra via these *Alexandrov gluings* was initiated in [LO96], and further explored in [DDLO00, DDLO02]. Theorem 1 in [DDLO02] established that every convex polygon can fold to a nondenumerably infinite number of incongruent convex polyhedra. Although this set is infinite, it arises from a finite collection of *gluing trees*, which record the combinatorially possible ways to glue up the perimeter. Enumerating the gluing trees leads to an inventory of the possible foldings of a given polygon to polyhedra. These ideas were implemented in two computer programs, developed independently by Anna Lubiw and Koishi Hirata.[1] These programs only list the gluings, not the polyhedra. Even though the polyhedra are uniquely determined by the gluings, there is no known practical algorithm for constructing the 3D shape of the polyhedra [O'R00].

The contribution of this paper is to construct all the polyhedra foldable from one particularly simple polygon: a unit square. The polyhedra have from 3 to 6 vertices, and we show they fall into nine distinct combinatorial classes: tetrahedra, two different pentahedra, hexahedra, and octahedra; and a flat triangle, square, rectangle, and pentagon. Each achievable shape can be continuously deformed into any other through intermediate foldings of the square, i.e., no shape is isolated. The continua fall into four distinct rings (A, B, C, D), each corresponding to a single parameter change in the gluing, which join together topologically as depicted in Figure 1. Three of the rings (A, B, D) share and join at the flat $1 \times \frac{1}{2}$ rectangle; rings A and C join at a symmetric tetrahedron.

Constructing the entire set of polyhedra permits us to answer a special case of a question posed by Joseph Malkevitch[2]: What is the maximum volume polyhedron foldable from a given polygon? We found (by numerical search) that the maximum is achieved by an octahedron along the A-ring (at about the 4-5 o'clock position), with a volume of approximately 0.055849.

References

[DDLO00] E. D. Demaine, M. L. Demaine, A. Lubiw, and J. O'Rourke. Examples, counterexamples, and enumeration results for foldings and unfoldings between polygons and polytopes. Technical Report 069, Smith College, Northampton, MA, July 2000. arXive:cs.CG/0007019.

[DDLO02] E. D. Demaine, M. L. Demaine, A. Lubiw, and J. O'Rourke. Enumerating foldings and unfoldings between polygons and polytopes. *Graphs and Combinatorics*, 18(1):93–104, 2002.

[LO96] A. Lubiw and J. O'Rourke. When can a polygon fold to a polytope? Technical Report 048, Dept. Comput. Sci., Smith College, June 1996. Presented at *AMS Conf.*, 5 Oct. 1996.

[O'R00] J. O'Rourke. Folding and unfolding in computational geometry. In *Discrete Comput. Geom.*, volume 1763 of *Lecture Notes Comput. Sci.*, pages 258–266. Springer-Verlag, 2000. Papers from the *Japan Conf. Discrete Comput. Geom.*, Tokyo, Dec. 1998.

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[1]Personal communications, Fall 2000. Hirata's program is available at `http://weyl.ed.ehime-u.ac.jp/cgi-bin/WebObjects/Polytope2`.

[2]Personal communication, Feb. 2002.

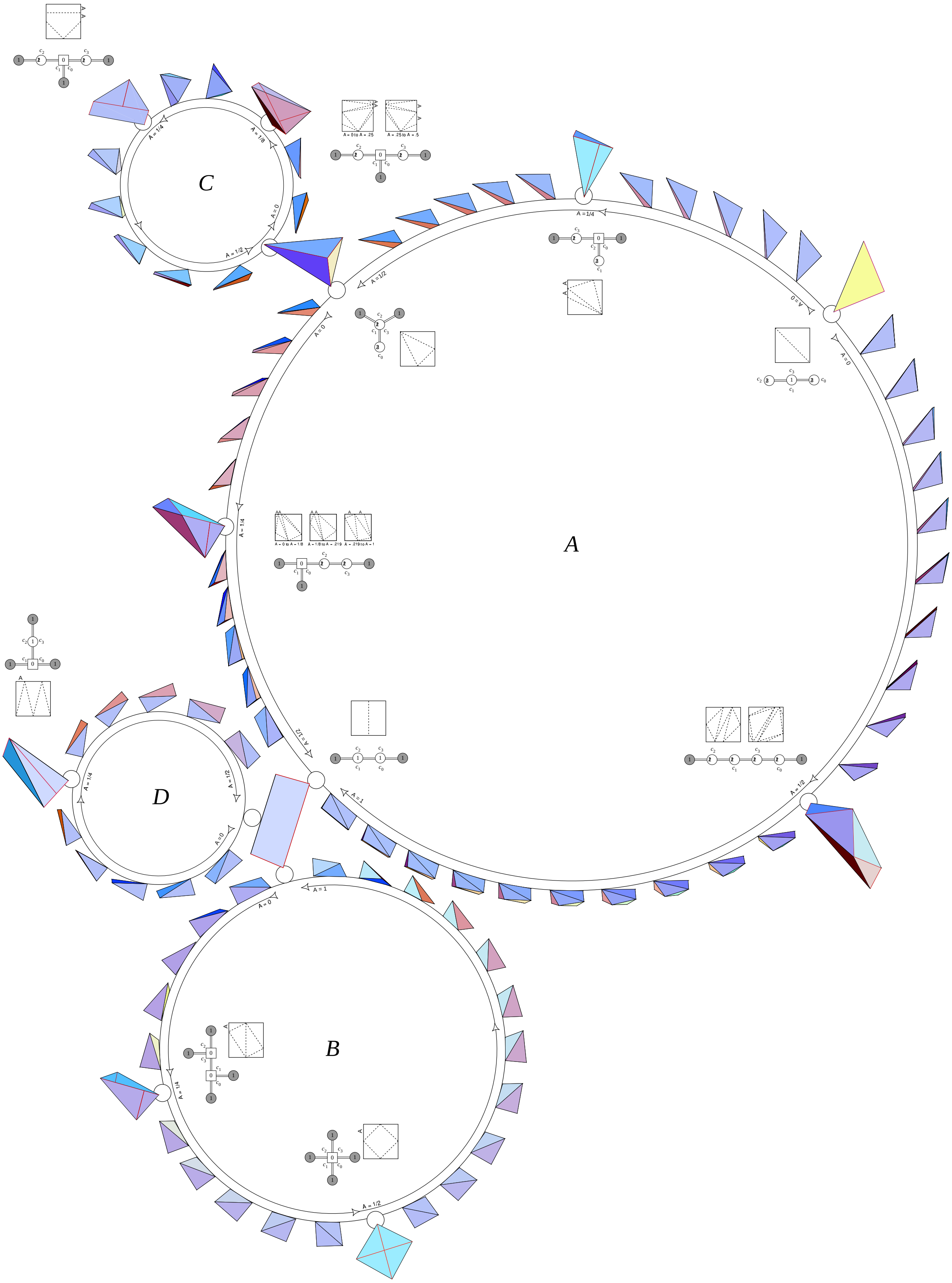

Figure 1: The foldings of a square to convex polyhedra. Selected polyhedra are shown enlarged, together with their crease patterns and corresponding gluing trees.