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CONVERGENCE OF GRAPHS AND GENERALIZED QUASIRANDOM GRAPHS

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When are two (large) graphs

- similar
- close to each other
- local $\longleftrightarrow$ global properties
- approximation of large g. by small g.

Generalized quasirandom graph

Convergence of graph sequences

Limit objects

Distance, completion

Approximation; Szemerédi - lemmas
tas

(G_n) convergent $\sim$

$\forall F$ "density" of F in (G_n) converges for $n \to \infty$

$(G(n;p))$ sequence of p-random graphs

$m(F,G) = \#$ labeled copies of F in G

$k = |V_F|$
$e = |E_F|$
$n = |V_G|$

$$m(F, G_n) \sim n^k p^e$$

<u>Def.</u> (G_n) is a sequenc of - quasirandom graphs if $\forall F$

$$\frac{m(F, G_n)}{n^k} \longrightarrow p^e \qquad (*)$$

<u>Theorem</u> (Chung, Graham, Wilson)

$$\left.\begin{array}{l} \dfrac{m(\bullet\!\!-\!\!\bullet, G_n)}{n^2} \longrightarrow p \\[2em] \dfrac{m(\square, G_n)}{n^4} \longrightarrow p^4 \end{array}\right\} \Longrightarrow \begin{array}{l} (G_n) \text{ is} \\ p\text{-quasirando} \\[1em] \underline{(*) \text{ holds } \forall F} \end{array}$$

Generalized random graphs

H weighted graph

$V(H) = \{1, \dots, q\}$, $\alpha_i > 0$,

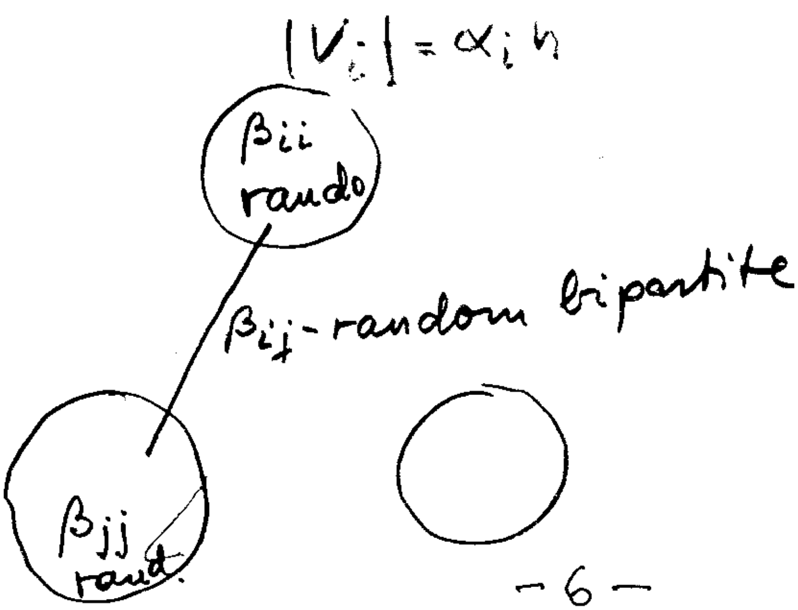
vertex weights $\alpha_i > 0$,

edges-weights $0 \le \beta_{ij} \le 1$

We may assume: H complete, with loops

Generalized random graph with model H, $\sum_1^k \alpha_i = 1$

$1 \le i \le q$



$G(n; H)$

Def

(G_n)

generalized H-quasirandom

$\forall F$

$$m(F, G_n) \sim m(F, G(n;H)) \quad (*)$$

$\#$ copies of $F \subset G_n$

Questions

Is the structure of G_n similar to $G(n;H)$?

Is it enough to require $(*)$ for a finite set $\{F_i\}$ (depending on H)?

F simple, unweighted, H weighted

$$\hom(F,H) = \sum_{\varphi: V_F \to V_H} \prod_{i \in V_F} \alpha_{\varphi(i)} \prod_{ij \in E_F} \beta_{\varphi(i)\varphi(j)}$$

If all the vertex-weights and edge-weights are 1, (no edge: 0) :

$$\hom(F,H) = \# \varphi : V_F \to V_H$$

homomorphism, - edg-preserving φ

$$\boxed{t(F,H) = \frac{\hom(F,H)}{(\sum \alpha_i)^{|V_F|}}}$$

F simple, G simple, unweighted
small, large

$$t(F,G) = \frac{\hom(F,G)}{n^k} \qquad n = |V_G| \quad k = |V_F|$$

density of F in G

Def

H "small", weighted

(G_n) is H-quasirandom, if

$\forall$ F

$$t(F, G_n) \longrightarrow t(F, H)$$

Exp $G(n; H)$ H-random sequence

$$t(F, G(n;H)) \longrightarrow t(F, H)$$

with prob. 1

<u>Theorem</u> (Lovász – S.)

(G_n) H-quasirandom

structure of G_n $\sim$ structure of $\mathbb{G}(n; H)$

(G_n) H-quasirandom. Then

$\forall n$ $\exists$ $V_{G_n} = \bigcup_{i=1}^{q} V_i$ such that

- $\dfrac{|V_i|}{|V_{G_n}|} \to \alpha_i$ $\quad 1 \le i \le q$

- $G_n(V_i)$ is β_{ii}-quasirandom

- $G_n(V_i, V_j)$ is β_{ij}-quasirandom bipartite

$\exists$ finite test-class

(G_m) is H-quasirandom iff

$\Updownarrow$

$$t(F, G_n) \to t(F, H)$$

for $\forall$ F with $|V_F| < (10q)^q$

where $q = |V_H|$

¿ minimal finite family $\mathcal{F}$
structure,
$|\mathcal{F}|$?

CONVERGENCE OF GRAPHS

(G_n), F, G_n simple

$$t(F, G_n) = \frac{\hom(F, G_n)}{|V_{G_n}|^{|V_F|}}$$

<u>Def</u> (G_n) is <u>convergent</u>, if

$\forall F$ $\quad$ <u>$t(F, G_n)$ is convergent</u>

<u>Exp</u> $(G(n;p))$ p-random

$$t(F, G_n) \longrightarrow p^{|E_F|} \quad \text{with prob } 1$$

$(G(n;p))$ p-quasirandom

$$t(F, G_n) \longrightarrow p^{|E_F|}$$

<u>Exp</u> (G_n) generalized H-quasirandom

$$t(F, G_n) \longrightarrow t(F, H)$$

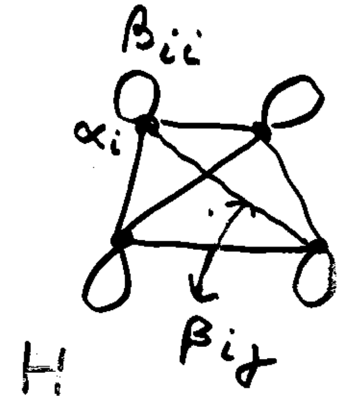
¿ WHAT IS THE LIMIT ?

(G_n) p - random

p - quasirandom

(G_n) H - random

(G_n) H - quasirandom



$$t(F, G_n) \longrightarrow t(F, H)$$

¿ is H the limit ?

¿ (G_n) convergent, but <u>not</u> H-quasir. ?

Lovász - B. Szegedy

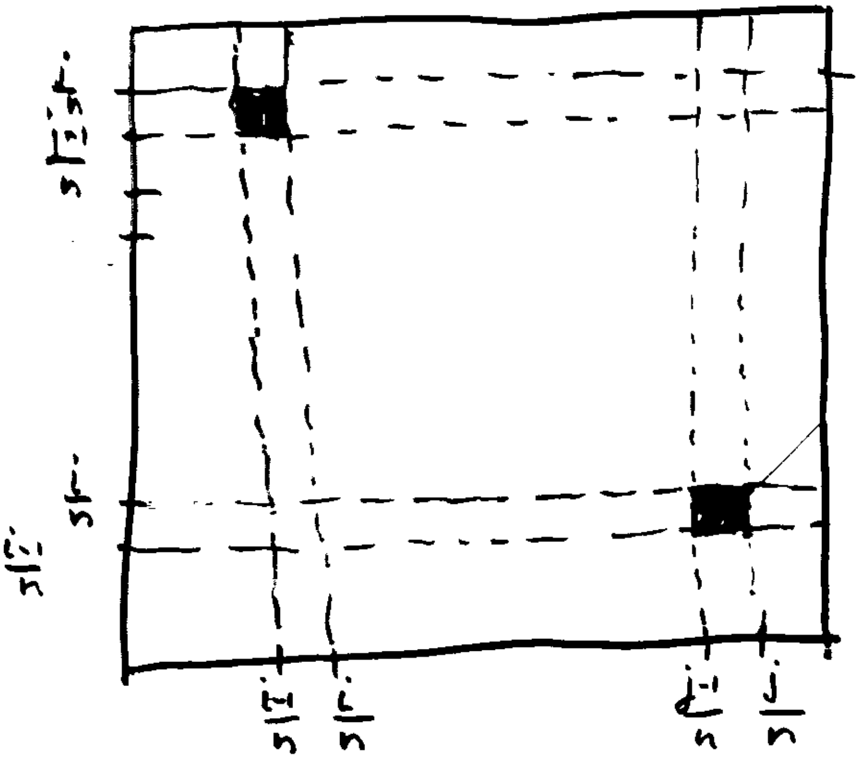
Let

$$\mathcal{W} = \{ w : [0,1]^2 \to [0,1] \text{ , meas., symm.} \}$$

$$t(F,w) := \int_{[0,1]^k} \prod_{ij \in E_F} w(x_i, x_j)\, dx$$

$k = |E_F|$

$G_n \longleftrightarrow \omega_{G_n}$



$$\omega = \begin{cases} 1 & ij \in E_G \\ 0 & ij \notin E_G \end{cases}$$

$$t(F, G) = t(F, \omega_G)$$

<u>Theorem</u> (Lovász - B. Szegedy)

- For <u>every</u> convergent graph sequence <u>(G_n)</u> there is a <u>$w \in \mathcal{W}$</u> such that $\forall F$

$$\lim_{n\to\infty} t(F, G_n) = t(F, w)$$

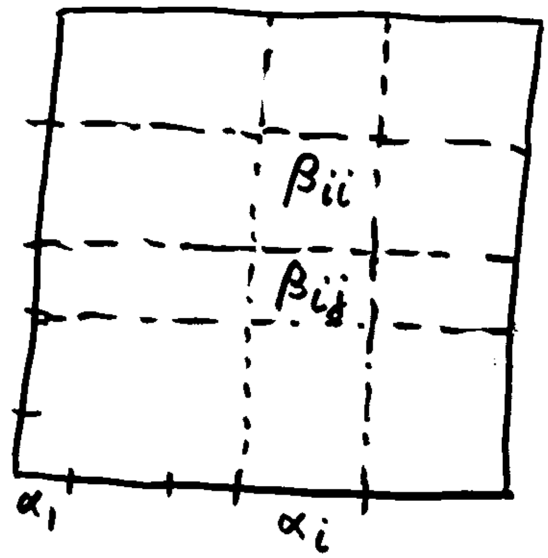
- $\forall\, w \in \mathcal{W}$ arises as a limit of some sequence (G_n)

Borgs-Chayes-Lovász
w determined upto meas. pres. transf.

$$\boxed{\begin{gathered}\forall F \quad \lim t(F, G_n) = t(F, w) \\ \Updownarrow \\ \lim G_n = w\end{gathered}}$$

Let H be a weighted graph

$\omega_H \in W$ $\qquad \sum_1^k \alpha_i = 1$



step-functions

$$\left.\begin{array}{l} G(n,p), \\ (G_n)\ p\text{-quasirandom} \end{array}\right\} \longrightarrow \lim = W_p = p \text{ const. f}$$

$$\left.\begin{array}{l} G(n;H)\quad H\text{-random} \\ (G_n)\quad H\text{-qusirandom} \end{array}\right\} \longrightarrow \lim = \omega_H \text{ step f.}$$

Szemerédi-lemma ~

approximation by step-function

DISTANCE OF GRAPHS, W ~ METRIC SPACE

① $V_G = V_{G'}$

$$d_\square(G,G') = \max_{S,T\subseteq V} \frac{1}{|V_G|^2} \left| e_G(S,T) - e_{G'}(S,T) \right|$$

② "best overlay", $|V_G| = |V_{G'}|$

$$\hat{\delta}_\square(G,G') = \min_{\substack{\tilde{G} \sim G \\ \text{isom.}}} d_\square(\tilde{G}, G')$$

③ "best fractional overlay"

also for $|V_G| \neq |V_{G'}|$,

$$\delta_\square(G,G') = \min_X d_\square(G(X), G'(X)) \quad G, G' \text{ weighted}$$

where

$$\sum_{u=1}^{n'} X_{iu} = \alpha_i(G), \quad \sum_{i=1}^{n} X_{iu} = \alpha_u(G')$$

(G_n) is convergent $\qquad$ ($\exists \lim t(F, G_n)$ $\forall F$

$$\Updownarrow$$

(G_n) is Cauchy in $\delta_\square$ - metric

G similar G'

$\sim$ close in $\delta_\square$

Szemerédi - lemma

$\sim$

approximation by stepfuction