Parity Edge-Coloring of Graphs

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu

(Joint with David Bunde, Kevin Milans, Hehui Wu)

Motivation

Ques. What graphs embed in a k-dimensional cube?

Motivation

Ques. What graphs embed in a k-dimensional cube?

- k-coloring the edges by the k coordinates yields natural necessary conditions. In this coloring:
  (1) On every cycle, every color appears even # times.
  (2) On every path, some color appears odd # times.

Motivation

Ques. What graphs embed in a k-dimensional cube?

- k-coloring the edges by the k coordinates yields natural necessary conditions. In this coloring:
  (1) On every cycle, every color appears even # times.
  (2) On every path, some color appears odd # times.

- Some graphs (C_{2m+1}, $K_{2,3}$, etc.) occur in no cube, but every graph has a coloring satisfying (2).

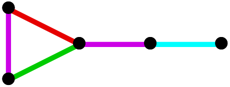
Motivation

Ques. What graphs embed in a k-dimensional cube?

• k-coloring the edges by the k coordinates yields natural necessary conditions. In this coloring:
(1) On every cycle, every color appears even # times.
(2) On every path, some color appears odd # times.

• Some graphs (C_{2m+1}, $K_{2,3}$, etc.) occur in no cube, but every graph has a coloring satisfying (2).

Def. Parity edge-coloring = edge-coloring having (2).
Parity edge-chrom. num. $p(G)$ = min # colors needed.



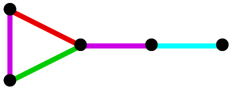
Motivation

Ques. What graphs embed in a k-dimensional cube?

• k-coloring the edges by the k coordinates yields natural necessary conditions. In this coloring:
(1) On every cycle, every color appears even # times.
(2) On every path, some color appears odd # times.

• Some graphs (C_{2m+1}, $K_{2,3}$, etc.) occur in no cube, but every graph has a coloring satisfying (2).

Def. Parity edge-coloring = edge-coloring having (2).
Parity edge-chrom. num. $p(G)$ = min # colors needed.



Obs. $p(G) \geq \chi'(G)$, and $H \subseteq G \Rightarrow p(H) \leq p(G)$.

Pf. Every parity edge-coloring is a proper edge-coloring.
Every parity edge-col. of G is a parity edge-col. of H.

A Related Parameter

Def. Parity walk = walk using each color even #times.
Strong parity edge-coloring (spec) = edge-coloring such that every parity walk is closed.
spec number $\hat{p}(G)$ = least #colors in a spec.

A Related Parameter

Def. Parity walk = walk using each color even #times.
Strong parity edge-coloring (spec) = edge-coloring such that every parity walk is closed.
spec number $\hat{p}(G)$ = least #colors in a spec.

Obs. $\hat{p}(G) \geq p(G)$.

A Related Parameter

Def. Parity walk = walk using each color even #times.
Strong parity edge-coloring (spec) = edge-coloring such that every parity walk is closed.
spec number $\hat{p}(G)$ = least #colors in a spec.

Obs. $\hat{p}(G) \geq p(G)$.

Thm. $\hat{p}(K_n) = p(K_n) = \chi'(K_n) = n - 1$ when $n = 2^k$, with a unique coloring.

A Related Parameter

Def. Parity walk = walk using each color even #times.
Strong parity edge-coloring (spec) = edge-coloring such that every parity walk is closed.
spec number $\hat{p}(G)$ = least #colors in a spec.

Obs. $\hat{p}(G) \geq p(G)$.

Thm. $\hat{p}(K_n) = p(K_n) = \chi'(K_n) = n - 1$ when $n = 2^k$, with a unique coloring.

Thm. [Main Result] $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n.

A Related Parameter

Def. Parity walk = walk using each color even #times.
Strong parity edge-coloring (spec) = edge-coloring such that every parity walk is closed.
spec number $\hat{p}(G)$ = least #colors in a spec.

Obs. $\hat{p}(G) \geq p(G)$.

Thm. $\hat{p}(K_n) = p(K_n) = \chi'(K_n) = n - 1$ when $n = 2^k$, with a unique coloring.

Thm. [Main Result] $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n.

Conj. $p(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n. (Known for $n \leq 16$.)

Motivating Application

Thm. (Daykin-Lovász [1975]) If S is a family of n finite sets, and B is a nontrivial Boolean function, then $\#\{B(u,v)\colon\ u,v \in S\} \geq n$.

- Marica-Schönheim [1969] proved it for B = set diff.

Motivating Application

Thm. (Daykin-Lovász [1975]) If S is a family of n finite sets, and B is a nontrivial Boolean function, then $\#\{B(u,v)\colon\ u,v \in S\} \geq n$.

- Marica-Schönheim [1969] proved it for B = set diff.

Thm. If S is a family of n finite sets, and $\oplus$ is symmetric diff., then $\#\{u \oplus v\colon\ u,v \in S\} \geq 2^{\lceil \lg n \rceil}$.

Motivating Application

Thm. (Daykin-Lovász [1975]) If S is a family of n finite sets, and B is a nontrivial Boolean function, then $\#\{B(u,v)\colon\ u,v\in S\}\ge n$.

- Marica-Schönheim [1969] proved it for B = set diff.

Thm. If S is a family of n finite sets, and $\oplus$ is symmetric diff., then $\#\{u\oplus v\colon\ u,v\in S\}\ge 2^{\lceil \lg n\rceil}$.

Pf. View S as $V(K_n)$. For $uv\in E(K_n)$, let $f(uv)=u\oplus v$.

Motivating Application

Thm. (Daykin-Lovász [1975]) If S is a family of n finite sets, and B is a nontrivial Boolean function, then $\#\{B(u,v)\colon u,v\in S\}\ge n$.

- Marica-Schönheim [1969] proved it for B = set diff.

Thm. If S is a family of n finite sets, and $\oplus$ is symmetric diff., then $\#\{u\oplus v\colon u,v\in S\}\ge 2^{\lceil \lg n\rceil}$.

Pf. View S as $V(K_n)$. For $uv\in E(K_n)$, let $f(uv)=u\oplus v$.
In traversing an edge, the color is the set of elements added or deleted to get the name of the next vertex.

Motivating Application

Thm. (Daykin-Lovász [1975]) If S is a family of n finite sets, and B is a nontrivial Boolean function, then $\#\{B(u,v)\colon u,v \in S\} \ge n$.

- Marica-Schönheim [1969] proved it for B = set diff.

Thm. If S is a family of n finite sets, and $\oplus$ is symmetric diff., then $\#\{u \oplus v\colon u,v \in S\} \ge 2^{\lceil \lg n \rceil}$.

Pf. View S as $V(K_n)$. For $uv \in E(K_n)$, let $f(uv) = u \oplus v$.
In traversing an edge, the color is the set of elements added or deleted to get the name of the next vertex.
$\therefore$ a parity walk must end where it starts.
$\therefore$ f is a spec, and the number of colors (symmetric differences) is at least $2^{\lceil \lg n \rceil} - 1$. Add $\varnothing$ for $u \oplus u$. ∎

Embedding Trees in k-cubes

Prop. A tree T is a subgraph of Q_k $\Leftrightarrow$ $p(T) \leq k$.

Embedding Trees in k-cubes

Prop. A tree T is a subgraph of $Q_k \iff p(T) \leq k$.

Pf. It suffices to show $p(T) = k \Rightarrow T$ embeds in Q_k.

Embedding Trees in k-cubes

Prop. A tree T is a subgraph of Q_k $\iff$ $p(T) \leq k$.

Pf. It suffices to show $p(T) = k \Rightarrow T$ embeds in Q_k.

Fix $r \in V(T)$. Define $f(v) \in V(Q_k)$ by letting bit i be the parity of color i usage on the r, v-path in T.

Embedding Trees in k-cubes

Prop. A tree T is a subgraph of Q_k $\Leftrightarrow$ $p(T) \leq k$.

Pf. It suffices to show $p(T) = k$ $\Rightarrow$ T embeds in Q_k.

Fix $r \in V(T)$. Define $f(v) \in V(Q_k)$ by letting bit i be the parity of color i usage on the r, v-path in T.

The image of each edge in T is an edge in Q_k.

Embedding Trees in k-cubes

Prop. A tree T is a subgraph of Q_k $\Leftrightarrow$ $p(T) \le k$.

Pf. It suffices to show $p(T) = k$ $\Rightarrow$ T embeds in Q_k.

Fix $r \in V(T)$. Define $f(v) \in V(Q_k)$ by letting bit i be the parity of color i usage on the r, v-path in T.

The image of each edge in T is an edge in Q_k.

$\exists$ color with odd usage on the u, v-path, so $f(u) \ne f(v)$. ■

Embedding Trees in k-cubes

Prop. A tree T is a subgraph of Q_k $\Leftrightarrow$ $p(T) \le k$.

Pf. It suffices to show $p(T) = k \Rightarrow T$ embeds in Q_k.

Fix $r \in V(T)$. Define $f(v) \in V(Q_k)$ by letting bit i be the parity of color i usage on the r, v-path in T.

The image of each edge in T is an edge in Q_k.

$\exists$ color with odd usage on the u, v-path, so $f(u) \ne f(v)$. ■

- Embeddability in hypercubes is NP-complete for trees (Wagner–Corneil [1990]), so computing $p(G)$ is also.

Embedding Trees in k-cubes

Prop. A tree T is a subgraph of Q_k $\Leftrightarrow$ $p(T) \le k$.

Pf. It suffices to show $p(T) = k$ $\Rightarrow$ T embeds in Q_k.

Fix $r \in V(T)$. Define $f(v) \in V(Q_k)$ by letting bit i be the parity of color i usage on the r, v-path in T.

The image of each edge in T is an edge in Q_k.

$\exists$ color with odd usage on the u, v-path, so $f(u) \neq f(v)$. ■

Cor. (Havel-Movárek [1972]) A graph G embeds in Q_k $\Leftrightarrow$ G has a k-pec where every cycle is a parity walk.

Embedding Trees in k-cubes

Prop. A tree T is a subgraph of Q_k $\Leftrightarrow$ $p(T) \le k$.

Pf. It suffices to show $p(T) = k$ $\Rightarrow$ T embeds in Q_k.

Fix $r \in V(T)$. Define $f(v) \in V(Q_k)$ by letting bit i be the parity of color i usage on the r, v-path in T.

The image of each edge in T is an edge in Q_k.

$\exists$ color with odd usage on the u,v-path, so $f(u) \neq f(v)$. ■

Cor. (Havel-Movárek [1972]) A graph G embeds in Q_k $\Leftrightarrow$ G has a k-pec where every cycle is a parity walk.

Pf. Embed a spanning tree T of G in Q_k as done above.

Embedding Trees in k-cubes

Prop. A tree T is a subgraph of Q_k $\Leftrightarrow$ $p(T) \le k$.

Pf. It suffices to show $p(T) = k$ $\Rightarrow$ T embeds in Q_k.

Fix $r \in V(T)$. Define $f(v) \in V(Q_k)$ by letting bit i be the parity of color i usage on the r, v-path in T.

The image of each edge in T is an edge in Q_k.

$\exists$ color with odd usage on the u, v-path, so $f(u) \ne f(v)$. ■

Cor. (Havel-Movárek [1972]) A graph G embeds in Q_k $\Leftrightarrow$ G has a k-pec where every cycle is a parity walk.

Pf. Embed a spanning tree T of G in Q_k as done above.

Each remaining edge e completes a cycle. When $e = uv$, the color on e is the only color with odd usage on the u, v-path in T. Hence $f(u) \leftrightarrow f(v)$ in Q_k. ■

All Graphs, Paths, Cycles

Cor. If G is connected, then $p(G) \geq \lceil \lg n(G) \rceil$, with equality for paths and even cycles.

All Graphs, Paths, Cycles

Cor. If G is connected, then $p(G) \geq \lceil \lg n(G) \rceil$, with equality for paths and even cycles.

Pf. If T is a spanning tree of G, then $p(G) \geq p(T)$.

Since $T \subseteq Q_{p(T)}$, we have $n(G) = n(T) \leq n(Q_{p(T)}) = 2^{p(T)}$.

All Graphs, Paths, Cycles

Cor. If G is connected, then $p(G) \geq \lceil \lg n(G) \rceil$, with equality for paths and even cycles.

Pf. If T is a spanning tree of G, then $p(G) \geq p(T)$.

Since $T \subseteq Q_{p(T)}$, we have $n(G) = n(T) \leq n(Q_{p(T)}) = 2^{p(T)}$.

Equality: P_n and C_n embed in $Q_{\lceil \lg n \rceil}$. ■

All Graphs, Paths, Cycles

Cor. If G is connected, then $p(G) \geq \lceil \lg n(G) \rceil$,
with equality for paths and even cycles.

Pf. If T is a spanning tree of G, then $p(G) \geq p(T)$.

Since $T \subseteq Q_{p(T)}$, we have $n(G) = n(T) \leq n(Q_{p(T)}) = 2^{p(T)}$.

Equality: P_n and C_n embed in $Q_{\lceil \lg n \rceil}$. ∎

- Odd cycles will need one more!

All Graphs, Paths, Cycles

Cor. If G is connected, then $p(G) \geq \lceil \lg n(G) \rceil$, with equality for paths and even cycles.

Pf. If T is a spanning tree of G, then $p(G) \geq p(T)$.

Since $T \subseteq Q_{p(T)}$, we have $n(G) = n(T) \leq n(Q_{p(T)}) = 2^{p(T)}$.

Equality: P_n and C_n embed in $Q_{\lceil \lg n \rceil}$. ■

- Odd cycles will need one more!

Obs. Always $p(G) \leq p(G - e) + 1$.

Pf. Put optimal pec on $G - e$; add new color on e. Each path is okay in G whether it uses e or not. ■

All Graphs, Paths, Cycles

Cor. If G is connected, then $p(G) \geq \lceil \lg n(G) \rceil$, with equality for paths and even cycles.

Pf. If T is a spanning tree of G, then $p(G) \geq p(T)$.

Since $T \subseteq Q_{p(T)}$, we have $n(G) = n(T) \leq n(Q_{p(T)}) = 2^{p(T)}$.

Equality: P_n and C_n embed in $Q_{\lceil \lg n \rceil}$. ■

- Odd cycles will need one more!

Obs. Always $p(G) \leq p(G - e) + 1$.

Pf. Put optimal pec on $G - e$; add new color on e. Each path is okay in G whether it uses e or not. ■

Cor. If n is odd, then $\lceil \lg n \rceil \leq p(C_n) \leq \lceil \lg n \rceil + 1$.

Lower Bound for Odd Cycles

Lem. Every pec of C_n is a spec, so $p(C_n) = \hat{p}(C_n)$.

Pf. The edges with odd usage in an open walk W form a path P joining the ends of W.
P has some odd-used color; $\therefore$ W is not a parity walk. ∎

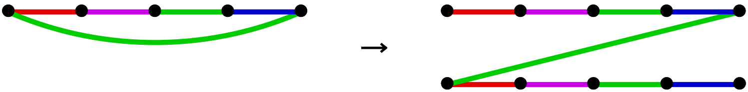
Lower Bound for Odd Cycles

Lem. Every pec of C_n is a spec, so $p(C_n) = \hat{p}(C_n)$.

Pf. The edges with odd usage in an open walk W form a path P joining the ends of W.
P has some odd-used color; $\therefore$ W is not a parity walk. ∎

Lem. If n is odd, then $\hat{p}(C_n) \geq p(P_{2n})$.

Pf. Spec of C_n yields pec of P_{2n}.



Each path in P_{2n} arises from an open walk in C_n
or one trip around the cycle (which is odd length). ∎

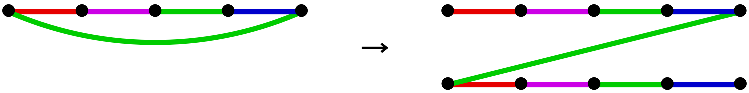
Lower Bound for Odd Cycles

Lem. Every pec of C_n is a spec, so $p(C_n) = \hat{p}(C_n)$.

Pf. The edges with odd usage in an open walk W form a path P joining the ends of W.
P has some odd-used color; $\therefore$ W is not a parity walk. ■

Lem. If n is odd, then $\hat{p}(C_n) \geq p(P_{2n})$.

Pf. Spec of C_n yields pec of P_{2n}.

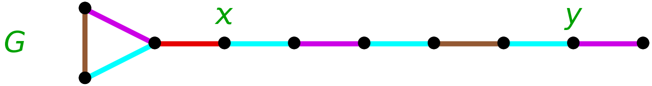


Each path in P_{2n} arises from an open walk in C_n
or one trip around the cycle (which is odd length). ■

Thm. If n is odd, then $p(C_n) = \lceil \lg n \rceil + 1$.

Example Showing $p \neq \hat{p}$

- Unrolling technique (like lower bound for odd cycle)

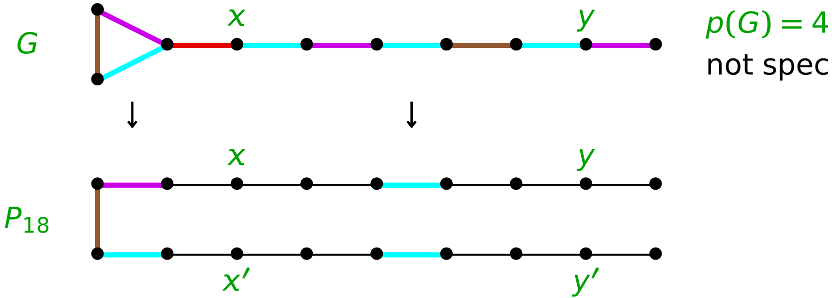


$p(G) = 4$

not spec

Example Showing $p \neq \hat{p}$

- Unrolling technique (like lower bound for odd cycle)

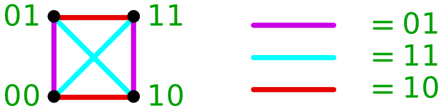


Obs. $\hat{p}(G) \geq p(P_{18}) = 5$.

Pf. Copy a spec of G onto P_{18} (path edges doubled).
An x, y'-subpath of P_{18} comes from an open walk in G.
An x, x'-subpath of P_{18} comes from an odd walk in G. ∎

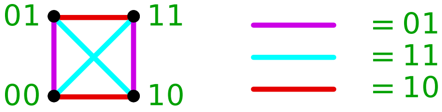
Complete Graphs, $n = 2^k$

Def. canonical coloring of K_{2^k} = edge-coloring f defined by $f(uv) = u + v$, where $V(K_{2^k}) = \mathbf{F}_2^k$.



Complete Graphs, $n = 2^k$

Def. canonical coloring of K_{2^k} = edge-coloring f defined by $f(uv) = u + v$, where $V(K_{2^k}) = \mathbf{F}_2^k$.



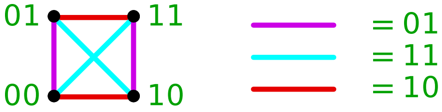
Prop. If $n = 2^k$, then $p(K_n) = \hat{p}(K_n) = \chi'(K_n) = n - 1$.

Pf. Canonical coloring uses $n - 1$ colors (0^k not used).

It is a spec: When the ends of a walk W differ in bit i, the total usage of colors flipping bit i is odd, so $\exists$ odd-usage color on W. ∎

Complete Graphs, $n = 2^k$

Def. canonical coloring of K_{2^k} = edge-coloring f defined by $f(uv) = u + v$, where $V(K_{2^k}) = \mathbf{F}_2^k$.



Prop. If $n = 2^k$, then $p(K_n) = \hat{p}(K_n) = \chi'(K_n) = n - 1$.

Pf. Canonical coloring uses $n - 1$ colors (0^k not used).

It is a spec: When the ends of a walk W differ in bit i, the total usage of colors flipping bit i is odd, so $\exists$ odd-usage color on W. ■

Cor. $\hat{p}(K_n) \leq 2^{\lceil \lg n \rceil} - 1 \leq 2n - 3$.

Conj. $p(K_n) = 2^{\lceil \lg n \rceil} - 1$. (**Thm.** $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$.)

Just Above the Threshold: K_2, K_3, K_5

- It suffices to prove $p(K_{2^k+1}) = 2^{k+1} - 1$.

$k = 0$: $p(K_2) = 1$; $k = 1$: $p(K_3) = 3$; $k = 2$?

Prop. $p(K_5) = 7$.

Just Above the Threshold: K_2, K_3, K_5

- It suffices to prove $p(K_{2^k+1}) = 2^{k+1} - 1$.

$k = 0$: $p(K_2) = 1$; $k = 1$: $p(K_3) = 3$; $k = 2$?

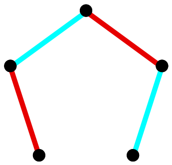
Prop. $p(K_5) = 7$.

Pf. Each color forms a matching $\Rightarrow$ used at most twice.

10 edges, ≤ 6 colors $\Rightarrow$ at least four colors used twice.

Two colors used twice must not form parity path P_5.

$\therefore$ colors of size two are used at the same four vertices, but then only three can be used twice. ■



Just Above the Threshold: K_2, K_3, K_5

- It suffices to prove $p(K_{2^k+1}) = 2^{k+1} - 1$.

$k = 0$: $p(K_2) = 1$; $k = 1$: $p(K_3) = 3$; $k = 2$?

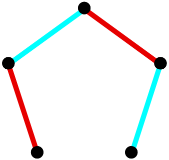
Prop. $p(K_5) = 7$.

Pf. Each color forms a matching $\Rightarrow$ used at most twice.

10 edges, ≤ 6 colors $\Rightarrow$ at least four colors used twice.

Two colors used twice must not form parity path P_5.

$\therefore$ colors of size two are used at the same four vertices, but then only three can be used twice. ∎

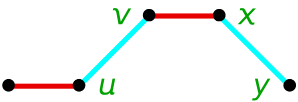


Prop. $p(K_9) = 15$. (Longer ad hoc argument.)

Structure of colorings

Thm. If f is a spec of K_n with every color class a perfect matching, then f is canonical & n is a 2-power.

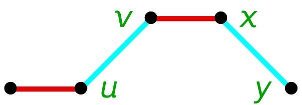
Pf. 4-constraint: If $f(uv) = f(xy)$, then $f(uy) = f(vx)$ (since every color is at every vertex).



Structure of colorings

Thm. If f is a spec of K_n with every color class a perfect matching, then f is canonical & n is a 2-power.

Pf. 4-constraint: If $f(uv) = f(xy)$, then $f(uy) = f(vx)$ (since every color is at every vertex).



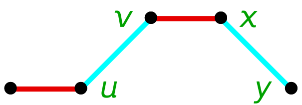
Aim: Map $V(K_n)$ to $\mathbf{F}_2^k$ so f is the canonical coloring.

Every edge is a canonically colored K_2. Let R be a largest vertex set on which f restricts to a canonical coloring. If $R \neq V(K_n)$, we obtain a larger such set.

Structure of colorings

Thm. If f is a spec of K_n with every color class a perfect matching, then f is canonical & n is a 2-power.

Pf. 4-constraint: If $f(uv) = f(xy)$, then $f(uy) = f(vx)$ (since every color is at every vertex).



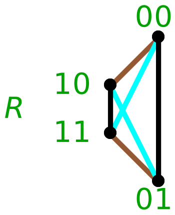
Aim: Map $V(K_n)$ to $\mathbf{F}_2^k$ so f is the canonical coloring.

Every edge is a canonically colored K_2. Let R be a largest vertex set on which f restricts to a canonical coloring. If $R \neq V(K_n)$, we obtain a larger such set.

With $|R| = 2^{j-1}$, we are given a bijection from R to $\mathbf{F}_2^{j-1}$ under which f is the canonical coloring.

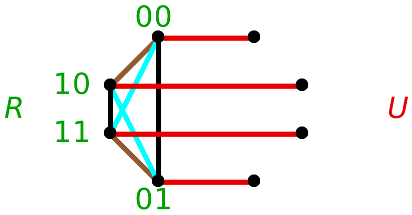
Expanding the Canonical Portion

f canonical on R $\Rightarrow$ any color used within R pairs up R.



Expanding the Canonical Portion

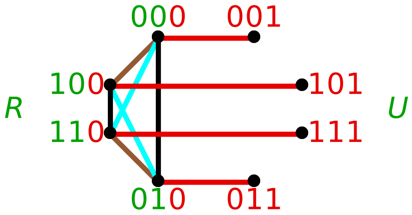
f canonical on R $\Rightarrow$ any color used within R pairs up R.



New color c pairs R to some set U; set $R' = R \cup U$.

Expanding the Canonical Portion

f canonical on R $\Rightarrow$ any color used within R pairs up R.

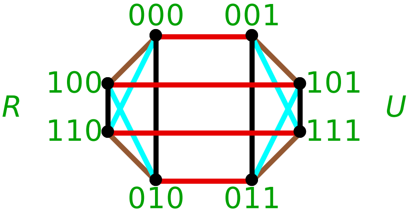


New color c pairs R to some set U; set $R' = R \cup U$.

Map R' to $\mathbf{F}_2^j$ by appending 0 to the codes in R and appending 1 instead to their c-mates in U.

Expanding the Canonical Portion

f canonical on R $\Rightarrow$ any color used within R pairs up R.



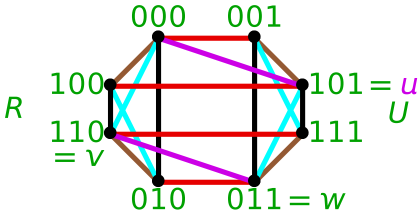
New color c pairs R to some set U; set $R' = R \cup U$.

Map R' to $\mathbf{F}_2^j$ by appending 0 to the codes in R and appending 1 instead to their c-mates in U.

The 4-constraint copies the coloring from R to U,
so $f(uu') = f(vv') = v + v' = u + u'$.

Expanding the Canonical Portion

f canonical on R $\Rightarrow$ any color used within R pairs up R.



New color c pairs R to some set U; set $R' = R \cup U$.

Map R' to $\mathbf{F}_2^j$ by appending 0 to the codes in R and appending 1 instead to their c-mates in U.

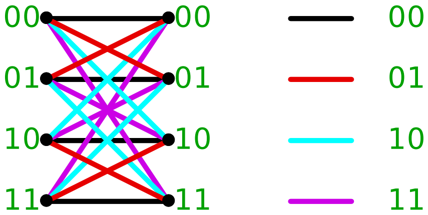
The 4-constraint copies the coloring from R to U,
so $f(uu') = f(vv') = v + v' = u + u'$.

Use u to name the color on $0^j u$, so $f(0^j u) = u = 0^j + u$.
The rest: $v \in R$ & $w = u + v \in U \Rightarrow f(v0^j) = f(uw) = v$;
4-constraint $\Rightarrow$ $f(uv) = f(0^j w) = w = u + v$. ∎

Complete Bipartite Graph $K_{n,n}$

Prop. If $n = 2^k$, then $p(K_{n,n}) = \hat{p}(K_{n,n}) = \chi'(K_{n,n}) = n$.

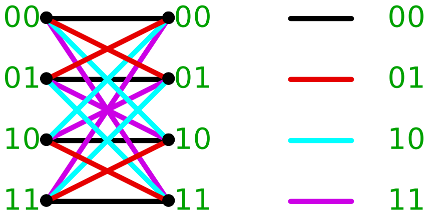
Pf. Label each partite set with $\mathbf{F}_2^k$. Let $f(uv) = u + v$.



Complete Bipartite Graph $K_{n,n}$

Prop. If $n = 2^k$, then $p(K_{n,n}) = \hat{p}(K_{n,n}) = \chi'(K_{n,n}) = n$.

Pf. Label each partite set with $\mathbf{F}_2^k$. Let $f(uv) = u + v$.



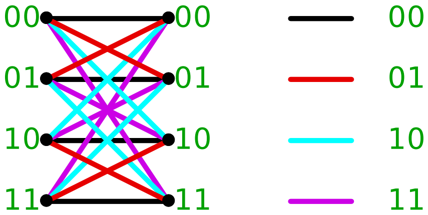
A parity walk (even usage each color) has even length.
Even usage $\Rightarrow$ bits flipped evenly often by each color.

$\therefore$ a parity walk ends at same label on the same side.
That is, every parity walk is a closed walk. ∎

Complete Bipartite Graph $K_{n,n}$

Prop. If $n = 2^k$, then $p(K_{n,n}) = \hat{p}(K_{n,n}) = \chi'(K_{n,n}) = n$.

Pf. Label each partite set with $\mathbf{F}_2^k$. Let $f(uv) = u + v$.



A parity walk (even usage each color) has even length.
Even usage $\Rightarrow$ bits flipped evenly often by each color.

$\therefore$ a parity walk ends at same label on the same side.
That is, every parity walk is a closed walk. ■

Conj. $p(K_{n,n}) = \hat{p}(K_{n,n}) = 2^{\lceil \lg n \rceil}$ for all n.

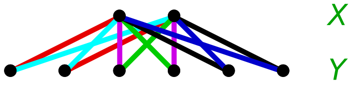
Other Complete Bipartite Graphs

Thm. $m = 2^k$ and $m \mid n \Rightarrow \hat{p}(K_{m,n}) = \Delta(K_{m,n}) = n$.

Other Complete Bipartite Graphs

Thm. $m = 2^k$ and $m \mid n \Rightarrow \hat{p}(K_{m,n}) = \Delta(K_{m,n}) = n$.

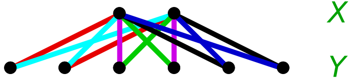
Pf. Let $r = n/m$, with $X = \mathbf{F}_2^k$ and $Y = \mathbf{F}_2^k \times [r]$.
Color $f(uv) = (u + v', j)$, where $v = (v', j)$
(r edge-disjoint copies of canonical coloring).



Other Complete Bipartite Graphs

Thm. $m = 2^k$ and $m \mid n \Rightarrow \hat{p}(K_{m,n}) = \Delta(K_{m,n}) = n$.

Pf. Let $r = n/m$, with $X = \mathbf{F}_2^k$ and $Y = \mathbf{F}_2^k \times [r]$.
Color $f(uv) = (u + v', j)$, where $v = (v', j)$
(r edge-disjoint copies of canonical coloring).

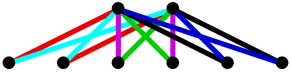


Claim: f is a spec. For a parity walk W, erasing second color coordinate maps W to a walk W' in $K_{m,m}$.

Other Complete Bipartite Graphs

Thm. $m = 2^k$ and $m \mid n \Rightarrow \hat{p}(K_{m,n}) = \Delta(K_{m,n}) = n$.

Pf. Let $r = n/m$, with $X = \mathbf{F}_2^k$ and $Y = \mathbf{F}_2^k \times [r]$.
Color $f(uv) = (u + v', j)$, where $v = (v', j)$
(r edge-disjoint copies of canonical coloring).



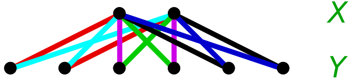
Claim: f is a spec. For a parity walk W, erasing second color coordinate maps W to a walk W' in $K_{m,m}$.

W' is a parity walk, so W' is closed. If W is open, then W ends in different $K_{m,m}$s; they have odd usage. ■

Other Complete Bipartite Graphs

Thm. $m = 2^k$ and $m \mid n \Rightarrow \hat{p}(K_{m,n}) = \Delta(K_{m,n}) = n$.

Pf. Let $r = n/m$, with $X = \mathbf{F}_2^k$ and $Y = \mathbf{F}_2^k \times [r]$.
Color $f(uv) = (u + v', j)$, where $v = (v', j)$
(r edge-disjoint copies of canonical coloring).



Claim: f is a spec. For a parity walk W, erasing second color coordinate maps W to a walk W' in $K_{m,m}$.

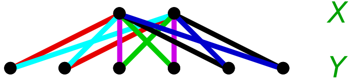
W' is a parity walk, so W' is closed. If W is open, then W ends in different $K_{m,m}$s; they have odd usage. ■

Cor. $m \le n$ and $m' = 2^{\lceil \lg m \rceil} \Rightarrow \hat{p}(K_{m,n}) \le m' \lceil n/m' \rceil$.

Other Complete Bipartite Graphs

Thm. $m = 2^k$ and $m \mid n \Rightarrow \hat{p}(K_{m,n}) = \Delta(K_{m,n}) = n$.

Pf. Let $r = n/m$, with $X = \mathbf{F}_2^k$ and $Y = \mathbf{F}_2^k \times [r]$.
Color $f(uv) = (u + v', j)$, where $v = (v', j)$
(r edge-disjoint copies of canonical coloring).



Claim: f is a spec. For a parity walk W, erasing second color coordinate maps W to a walk W' in $K_{m,m}$.

W' is a parity walk, so W' is closed. If W is open, then W ends in different $K_{m,m}$s; they have odd usage. ∎

Cor. $m \le n$ and $m' = 2^{\lceil \lg m \rceil} \Rightarrow \hat{p}(K_{m,n}) \le m' \lceil n/m' \rceil$.

Cor. $p(K_{2,2r+1}) = \hat{p}(K_{2,2r+1}) = 2r + 2$.

Pf. $p(K_{2,n}) = n \Rightarrow$ every color is at both vertices of X
$\Rightarrow$ 4-constraint holds $\Rightarrow$ Y in pairs $\Rightarrow$ n is even. ∎

Algebraic Aspects of S.p.e.c.

Def. Given an edge-coloring f, the parity vector $\pi(W)$ sets bit i to the parity of the usage of color i on W.
Parity space L_f = set of parity vectors of closed walks.

Algebraic Aspects of S.p.e.c.

Def. Given an edge-coloring f, the parity vector $\pi(W)$ sets bit i to the parity of the usage of color i on W.
Parity space L_f = set of parity vectors of closed walks.

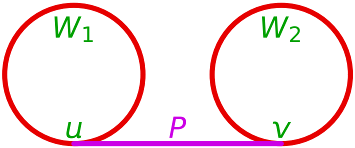
Lem. If f is an edge-coloring of a connected graph G, then L_f is a binary vector space.

Algebraic Aspects of S.p.e.c.

Def. Given an edge-coloring f, the parity vector $\pi(W)$ sets bit i to the parity of the usage of color i on W.
Parity space L_f = set of parity vectors of closed walks.

Lem. If f is an edge-coloring of a connected graph G, then L_f is a binary vector space.

Pf. When W is a u,u-walk and W' is a v,v-walk, let P be a u,v-path, with P' its reverse. Now W_1, P, W_2, P' is a u,u-walk with parity vector $\pi(W) + \pi(W')$. ■



Parity Space for Spec of K_n

Def. Let $w(L)$ denote the minimum weight of the nonzero vectors in L.

Prop. Edge-coloring f of K_n is a spec $\Leftrightarrow$ $w(L_f) \geq 2$.

Parity Space for Spec of K_n

Def. Let $w(L)$ denote the minimum weight of the nonzero vectors in L.

Prop. Edge-coloring f of K_n is a spec $\Leftrightarrow$ $w(L_f) \geq 2$.

Pf. $\exists\ \pi(W)$ with weight 1 for closed walk W

$\Leftrightarrow$ one color has odd usage in W (used on e)

$\Leftrightarrow$ $\exists$ open parity walk $W - e$

$\Leftrightarrow$ f is not a spec ■

Parity Space for Spec of K_n

Def. Let $w(L)$ denote the minimum weight of the nonzero vectors in L.

Prop. Edge-coloring f of K_n is a spec $\Leftrightarrow$ $w(L_f) \geq 2$.

Pf. $\exists\ \pi(W)$ with weight 1 for closed walk W

$\Leftrightarrow$ one color has odd usage in W (used on e)

$\Leftrightarrow$ $\exists$ open parity walk $W - e$

$\Leftrightarrow$ f is not a spec ∎

Lem. Given colors a and b in optimal spec f of K_n, some closed W has odd usage for a, b, and one other.

Parity Space for Spec of K_n

Def. Let $w(L)$ denote the minimum weight of the nonzero vectors in L.

Prop. Edge-coloring f of K_n is a spec $\Leftrightarrow$ $w(L_f) \geq 2$.

Pf. $\exists\ \pi(W)$ with weight 1 for closed walk W

$\Leftrightarrow$ one color has odd usage in W (used on e)

$\Leftrightarrow$ $\exists$ open parity walk $W - e$

$\Leftrightarrow$ f is not a spec ∎

Lem. Given colors a and b in optimal spec f of K_n, some closed W has odd usage for a, b, and one other.

Pf. Merging a and b into one color a' yields non-spec f'.
$\therefore$ some closed W has odd usage only on c under f'.
Since $c = a' \Rightarrow \mathrm{wt}(\pi_f(W)) = 1$, we have $c \neq a'$.
$\mathrm{wt}(\pi_f(W)) \geq 2 \Rightarrow a$ and b have odd usage in W. ∎

More on Parity Spaces

Lem. If G (colored by f) has a dominating vertex v, then $L_f = \text{span}\{\pi(T)\colon T \text{ is a triangle containing } v\}$.

Pf. The span is in L_f. Conversely, suppose $\pi(W) \in L_f$.

More on Parity Spaces

Lem. If G (colored by f) has a dominating vertex v, then $L_f = \mathrm{span}\{\pi(T)\colon\ T \text{ is a triangle containing } v\}$.

Pf. The span is in L_f. Conversely, suppose $\pi(W) \in L_f$.

Let S = $\{$edges with odd usage in $W\}$.
Let H = spanning subgraph of G with edge set S.

Since total usage at each vertex of W is even,
H is an even subgraph of G. Also, $\pi(H) = \pi(W)$.

More on Parity Spaces

Lem. If G (colored by f) has a dominating vertex v, then $L_f = \text{span}\{\pi(T)\colon\ T \text{ is a triangle containing } v\}$.

Pf. The span is in L_f. Conversely, suppose $\pi(W) \in L_f$.

Let S = {edges with odd usage in W}.
Let H = spanning subgraph of G with edge set S.

Since total usage at each vertex of W is even,
H is an even subgraph of G. Also, $\pi(H) = \pi(W)$.

$\therefore$ it suffices to show that S is the sum (mod 2) of the set of triangles formed by adding v to edges of $H - v$.

Each edge of $H - v$ is in one such triangle.

More on Parity Spaces

Lem. If G (colored by f) has a dominating vertex v, then $L_f = \mathrm{span}\{\pi(T)\colon\ T \text{ is a triangle containing } v\}$.

Pf. The span is in L_f. Conversely, suppose $\pi(W) \in L_f$.

Let S = {edges with odd usage in W}.
Let H = spanning subgraph of G with edge set S.

Since total usage at each vertex of W is even,
H is an even subgraph of G. Also, $\pi(H) = \pi(W)$.

$\therefore$ it suffices to show that S is the sum (mod 2) of the set of triangles formed by adding v to edges of $H - v$.

Each edge of $H - v$ is in one such triangle.

Edge vw is in odd number $\Leftrightarrow$ $d_{H-v}(w)$ is odd
$\Leftrightarrow$ $w \in N_H(v)$ (since $d_H(w)$ is even) $\Leftrightarrow$ $vw \in S$. ∎

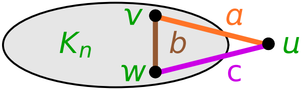
Lem. If optimal spec f of K_n uses some color α not on a perfect matching, then $\hat{p}(K_{n+1}) = \hat{p}(K_n)$.

Pf. Let v be a vertex missed by α; let u be a new vertex. We use f to define f' on the larger complete graph.

Lem. If optimal spec f of K_n uses some color a not on a perfect matching, then $\hat{p}(K_{n+1}) = \hat{p}(K_n)$.

Pf. Let v be a vertex missed by a; let u be a new vertex. We use f to define f' on the larger complete graph.

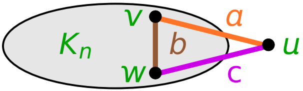
Let $f'(uv) = a$. For $w \notin \{u, v\}$, let $b = f(vw)$.
$\exists W$ with odd usage of a, b, and some c. Let $f'(uw) = c$.



Lem. If optimal spec f of K_n uses some color a not on a perfect matching, then $\hat{p}(K_{n+1}) = \hat{p}(K_n)$.

Pf. Let v be a vertex missed by a; let u be a new vertex. We use f to define f' on the larger complete graph.

Let $f'(uv) = a$. For $w \notin \{u, v\}$, let $b = f(vw)$.
$\exists W$ with odd usage of a, b, and some c. Let $f'(uw) = c$.

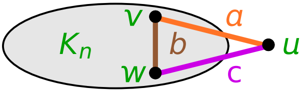


We show that $L_{f'} \subseteq L_f$ to get $w(L_{f'}) \geq 2$.
It suffices that $\pi(T) \in L_f$ when T is a triangle in K_{n+1} containing v, since these vectors span $L_{f'}$ (by lemma).

Lem. If optimal spec f of K_n uses some color a not on a perfect matching, then $\hat{p}(K_{n+1}) = \hat{p}(K_n)$.

Pf. Let v be a vertex missed by a; let u be a new vertex. We use f to define f' on the larger complete graph.

Let $f'(uv) = a$. For $w \notin \{u, v\}$, let $b = f(vw)$.
$\exists W$ with odd usage of a, b, and some c. Let $f'(uw) = c$.



We show that $L_{f'} \subseteq L_f$ to get $w(L_{f'}) \geq 2$.
It suffices that $\pi(T) \in L_f$ when T is a triangle in K_{n+1} containing v, since these vectors span $L_{f'}$ (by lemma).

If $u \notin T$, then $\pi(T) \in L_f$ by definition of L_f.
If $T = [u, v, w]$, then $\pi(T) = \pi(W) \in L_f$, where W was the walk in K_n used to specify $f'(uw)$. ■

Thm. $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$

Pf. Let $k = \hat{p}(K_n)$. Canonical coloring $\Rightarrow$ $k \le 2^{\lceil \lg n \rceil} - 1$.

Thm. $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$

Pf. Let $k = \hat{p}(K_n)$. Canonical coloring $\Rightarrow$ $k \le 2^{\lceil \lg n \rceil} - 1$.

Accumulate additional vertices without increasing $\hat{p}$ until every color class is a perfect matching.

This can't pass $2^{\lceil \lg n \rceil}$ vertices, since vertex degree then reaches $2^{\lceil \lg n \rceil} - 1$.

Thm. $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$

Pf. Let $k = \hat{p}(K_n)$. Canonical coloring $\Rightarrow$ $k \le 2^{\lceil \lg n \rceil} - 1$.

Accumulate additional vertices without increasing $\hat{p}$ until every color class is a perfect matching.

This can't pass $2^{\lceil \lg n \rceil}$ vertices, since vertex degree then reaches $2^{\lceil \lg n \rceil} - 1$.

$\therefore$ It stops with every color class a perfect matching. We showed this occurs only in the canonical coloring.

Hence $\hat{p}(K_n) = \hat{p}(K_{2^{\lceil \lg n \rceil}}) = 2^{\lceil \lg n \rceil} - 1$. ■

Thm. $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$

Pf. Let $k = \hat{p}(K_n)$. Canonical coloring $\Rightarrow$ $k \le 2^{\lceil \lg n \rceil} - 1$.

Accumulate additional vertices without increasing $\hat{p}$ until every color class is a perfect matching.

This can't pass $2^{\lceil \lg n \rceil}$ vertices, since vertex degree then reaches $2^{\lceil \lg n \rceil} - 1$.

$\therefore$ It stops with every color class a perfect matching. We showed this occurs only in the canonical coloring.

Hence $\hat{p}(K_n) = \hat{p}(K_{2^{\lceil \lg n \rceil}}) = 2^{\lceil \lg n \rceil} - 1$. ■

Cor. Every optimal spec of a complete graph is obtained by deleting vertices from a canonical coloring.

Other Related Parameters

Def. conflict-free coloring = edge-coloring s.t. each path has some color used once; $c(G)$ = least #colors.

edge-ranking = edge-coloring s.t. each path has the highest color used once; $\chi'_r(G)$ = least #colors.
Bodlaender-Deogun-Jansen-Kloks-Kratsch-Müller-Tuza 1998

Other Related Parameters

Def. conflict-free coloring = edge-coloring s.t. each path has some color used once; $c(G)$ = least #colors.

edge-ranking = edge-coloring s.t. each path has the highest color used once; $\chi'_r(G)$ = least #colors.
Bodlaender-Deogun-Jansen-Kloks-Kratsch-Müller-Tuza 1998

- $\chi'_r(G) \geq c(G) \geq p(G)$, and the difference can be large.
Indeed, $\chi'_r(K_n) \in \Theta(n^2)$ [BDJKKMT], but $p(K_n) \in \Theta(n)$.

- $c(C_8) = 4 > 3 = p(C_8)$ (by short case analysis).
Kinnersley constructed a tree where they differ.

Other Related Parameters

Def. conflict-free coloring = edge-coloring s.t. each path has some color used once; $c(G)$ = least #colors.

edge-ranking = edge-coloring s.t. each path has the highest color used once; $\chi'_r(G)$ = least #colors.
Bodlaender-Deogun-Jansen-Kloks-Kratsch-Müller-Tuza 1998

- $\chi'_r(G) \ge c(G) \ge p(G)$, and the difference can be large. Indeed, $\chi'_r(K_n) \in \Theta(n^2)$ [BDJKKMT], but $p(K_n) \in \Theta(n)$.

- $c(C_8) = 4 > 3 = p(C_8)$ (by short case analysis). Kinnersley constructed a tree where they differ.

Def. nonrepetitive edge-coloring = edge-coloring with no immed. repetition $c_1, \ldots, c_k, c_1, \ldots, c_k$ on any path; $\pi(G)$=least #colors.

- $p(G) \ge \pi(G) \ge \chi'(G)$.

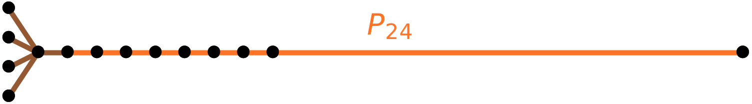
Examples Showing $c \neq \hat{p}$

Ex. $p(C_8) = \lceil \lg 8 \rceil = 3$. If conflict-free w. 3 colors, color used once $\Rightarrow$ parity 4-path.
$\therefore$ usage $(4, 2, 2)$ or $(3, 3, 2)$; kill edge of largest class.

Examples Showing $c \neq \hat{p}$

Ex. $p(C_8) = \lceil \lg 8 \rceil = 3$. If conflict-free w. 3 colors, color used once $\Rightarrow$ parity 4-path.
$\therefore$ usage $(4, 2, 2)$ or $(3, 3, 2)$; kill edge of largest class.

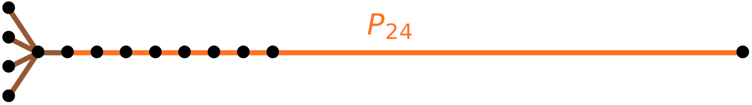
Ex. Let T_k = broom formed by identifying an end of $P_{2^k - 2k + 2}$ with a leaf of a k-edge star. (T_5 below.)



Examples Showing $c \neq \hat{p}$

Ex. $p(C_8) = \lceil \lg 8 \rceil = 3$. If conflict-free w. 3 colors, color used once $\Rightarrow$ parity 4-path.
$\therefore$ usage $(4, 2, 2)$ or $(3, 3, 2)$; kill edge of largest class.

Ex. Let T_k = broom formed by identifying an end of P_{2^k-2k+2} with a leaf of a k-edge star. (T_5 below.)



T_k embeds in Q_k, so $p(T_k) = k$. (Induct on k, using lemma that for $x, y \in V(Q_k)$ with equal parity, $\exists$ path of length $2^k - 3$ starting at x and avoiding y.)

Examples Showing $c \neq \hat{p}$

Ex. $p(C_8) = \lceil \lg 8 \rceil = 3$. If conflict-free w. 3 colors, color used once $\Rightarrow$ parity 4-path.
$\therefore$ usage $(4, 2, 2)$ or $(3, 3, 2)$; kill edge of largest class.

Ex. Let T_k = broom formed by identifying an end of P_{2^k-2k+2} with a leaf of a k-edge star. (T_5 below.)

T_k embeds in Q_k, so $p(T_k) = k$. (Induct on k, using lemma that for $x, y \in V(Q_k)$ with equal parity, $\exists$ path of length $2^k - 3$ starting at x and avoiding y.)

For $k \geq 5$, $c(T_k) = k + 1$. If conflict-free w. k colors, $P_{2^{k-1}+1}$ takes k colors, and $P_{2^{k-2}+1}$ takes $k - 1$.
All k colors appear at x, so the color missing on $P_{2^{k-2}+1}$ extends the path to one having all colors $\geq$ twice. ■

Open Problems

Conj. 1 $p(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n.
Known for $n \le 16$; proved $\hat{p}(K_n) = 2^{\lceil \lg n \rceil} - 1$ for all n.

Conj. 2 $p(K_{n,n}) = \hat{p}(K_{n,n}) = 2^{\lceil \lg n \rceil}$ for all n.

Conj. 3 $\hat{p}(G) = p(G)$ for every bipartite graph G.

Ques. 4 What is the $\max \hat{p}(G)$ when $p(G) = k$?

Ques. 5 How do $\hat{p}(K_{k,n})$ and $p(K_{k,n})$ grow with k?

Ques. 6 What is $\max p(T)$ when T is an n-vertex tree with maximum degree k? (That is, what cube contains all n-vertex trees with maximum degree k?)

Ques. 7 When does $p(G)$ equal $\lceil \lg n(G) \rceil$?

Ques. 8 Is $p(T)$ NP-hard on trees w. bounded degree?

Ques. 9 Stability . . . $\hat{p}(G \square H)$ . . . Digraphs . . .