ACCURACY AND STABILITY OF SWEPT VOLUME REPRESENTATIONS

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PROJECT OVERVIEW

GOALS

- Fundamental advances in state-of-the-art of computing and representing swept volumes and associated operations that are smoother than existing methods and incorporate effectively computable shape invariants.
- Application of results to some important problems that highlight the utility and advantages of the new algorithms.

OUTCOMES

- New algorithms for swept volume operations that are more efficient, smoother and capable of resolving accuracy, stability and consistency problems.
- Accurate and fast programs for use in virtual sculpting and tissue engineering that include shape verification capabilities.
- Techniques and insights helpful in rigorous formulation of the foundations of computational topology.

PRESENTATION OVERVIEW

- Some fundamental concepts and questions in computational topology
- Brief introduction to swept volumes and associated operations
- Smoother interpolation in object representation
- Using singularity theory to analyze and represent swept volumes
- Shape invariants and their applications to swept volumes

PRESENTATION OVERVIEW (continued)

- Applications of new swept volume algorithms to virtual sculpting
- Modeling heterogeneous structures arising in tissue engineering using swept volume techniques
- Initial results on smoother representation of swept volumes and their intersections
- Recapitulation of project goals, research plans and results

Computational topology fundamentals

Many fundamental questions in computational topology have not been answered in a broadly accepted way. Moreover, numerous foundational concepts have as yet not been delineated in an unambiguous and widely adopted manner.

For example, when do two objects M and N embedded in Euclidian n-space $\boldsymbol{R}^n$ have the same *shape*? Interpreted in the strictest possible sense, an appropriate answer seems to be the following:

M and N have the same shape if there is a Euclidian transformation $\Phi\mathbf{:}\boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$, with $\Phi \in Euc(n)$ such that $\Phi(M)=N$. This can be expressed precisely in the category of Euclidian embeddings, by saying that there is a Euclidian isomorphism Φ such that the diagram of embeddings:

$$\begin{array}{ccc} & & \boldsymbol{R}^n \\ & \overset{f}{\nearrow} & \\ \boldsymbol{X} & & \downarrow \boldsymbol{\Phi} \\ & \underset{g}{\searrow} & \\ & & \boldsymbol{R}^n \end{array} \qquad (1)$$

commutes, where $f,g{:}X \rightarrow \boldsymbol{R}^n$ are isometric embeddings with $F(X)=M$, $g(X)=N$. On the other hand, possibly the weakest reasonable interpretation of shape is:

M and N have the same shape if there is a homeomorphism $\Phi\mathbf{:}\boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$ such that the diagram (1) commutes.

Note that this weak form of shape characterization is not synonymous with homeomorphism type. For example, the two knots shown below are homeomorphic, but not isomorphic in category of continuous embeddings.

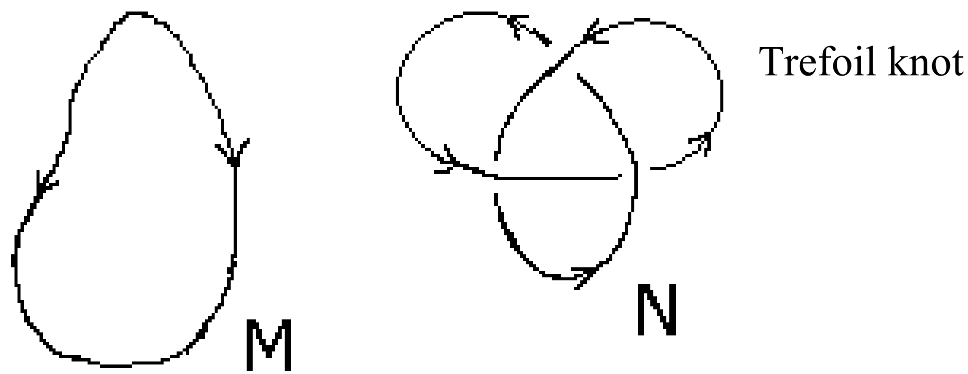


Fig 1. Homeomorphic objects of different shape

Equivalence in the category of embeddings involves more invariants than in the topological category – knotting and linking characteristics must also be computed.

As shape should be independent of size, a better strict definition may be the following: M and N have the same shape if there is a commutative diagram

$$\begin{array}{ccc} & & \mathbf{R}^n \\ & \overset{f}{\nearrow} & \\ X & & \downarrow \Psi \\ & \underset{g}{\searrow} & \\ & & \mathbf{R}^n \end{array} \qquad (2)$$

where $\Psi \in Sim(n)$ – the Lie group of similarities of $\mathbf{R}^n$. Of course, there is a whole range of intermediate definitions between this and the topological category.

For computational representations, the embeddings of interest, f and g, are close to one another (in an appropriate topology), so the question of shape can be reduced to the categories of topological spaces and homeomorphisms, smooth varieties and morphisms, etc. Then some of the key issues are:

Accuracy – If $M=f(X)$ is the exact object, and $N=g(X)$ is an algorithmically rendered approximation, how close are f and g in a chosen topology?

Consistency – Let g be an approximate embedding computed using an algorithm $\mathcal{A}$ and data $\mathcal{D}$ so that $g=g_{\mathcal{A},\mathcal{D}}$. When do M and $N=g_{\mathcal{A},\mathcal{D}}(X)$ have the same shape?

Stability (Robustness) – Do $f(X)$ and $g_{\mathcal{A},\mathcal{D}}(X)$ have the same shape when f and g are sufficiently close?

To algorithmically check for preservation of shape, one needs effectively computable shape characteristics (invariants) . A complete set of effectively computable shape invariants is available in some instances; for example, the Euler characteristic for closed surfaces in $\boldsymbol{R}^3$. However, in more complicated situations it is well known that even basic invariants such as the fundamental group are not effectively computable (Markov, Novikov).

Basic question : For what classes of objects is it possible to include sufficiently many effectively computable shape invariant subroutines in a representation algorithm to effectively resolve the questions of consistency and stability

Partial answer : It seems reasonable to begin the investigation with the class of swept volumes.

Smoother Interpolation

Can the current interpolation methods such as piecewise linear and NURBS be effectively supplanted by smoother procedures capable of incorporating more of the known object features in the next generation of representation programs?

Swept volumes may provide a clue to a possible affirmative answer to this question. The key here is that the boundary ∂M of a swept volume M has a natural description as a flow of a differential equation, namely the sweep-envelope differential equation.

Perhaps local flows of differential equations, smoothly joined over the entire boundary, can serve as the basis of a better interpolation scheme. For example, such a formulation is likely to lead to more efficient intersection schemes.

Introduction to Swept Volumes Operations

An *initial object* M is a compact, connected, n-dimensional, piecewise smooth submanifold of $\boldsymbol{R}^n$. This is acted upon by a *sweep* σ - a continuous function

$$\sigma\colon I=[0,1]\rightarrow Diff_c(\boldsymbol{R}^n),$$

taking values in the space of diffeomorphisms that are compactly different from the identity, with associated *sweep map* $\Sigma(x,t) := \sigma_t(x)$ and *swept volume*

$$S_\sigma(M):=im\Sigma = \Sigma(M\times I) \subset \boldsymbol{R}^n \qquad (3)$$

extended sweep map $\Sigma^*(x,t) := (\sigma_t(x),t)$ and *extended swept volume*

$$S_\sigma^*(M) := im\Sigma^* = \Sigma^*(M\times I) \subset \mathbf{R}^{n+1} \qquad (4)$$

The sweep and extended sweep are generated, respectively, by the *sweep differential equation* (SDE) and *extended sweep differential equation* (ESDE)

$$\dot{x} = X_\sigma(x,t) \qquad (5)$$

and

$$(\dot{x},\dot{t}) = (X_\sigma(x,t),1) \qquad (6)$$

and

$$P(S_\sigma^*(M)) = S_\sigma(M),$$

where $P(x,t)=x$ is the natural projection $\boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}$.

A swept volume is a variety as shown below and in the subsequent pictures.

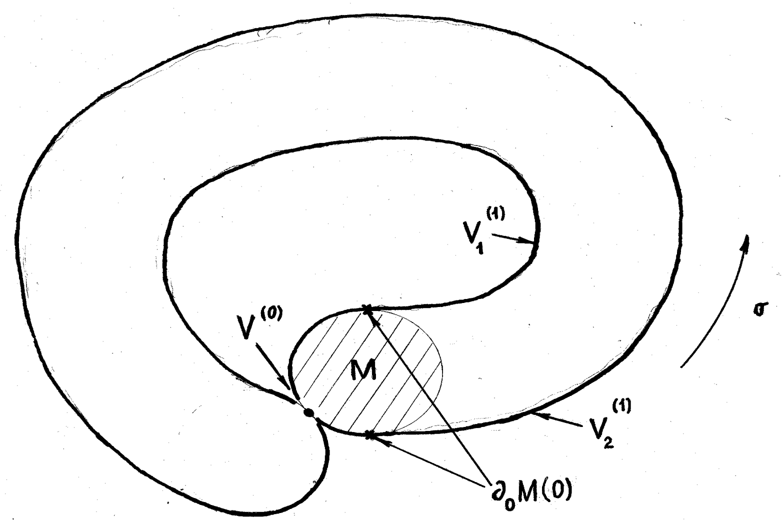
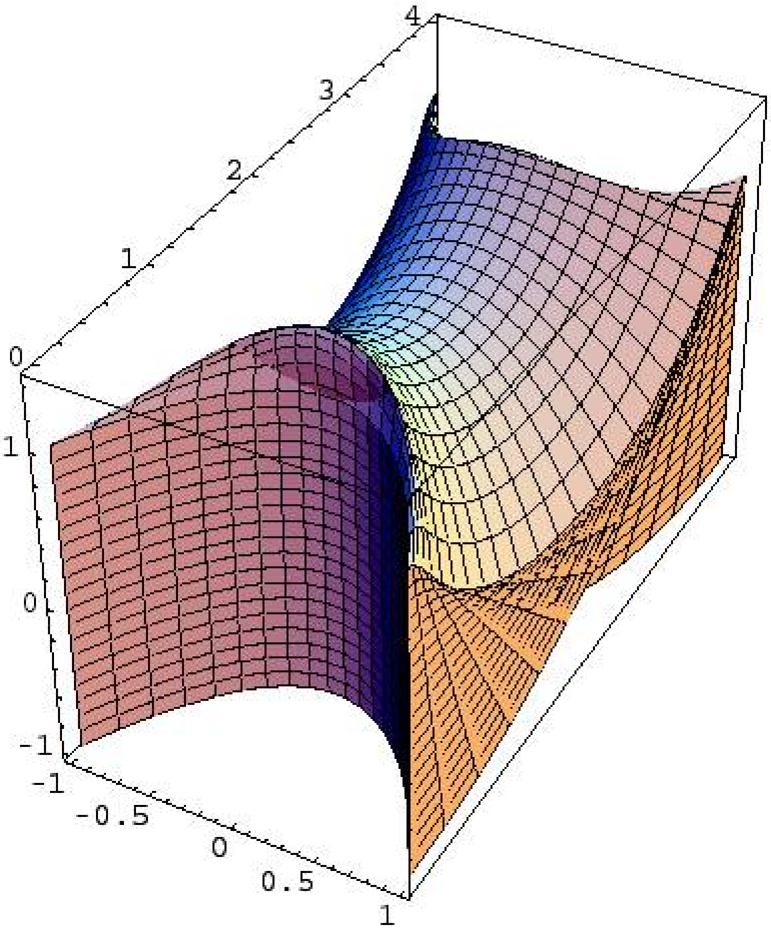
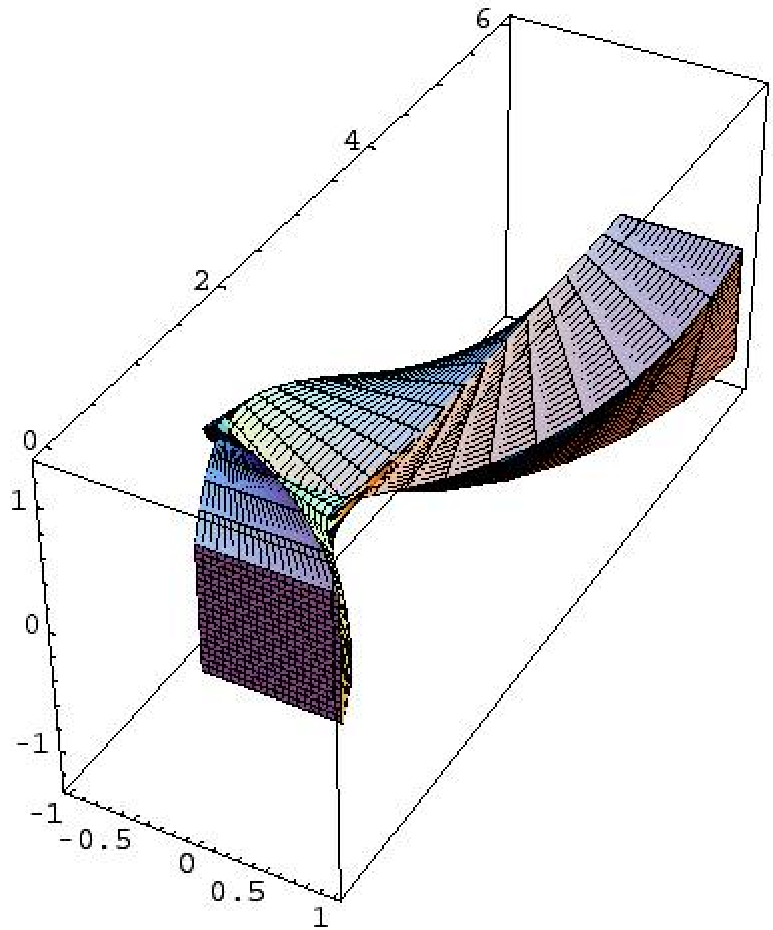


Fig. 2. Swept volume of a disk in $\boldsymbol{R}^3$
(with boundary stratification)


4
3
2
1
0
1
0
-1
-1
-0.5
0
0.5
1


6
4
2
0
1
0
-1
-1
-0.5
0
0.5
1

The SDE leads to a handy decomposition of the boundary of the swept volume via the *sweep flow formula*

$$\partial S_\sigma(M) = \partial_- M(0) \cup \partial_+ M(1) \cup \mathcal{G}_\sigma(M)/\mathcal{T}_\sigma , \qquad (7)$$

where $\partial_- M(0)$ are initial *ingress points* where (5) points into the interior of $M=M(0):=\sigma_0(M)$, $\partial_+ M(1)$ are the terminal *egress points* where (5) points out of the interior of $M(1):=\sigma_1(M)$, $\mathcal{G}_\sigma(M)$ are the *grazing points* where (5) points neither into nor out of the interior of $M(t):=\sigma_t(M), 0\leq t \leq 1,$ and $\mathcal{T}_\sigma$ is a *trim set* of interior self-intersection points.

There is a variant of the SDE called the *sweep envelope differential equation* (SEDE) of the form

$$\dot{x} = Y_{(\sigma,M)}(x,t) \qquad (8)$$

having the property that its trajectories starting on the initial grazing point set $\partial_0 M(0)$ generate all of $\mathcal{G}_\sigma(M)$, thereby providing the basis for very efficient swept volume algorithms.

Smoother Interpolation

It follows from the SEDE (8) that points on the boundary $\partial S_\sigma(M)$ of a swept volume are naturally represented by the local flow (generated by a differential equation) of a codimension-1 submanifold as shown below.

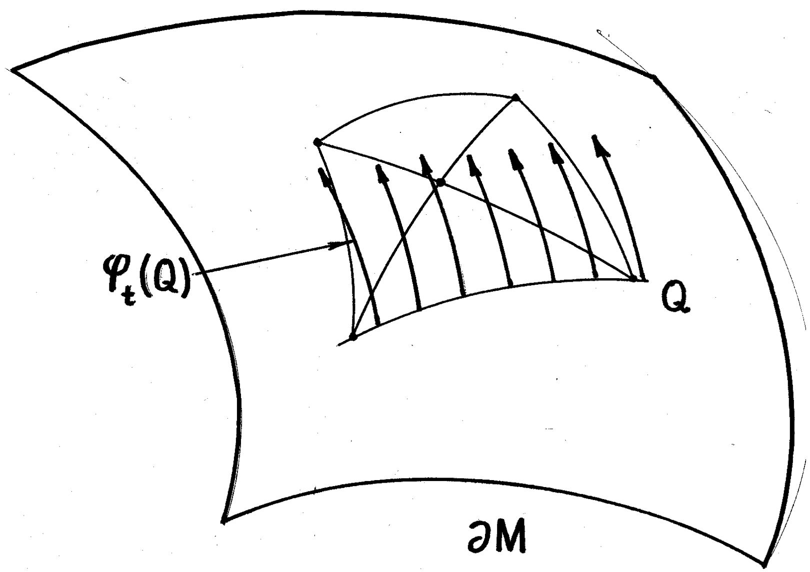


Fig 3. Local boundary sweep

A natural question is can this be extended to more general object boundaries and how can such local sweep representations be smoothly blended together? Preliminary results obtained concerning this question are quite promising, so smoother more versatile interpolation schemes may be feasible via this approach.

Stratification of Swept Volumes

There is a natural way of decomposing swept volumes based on singularity/stratification theory that begins with the sweep map

$$\Sigma : (M \times I) \rightarrow \boldsymbol{R}^n.$$

The image $\Sigma(M \times I) = S_\sigma(M)$ may be written in the form

$$S_\sigma(M) = V_1 \cup V_2 \cup \ldots \cup V_m \qquad (9)$$

Where the *strata* $\{V_k\}$ are submanifolds of $\boldsymbol{R}^n$ with dimensions raging from 0 to n. This *stratification* of the swept volume is of the Thom-Boardman type, wherein the strata of dimension less that n correspond to *singularities* of Σ, i.e. points where Σ' has less than maximal rank.

It can be proven that the stratification is *Whitney regular*, meaning roughly that all points in each stratum V_k are "equally singular" and each pair of abutting strata V_j , V_k join at well defined angles (see Fig.2).

A one-dimensional reduction in the singularity characterization of swept volumes is realized by using the flow of the SEDE (8) represented in the form

$$\Theta : \partial_0 M(0) \times I \to \mathbf{R}^n \qquad (10)$$

The Thom-Boardman classes of (10) generate the stratification

$$\mathcal{G}_{\sigma}(M) = W_1 \cup W_2 \cup \ldots \cup W_q \qquad (11)$$

Here the trade-off is that Θ is considerably more complicated than Σ. Nevertheless, the stratification (11) can also be shown to be regular.

Determining the strata tends to be computationally expensive, but useful local normal forms are readily obtained from this singularity theory approach (cf. Abdel-Malek, Blackmore, Shapiro,…). Regularity allows one to verify consistency and stability more qualitatively using Thom-Mather theory.

Computable Shape Invariants

One of the reasons that categorical (shape) invariants are rather accessible for swept volumes $S_\sigma(M)$ is that they are essentially isomorphic to $M \times I$ (modulo self intersection or trimming) in most of the shape categories of interest.

The most obvious shape invariants are the characteristic (cohomology) classes such as the Euler class, Pontryagin classes, and Stiefel-Whitney classes. These are invariants that can be used to check for consistency and stability, and they are effectively computable via simplicial construction.

They do not, in general provide a complete set of invariants, but in some cases they are sufficient as with the Euler class (characteristic) for embedded surfaces. Local versions of some of these invariants can also be helpful in detecting singular behavior such as self-intersection.

There are other related, possibly effectively computable, approaches to the questions of consistency and stability that look promising, especially for swept volumes.

For example, obstruction theory fits rather nicely into the structure of swept volumes owing to the fiber structure illustrated below in Fig 4.

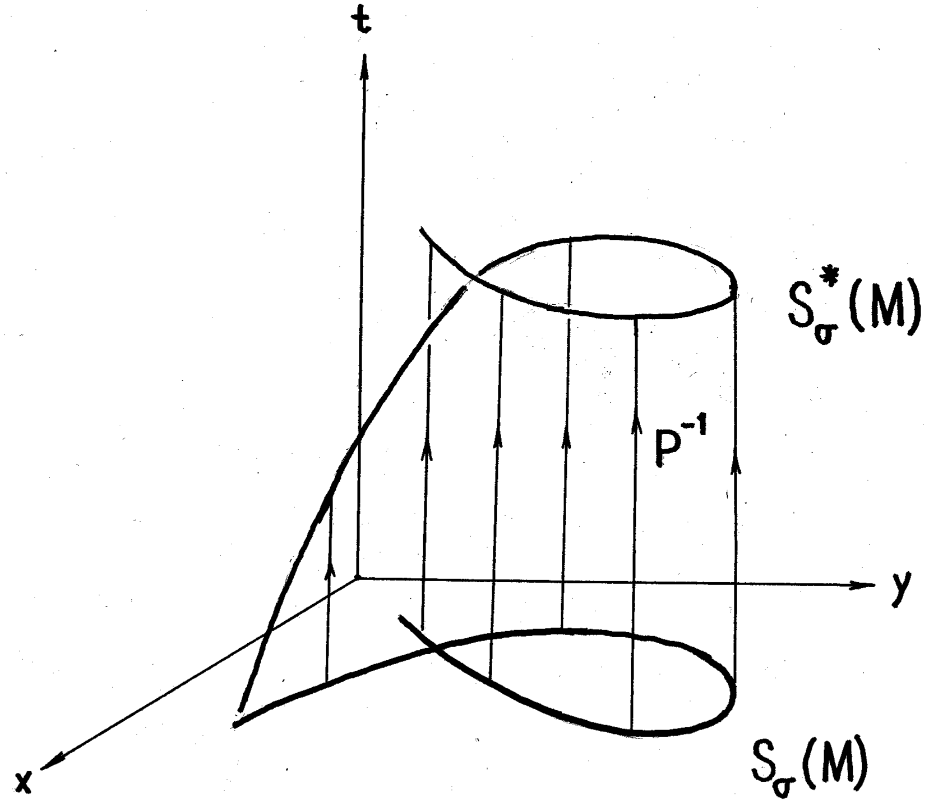
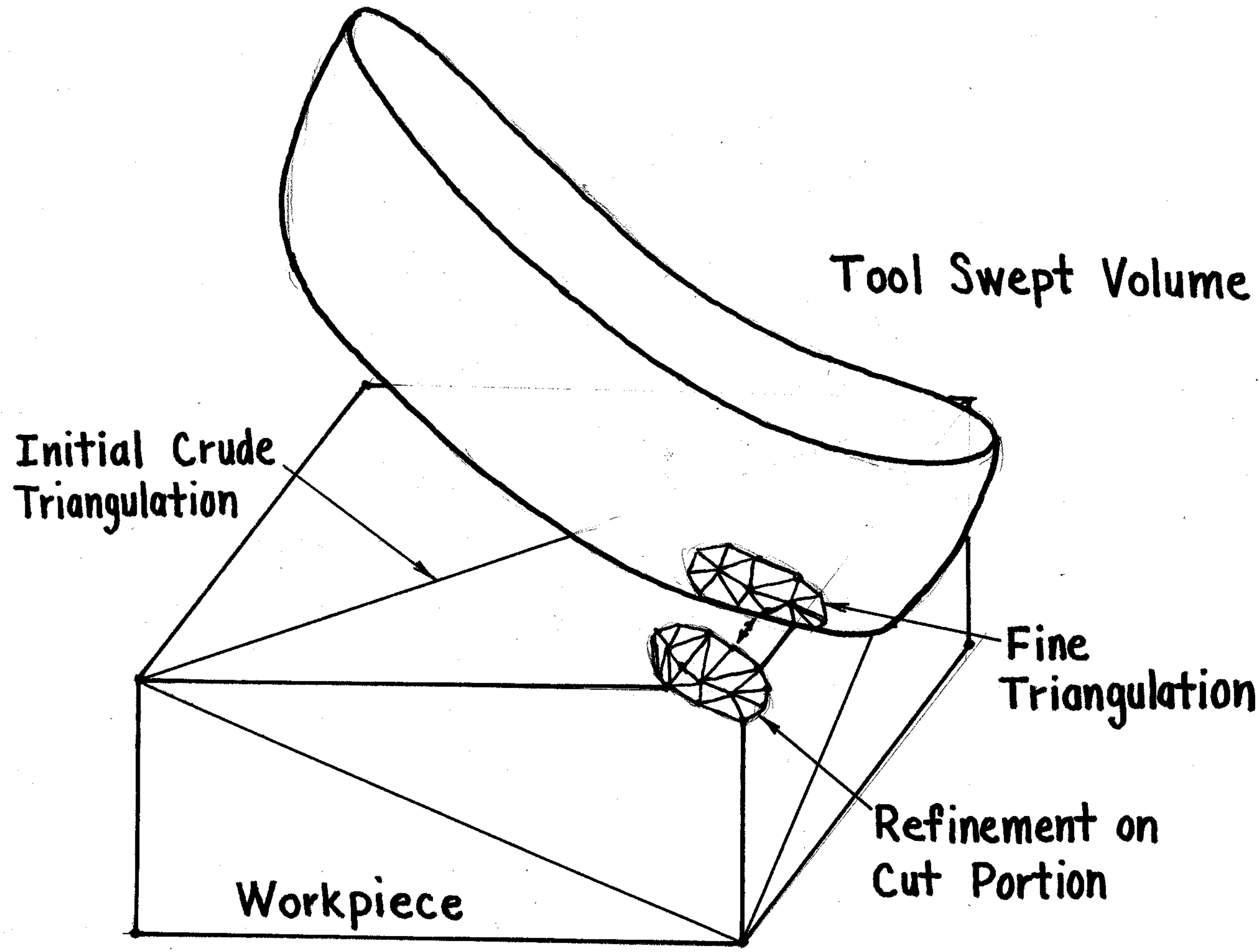


Fig 4. The fibration $P^{-1}: S_\sigma(M) \rightarrow S_\sigma^* (M)$.

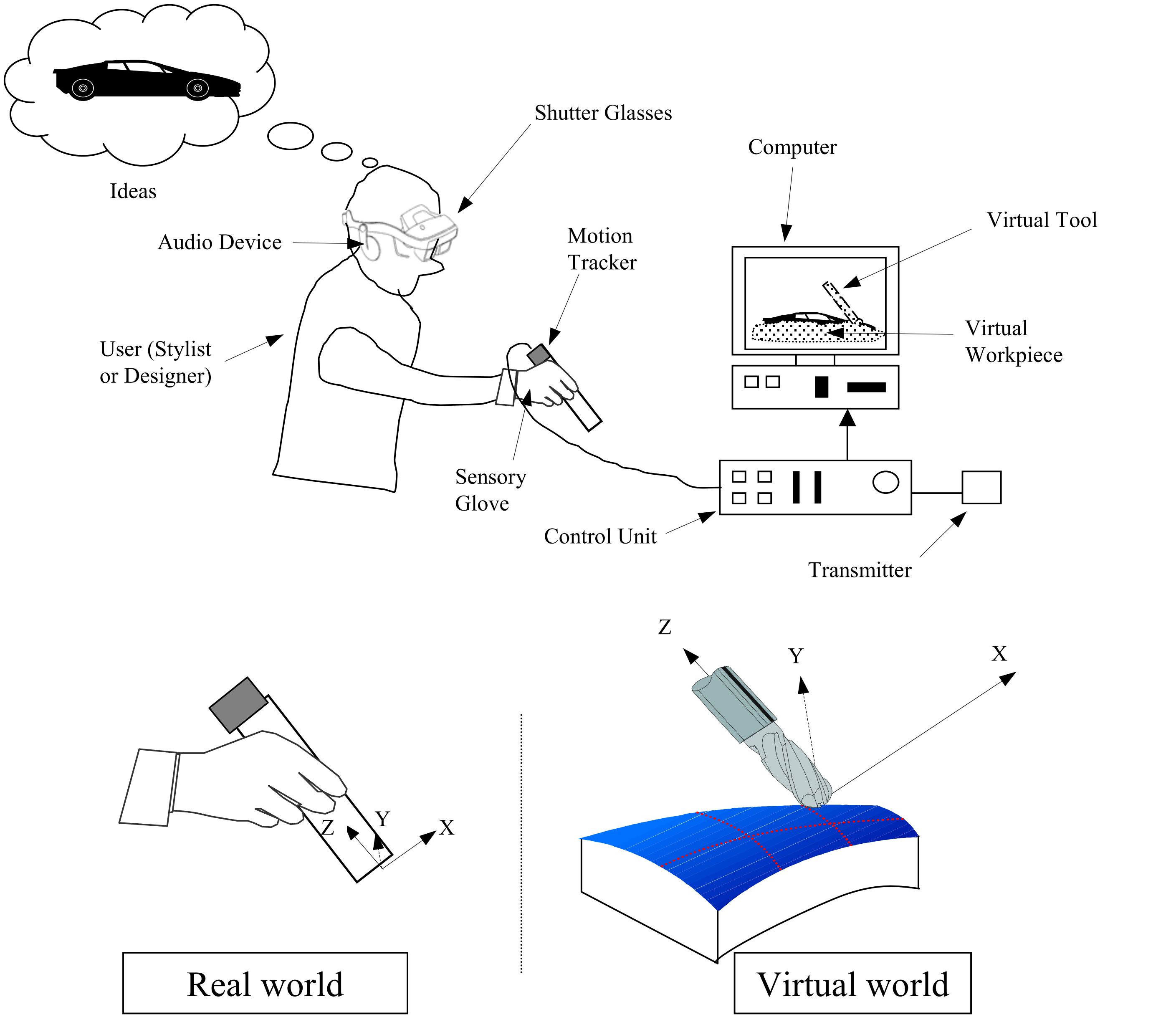
Is $S_{\sigma}(M)$ a singular (corresponding say to the adjunction of cells in a CW-complex structure) or nonsingular lifting of $S_{\sigma}^{*}(M)$? Obstruction theory (in particular Moore-Postnikov factorization) is a natural approach to resolving this question.

Application to Virtual Sculpting

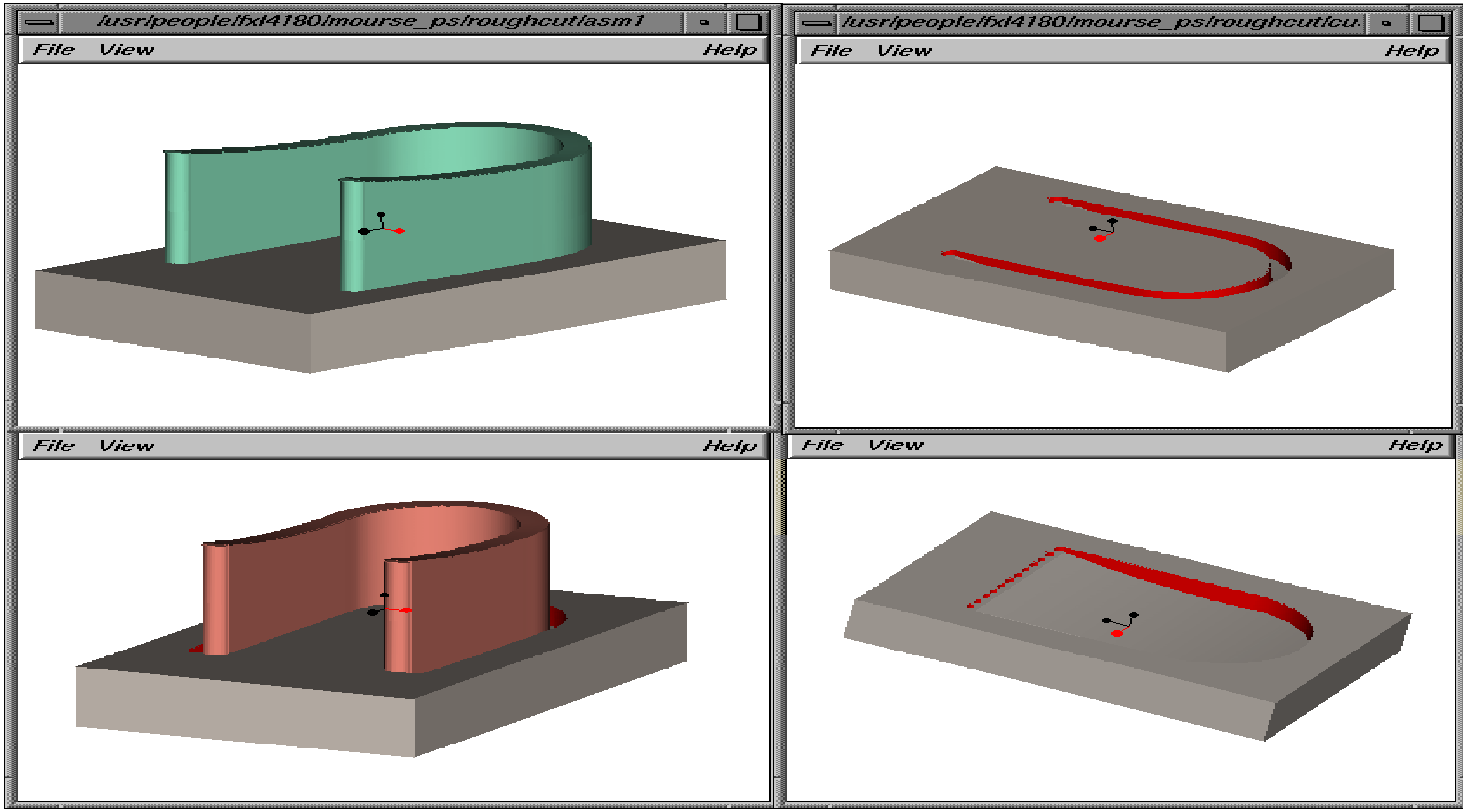
A new algorithm will be developed for use in virtual sculpting that improves on the SDE based scheme devised by Maiteh *et al*. To accomplish this, an SEDE base will be used together with ray-casting and more efficient localization and triangulation refinement procedures.


Tool Swept Volume
Initial Crude Triangulation
Fine Triangulation
Refinement on Cut Portion
Workpiece

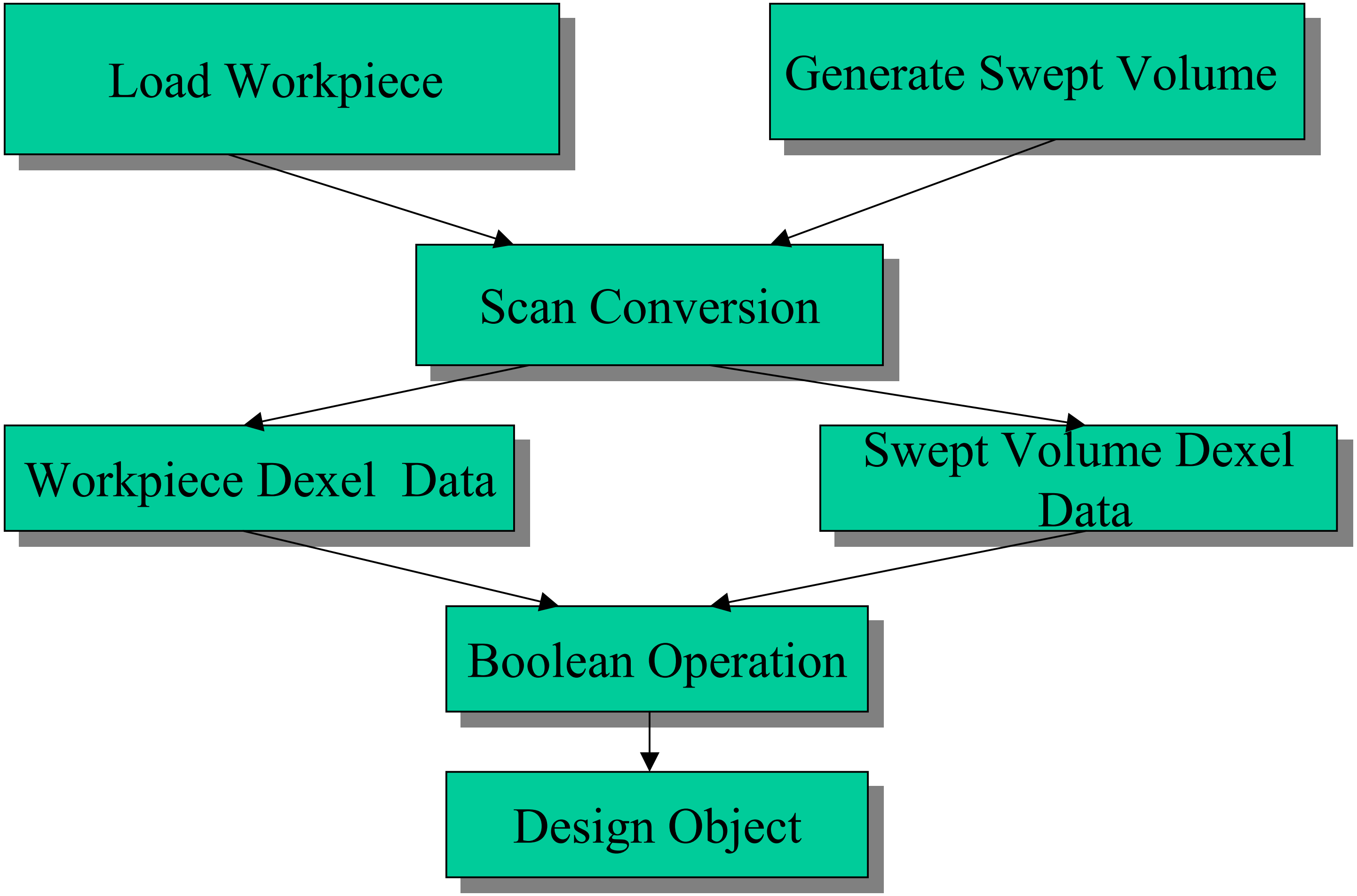
Main Ideas of Virtual Sculpting



Analogy: NC Machining Simulation



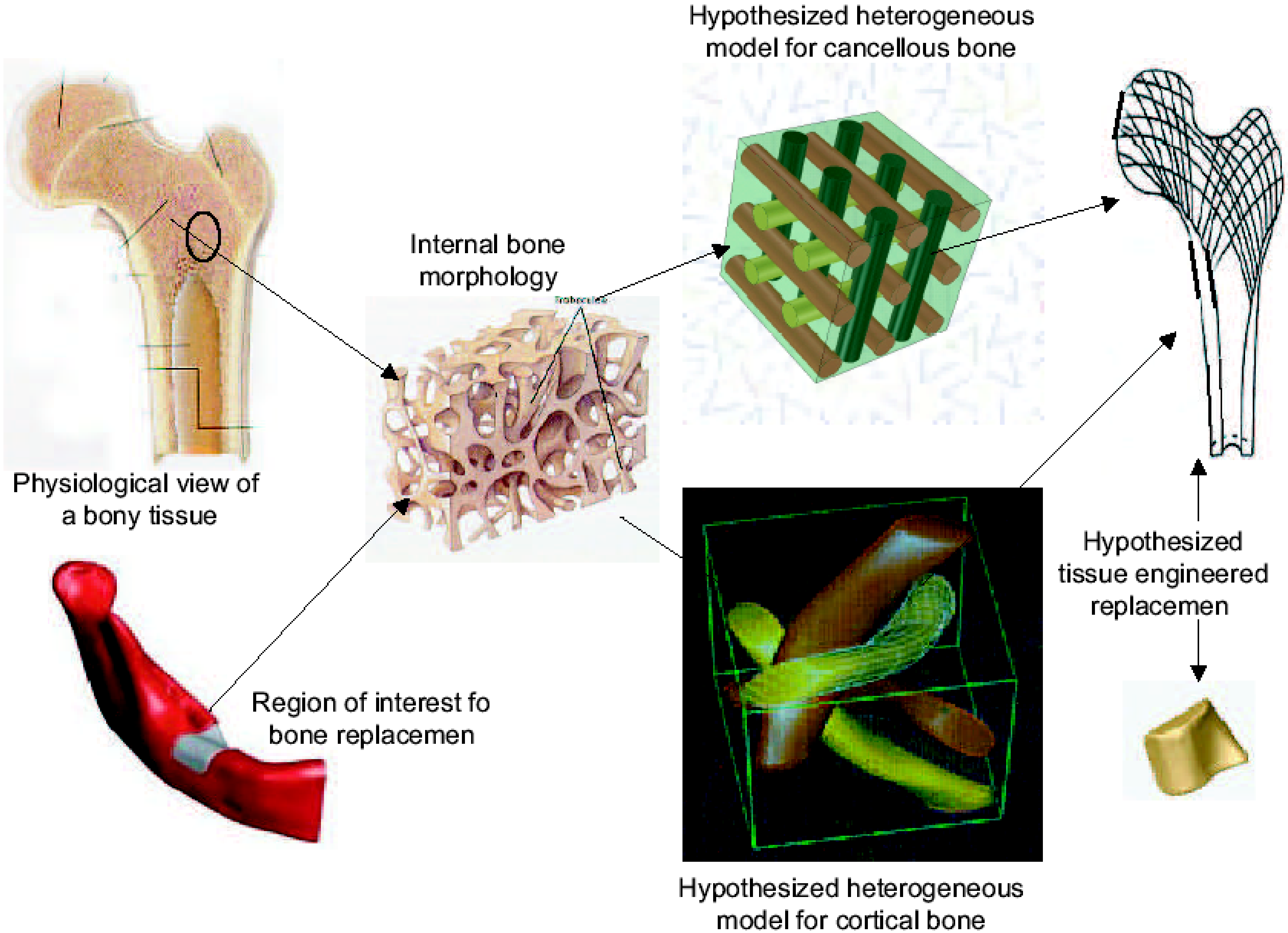
Solid Modeling Engine



Application to Tissue Engineering

The heterogeneous structures found in tissue engineering can be modeled as objects produced by swept volume operations.

- Examples include fiberous materials, bone, connective tissues, growth matrices, etc


Hypothesized heterogeneous model for cancellous bone
Internal bone morphology
Physiological view of a bony tissue
Hypothesized tissue engineered replacemen
Region of interest fo bone replacemen
Hypothesized heterogeneous model for cortical bone

Selected Goals for Tissue Engineering Application

- Find an efficient ways to represent complex object properties
  - density (studied much in current literature)
  - porosity (e.g., the air pockets in a loaf of bread or the cavities in a piece of bone; not studied much).
  - permeability (e.g. rate of air/liquid/etc able to pass through an object)
- Develop efficient algorithms to perform modeling and analysis operations on objects
- Develop manufacturing processes to create objects with these properties

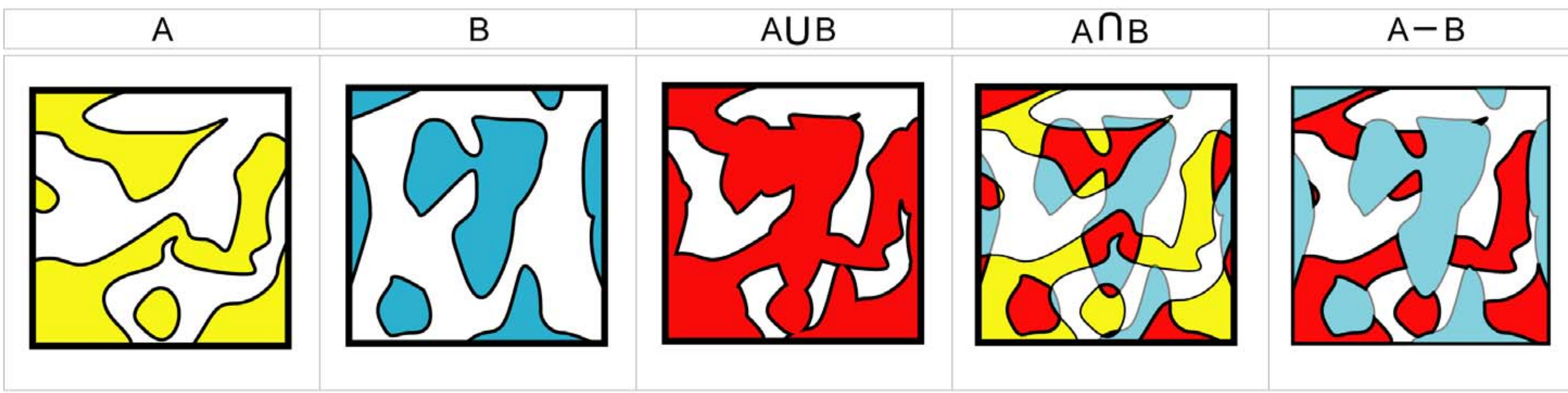
Approach

- Store the statistical properties of the object's interior rather than the exact internal geometry of each and every cavity or pocket in the object.
  - Integrate *Stochastic Geometry* with CAD and solid modeling
  - Model complex object properties as stochastic point processes, stochastic fiber processes, etc
  - Properties are captured as statistical distributions and property measures
- Develop operators work on statistical distributions and returns a distribution that would likely describe operations (e.g. union, intersection, or difference) between the original materials.

Boolean Operations on Stochastic Material Representations

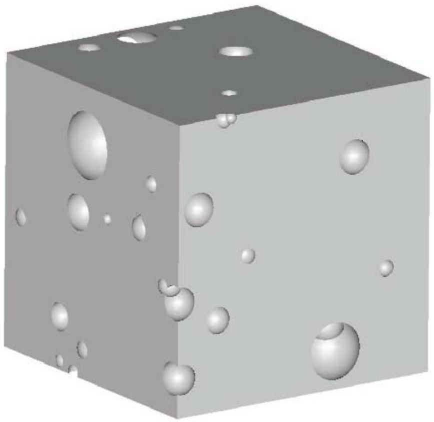
The probability of the object obtained after a Boolean operation containing material at a certain location is based on…

- Union: probability that A or B contain material there.
- Intersection: probability that both A and B contain material…
- Subtraction: probability that A contains material, B does not.

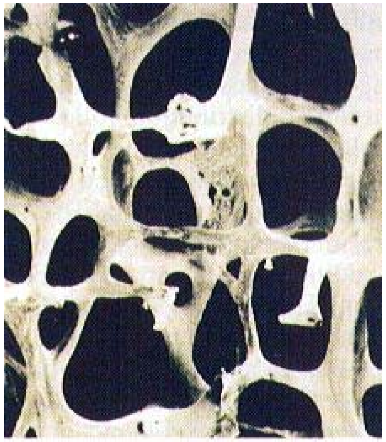


The red areas represent the combination of probability distributions from Boolean operations.

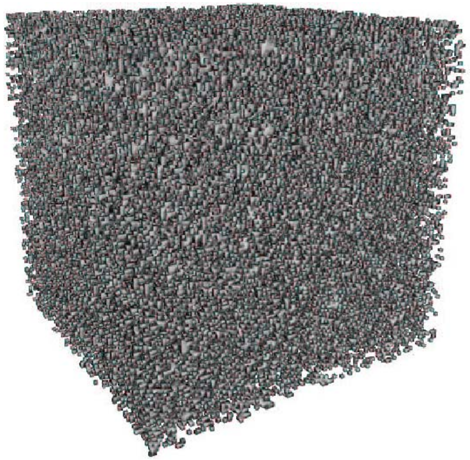
Example Porous Materials



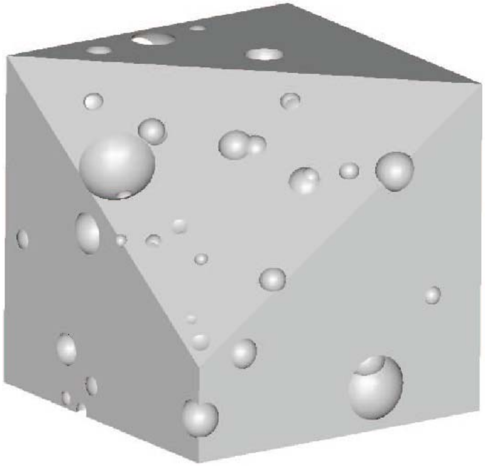
Cube generated by removing spheres



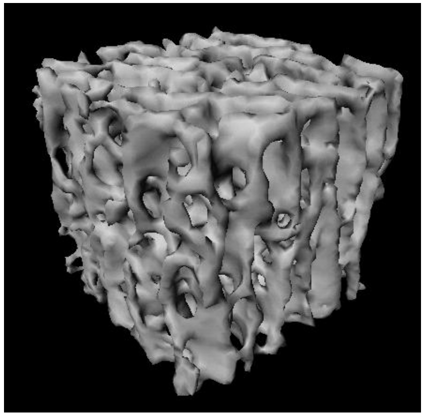
Bone matrix, courtesy of NASA.



Porous cube generated by simulation



Cross section



Trabecular bone, courtesy of Berkeley Univ.

Model with varying porosity

Activities under CARGO

- Integration of swept volume representations with stochastic properties
- Modeling attributes like "flow" and inter-material connectivity with sweep representations
- Derive manufacturing parameters
  - From sweep vols to SFF-manufactured prototypes and parts
  - Work with Therics, bio-material manufacturing company in Princeton, NJ
  - Work with NIST on heterogeneous model standards

Preliminary Work on Flow Representation and Intersections

Some progress has already been made on a couple of basic problems associated with the project, namely:

Problem A : How can smooth flow representations of object boundaries be effectively employed to determine intersections, and what type of shape invariants may prove useful?

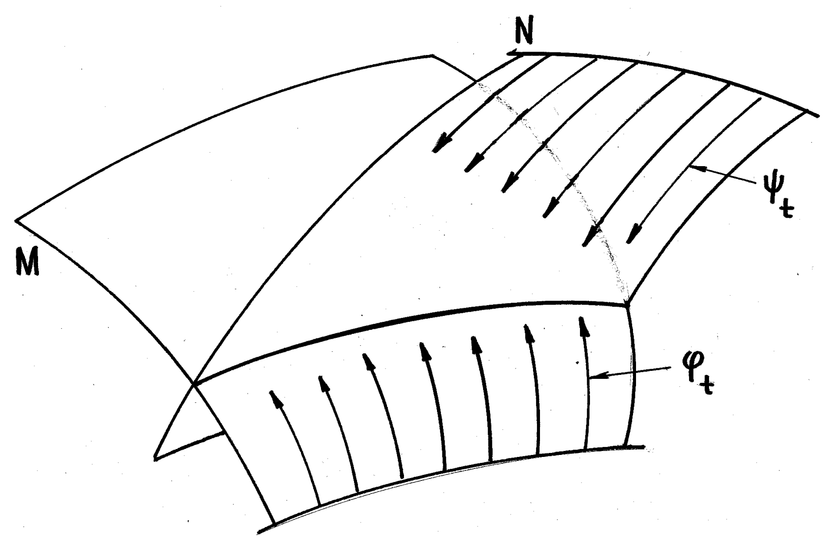


Fig 6. Intersection of objects

The intersection question – in various guises – has been and is being studied extensively (e.g., see the work of several CARGO grantees). Initial indications are that the flow approach can be effectively combined with several existing intersection algorithms and further improvements may be attainable though such innovations as smooth versions of Bezout's theorem.

Problem B : How can smooth flow interpolations be smoothly blended over a whole object, and how can additional information on object features such as curvature and various singular subsets be efficiently integrated into such interpolation programs?

It has been found that there are quite a few means available to resolve this question. However, considerably more research will be necessary to develop an "optimal" solution.

Project Flowchart