Random Polytopes

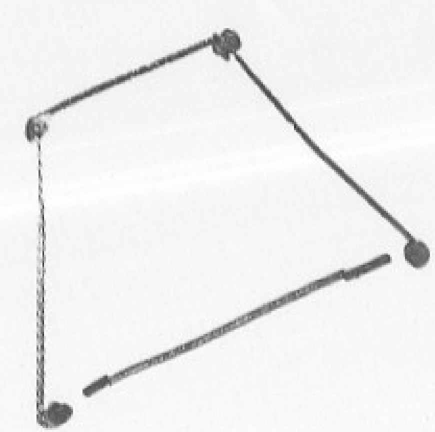
Van Vu. Rutgers Univ.

www.math.rutgers.edu/~vanvu

Sylvester's Four Point Problem (1865)

Choose 4 "RANDOM" points on the plane.

? Pr (Convex Hull is a quadrilateral)



Answer is NOT unique

Pfiefer
Blaschke
Kendall
Croft ....

Rényi-Sulanke

Efron (1960s)

Choose n random points in $\mathbb{R}^d$

K_n = convex hull ($n \to \infty$ d fixed)

What can one say about the distributions of the key functionals of K_n?

$\longrightarrow$ Theory of Random Polytopes

"Random" points: MODELS:

- Fix K convex (say, unit Ball)

  Choose the points inside K. UNIF, IND

- Normal Distribution: $\varphi_d = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$.

- Choose the points on the Boundary of K

  (good approximation).

  COMPUTATIONAL GEOMETRY

Key Functionals:

- Volume
- # of vertices
- # of faces of ANY Dimension.
- Surface area, MIXED volumes
- MEAN WIDTH ....

Questions:

- $\mathbb{E}(Y)$
- $\mathrm{Var}(Y)$, ~~thia~~ Fixed Moments.
- Central Limit Theorem

$$P\left(\frac{Y - \mathbb{E}(Y)}{\sqrt{\mathrm{Var}\, Y}} \le t \right) - \Phi(t) \underset{???}{\longrightarrow} 0$$

$$P(|Y - \mathbb{E}(Y)| \ge t\sqrt{\mathrm{Var}\, Y}) \simeq 2(1 - \Phi(t))$$

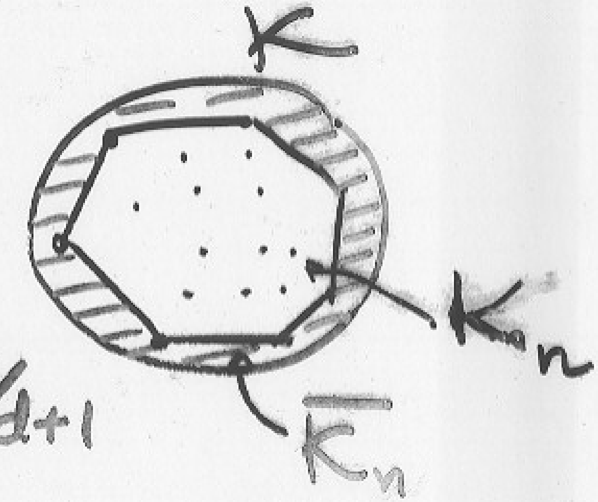
- Sharp concentration

$$P(|Y - \mathbb{E}(Y)| \ge \lambda\sqrt{\mathrm{Var}\, Y}) \overset{?}{\le} \exp(-c\lambda^2)$$

K smooth, convex (Ball)

Y = volume

Expectations have been calculated precisely



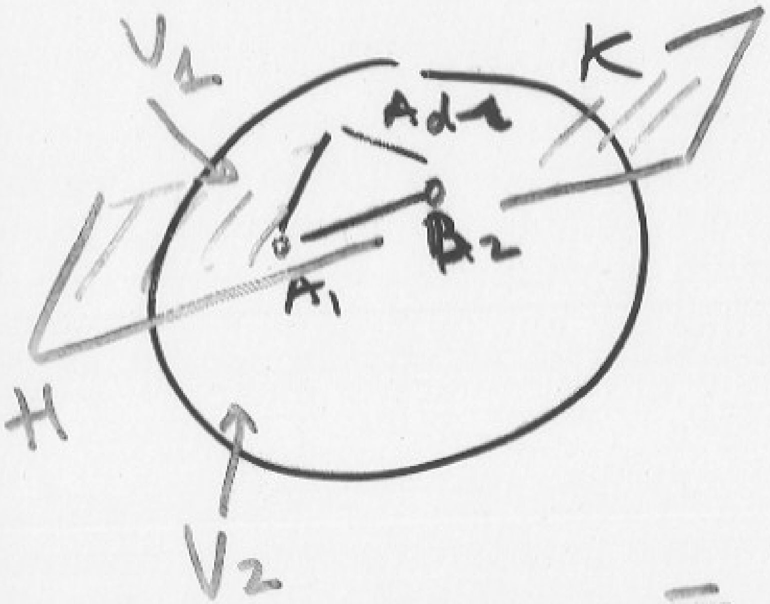
Thm

$$\mathbb{E}(\mathrm{vol}\,\bar{K}_n) = (c(K) + o(1))\, n^{-2/(d+1)}$$

(Buchta, Bárány, Schütt)

Integration Method

$Y = \#$ faces of dim $d-1$



$\mathbb{E}(Y) = \binom{n}{d} P(\cancel{A_1, \ldots, A_d}$ d random pts form a face$)$

$P(\overset{\text{Fixed}}{A_1} \ldots A_d$ form a face$)$

$= V_1^{n-d} + V_2^{n-d}$

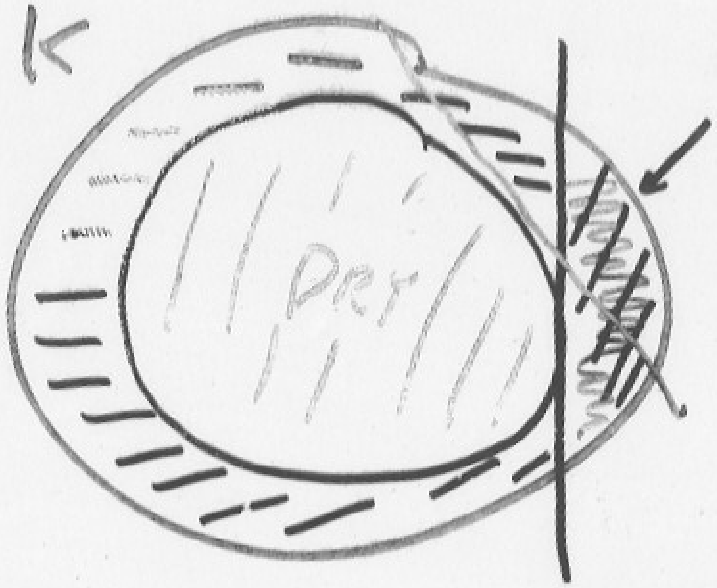
$$P\left(\begin{array}{c} d \text{ random pts} \\ \text{form a face} \end{array}\right) = \int_{K^d} V_1^{n-d}(z) + V_2^{n-d}(z) \, dz$$

$z = (A_1, \ldots, A_d)$.

OPEN QUESTION: GO BEYOND EXPECTATION:

A NEW, COMBINATORIAL WAY TO LOOK AT THE PROBLEM:

* Sharp concentration
* Any Fixed Moments
* Rate of Convergence (How Fast $\frac{Y_n}{E(Y_n)} \to 1$)
* Central limit Theorems



ε- CAP

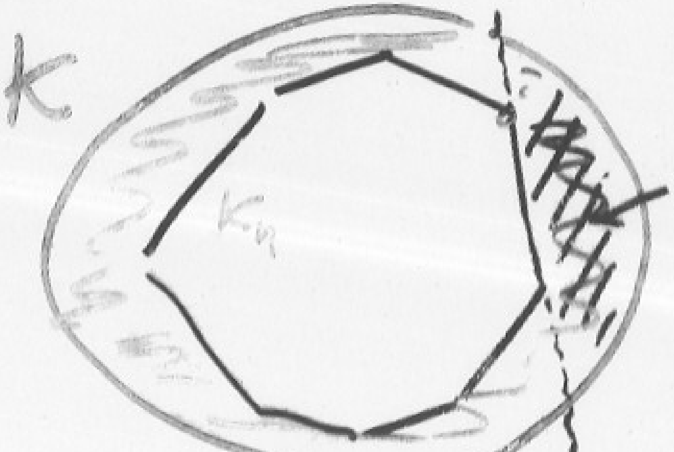
(Vol = ε)

ε-WET PART

= UNION OF ε-CAPS

Intuition

$K_n \sim \frac{1}{n}$ - ~~Wet~~ Dry PART



Vol = $\frac{1}{n}$

BÁRÁNY-LARMAN (88)

$E(\mathrm{vol}\,\bar{K}_n) = \Theta(\mathrm{vol}\ \frac{1}{n}$-W

$Y = 1 -$ [illegible]

(12)

$\mathrm{vol}(\bar{K}_n) = Y(t_1, \dots, t_n)$

random points

Sharp Concentration of Y.

Rényi - Sulanke (63)

Erdős - Rényi (60)

RANDOM POLYTOPES

RANDOM GRAPHS

(2005)

<u>Azuma</u> 60s

Changing any t_i changes Y by at most 1.

$Y(t_1, \ldots, t_n)$ is 1-Lipschitz

then $$P(|Y - E(Y)| \geq \lambda\sqrt{n}) \leq 2\exp(-\tfrac{\lambda^2}{4})$$

which has the form we want

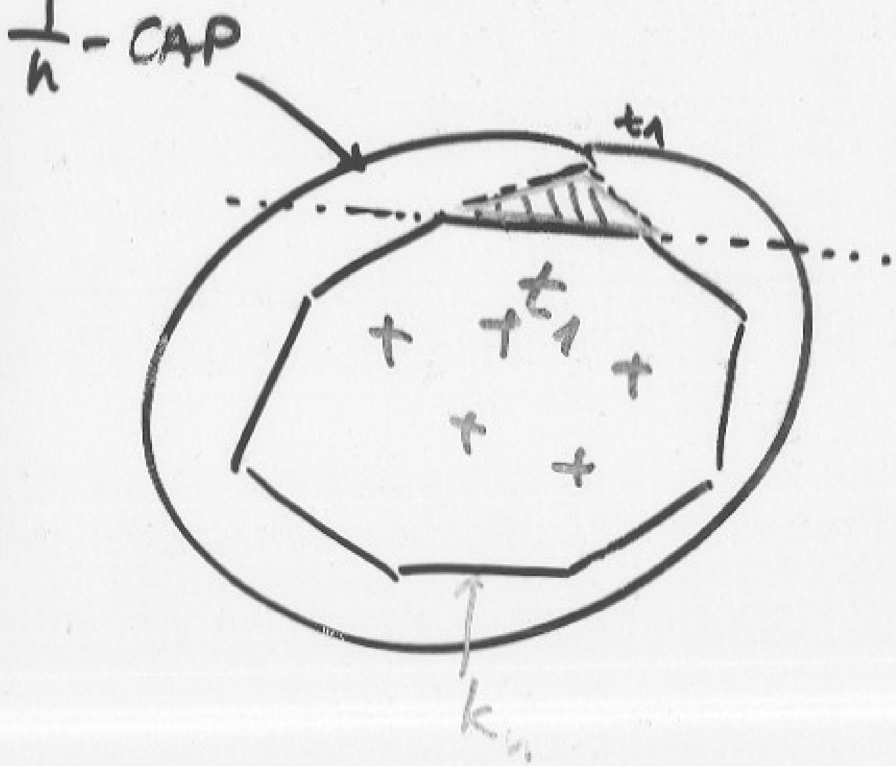
$$P(|Y - E(Y)| \geq \lambda\sqrt{\operatorname{Var} Y}) \leq 2\exp(-c\lambda^2)$$

Problem: $n \ggg \operatorname{Var} Y$

Recall: $\mathbb{E}(Y) = \Theta(n^{-2/d+1})$.

Refinement of Azuma

" ~~Worst case~~ Lipschitz coefficient "
Typical



Effect of moving t_1

0: If jumps back INSIDE.

$\frac{1}{n}$: If jumps outside

<u>Guess</u>

$$\mathrm{Var.} = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 P(t_i \text{ jumps OUTSIDE})$$

$$= \frac{1}{n^2} \cdot n \, \mathbb{E} \mathrm{Vol}(\bar{K}_n)$$

$$= \Theta\left(n^{-1} \cdot n^{-2/d+1}\right) = \Theta\left(n^{-\frac{d+3}{d+1}}\right)$$

AND (MORE CRITICALLY)

$$P\left(|Y - E(Y)| \geq \lambda \sqrt{n^{-d+3/d+1}}\right) \leq \exp(-c\lambda^2)$$

<u>Thm</u> (V. 2005) THE GUESS IS TRUE

(for $\lambda \le n^{\delta}$ $\quad \delta = \delta(d) \sim 1/3$)

MAIN TOOL: Kim-Vu Divide AND CONQUER MARTINGALE.

BÁRÁNY ECONOMICAL COVERING.

VC- Dimension.

(Vapnik- Chervonenkis)

Sharp concentration $\rightarrow$ Moments

$\rightarrow$ Central Limit Theorem

(Reitzner-V. 05)

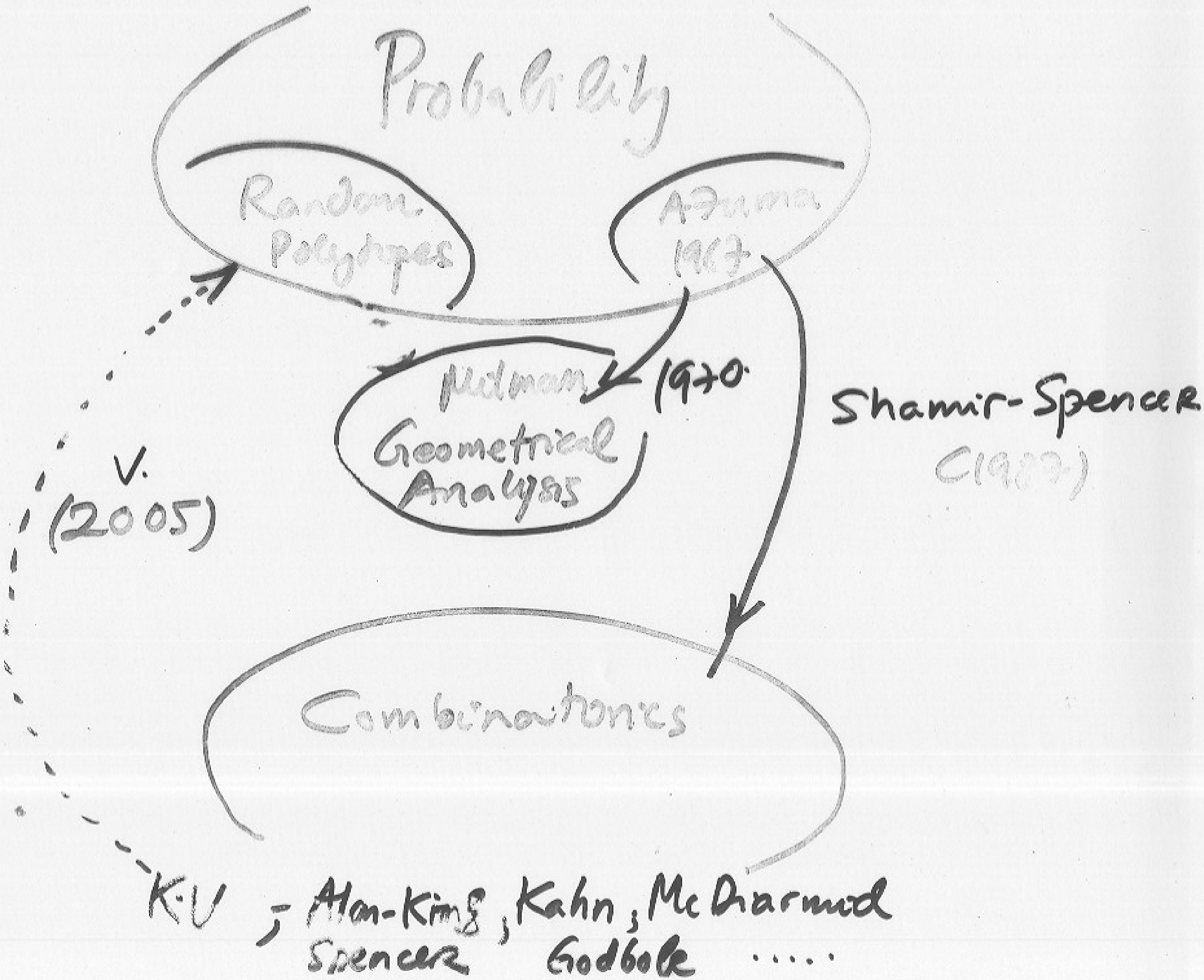
OTHER RESULTS:

K = Pop (BÁRÁNY - Reitzner)

Normdistribution (BÁRÁNY - V.)

Surfacemodel (Richardson - V. - Wu)

OPEN Q'S: A VERY LONG LIST.


Probability
Random Polytopes
Azuma 1967
Milman
1970
Geometrical Analysis
Shamir-Spencer
(1987)
V.
(2005)
Combinatorics
K-V ; Alon-Kim ; Kahn ; McDiarmid
Spencer Godbole .....