Choosability of Graphs With Infinite Sets of Forbidden Differences

Pavel Nejedlý

Georgia Institute of Technology, Atlanta

and

Charles University, Prague

Frequency Assignment Problem

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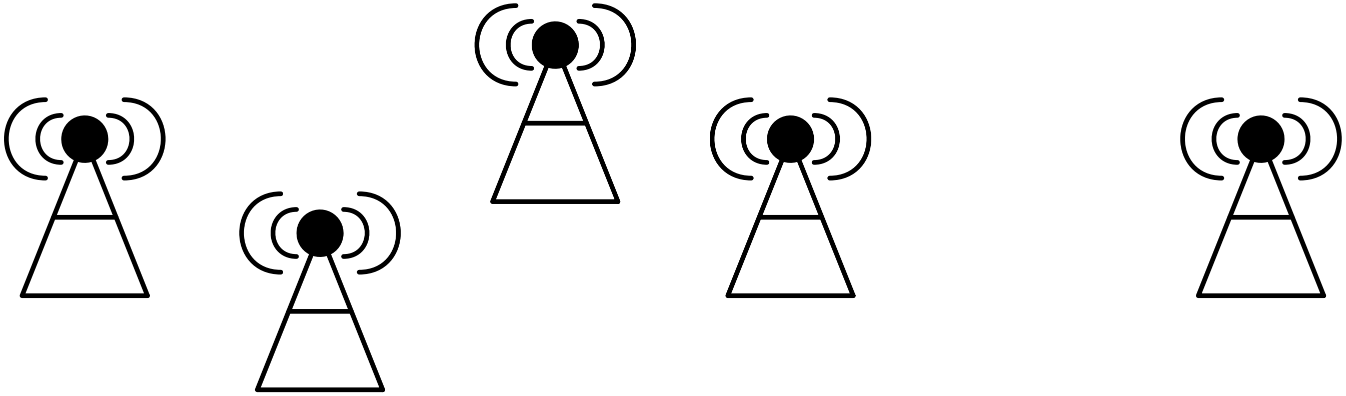
- Example:



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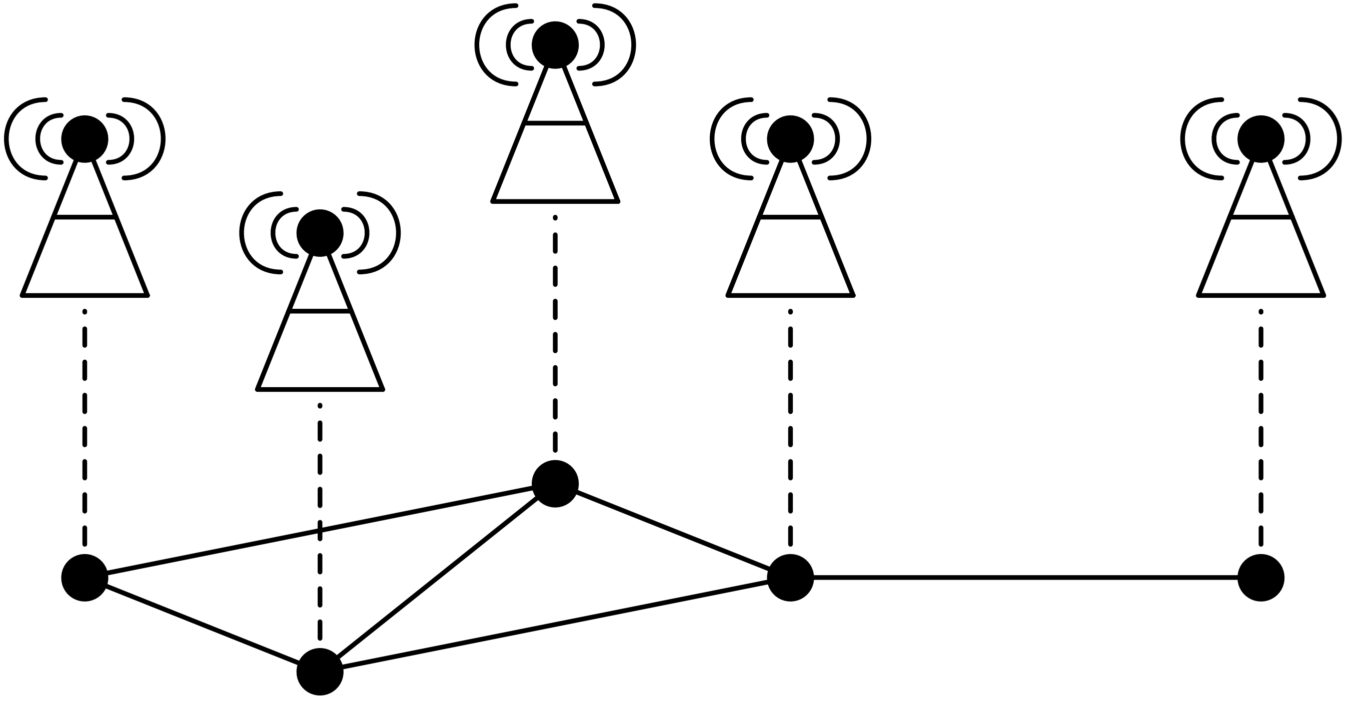

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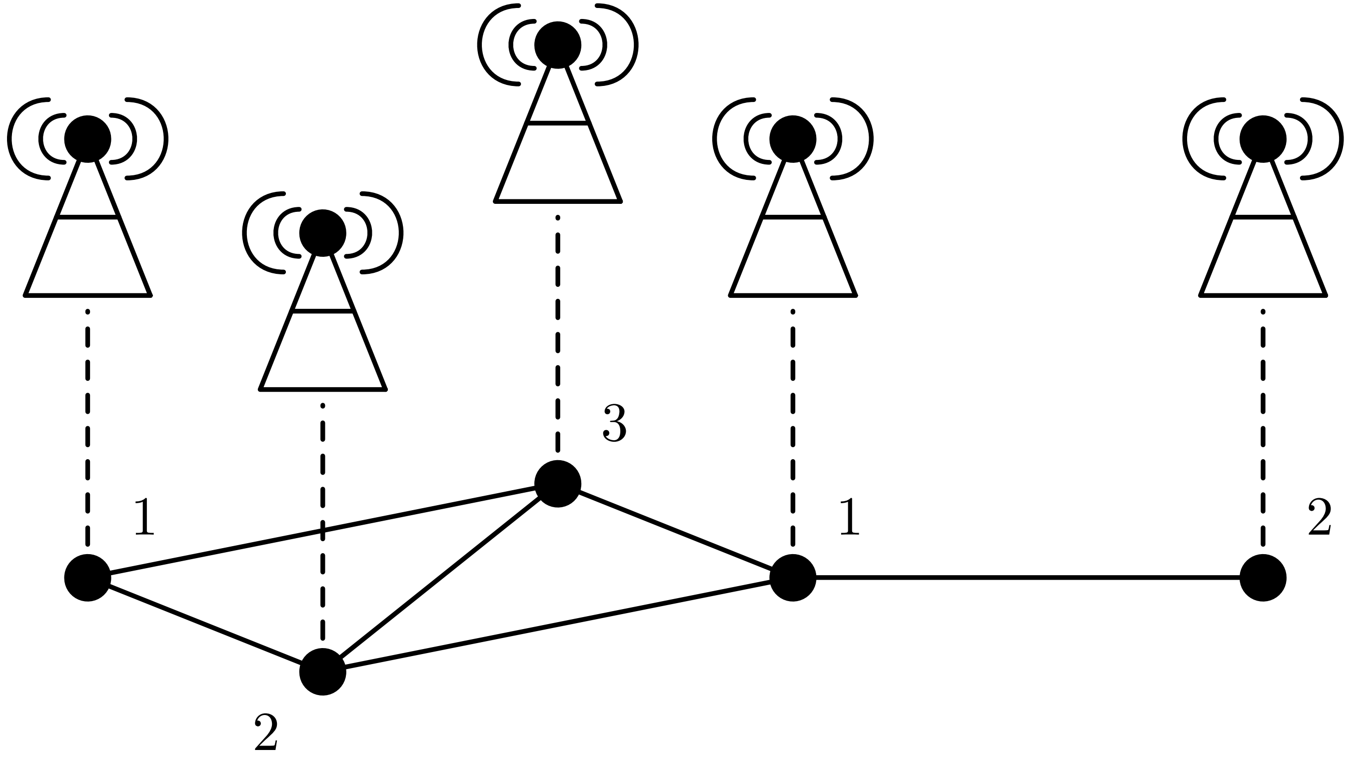

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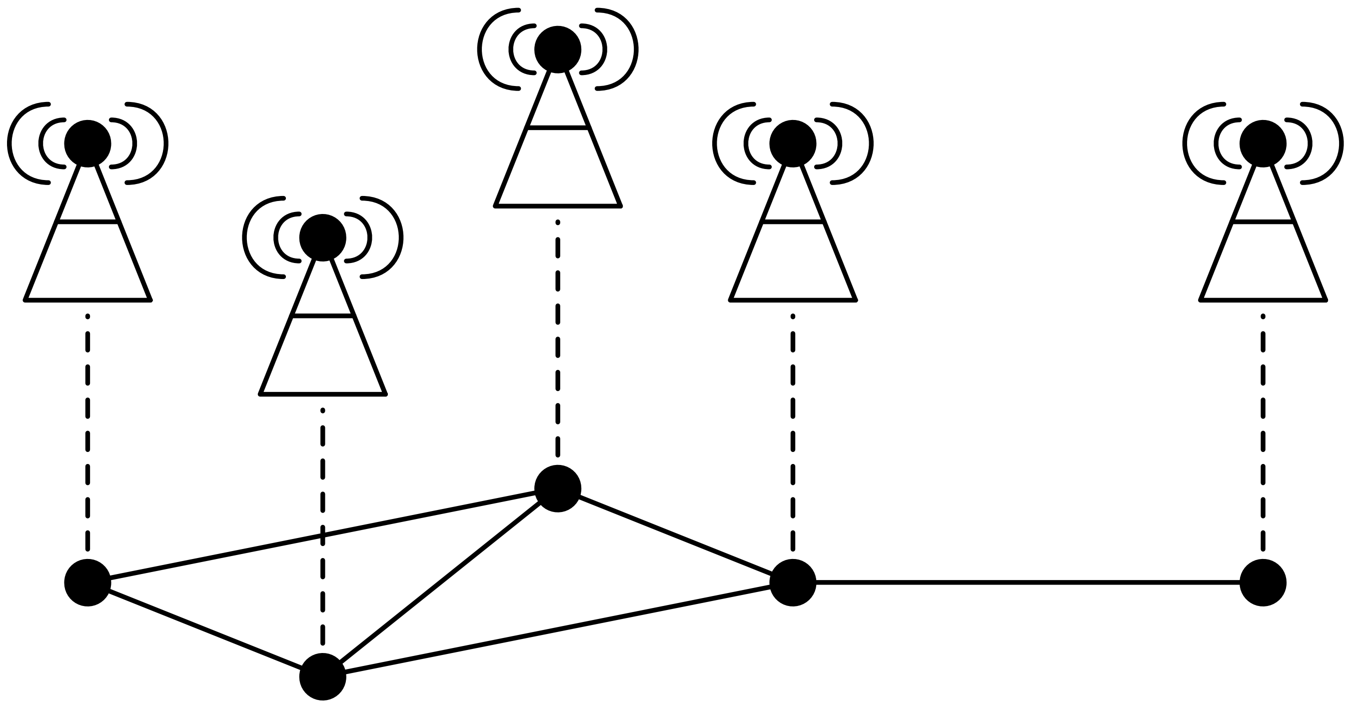

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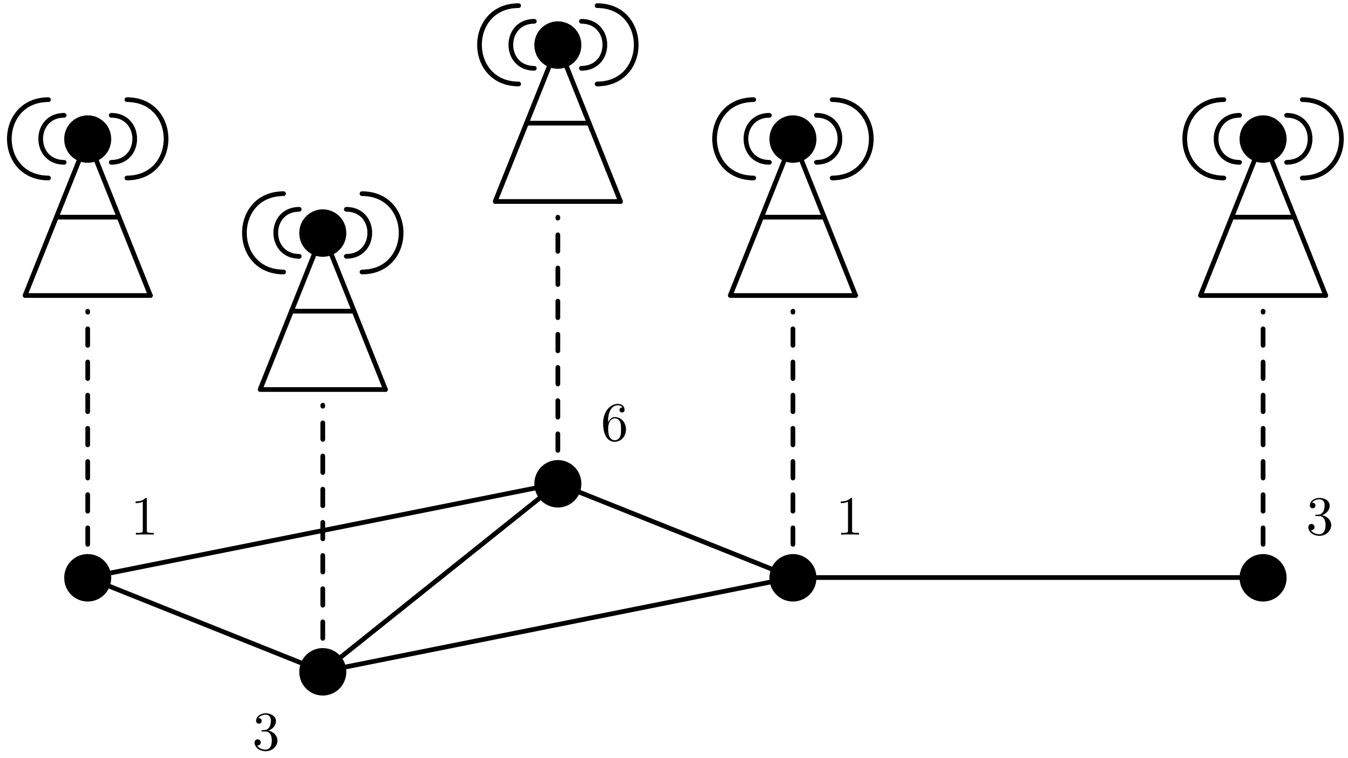

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- G is T-k-choosable if it is list-T-colorable for every L such that $|L(v)| \geq k$ for every vertex v.
- The T-choice number of G ($\mathrm{ch}_T(G)$) is the smallest k such that G is T-k-choosable.

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- The main result:
  - For every nonempty graph G, $\mathrm{ch}_T(G)$ is finite iff $\mathrm{ch}_T(K_2)$ is finite.

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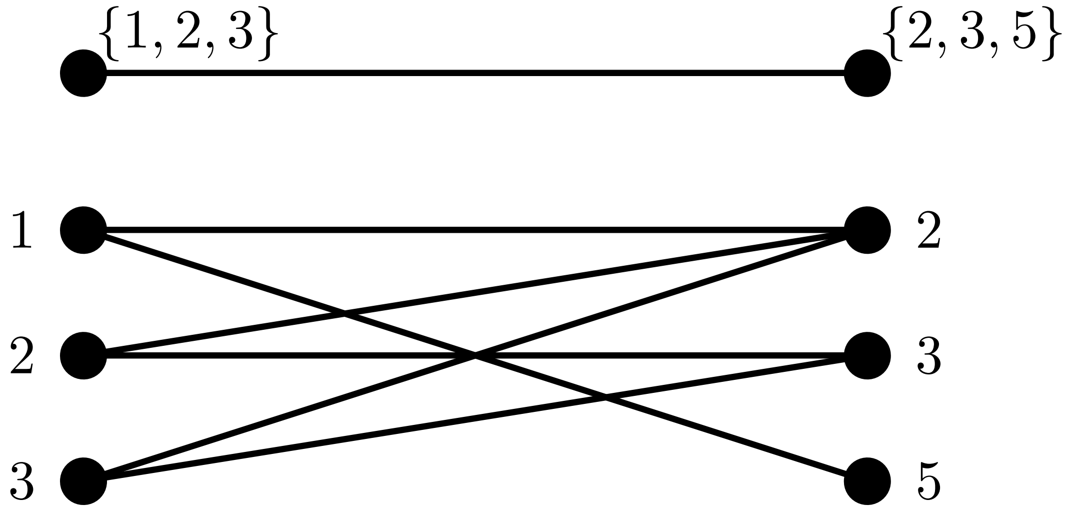

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- Using Lovász's Local Lemma, we have $p[G \text{ is not properly colored}] < 1$.

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- ... because the bound $(k-1)(4e\Delta)^k$ is not very nice.
- Can we find a bound which is not exponential in $\mathrm{ch}_T(K_2)$?

Idea of the second proof

- Key lemma:
  - Let T, S_1, and S_2 be sets of integers, $|S_1| = k$, $|S_2| \geq k$, $\mathrm{ch}_T(K_2) \leq k$.
  - Then there exists $c \in S_1$ such that
    $$|S_2 \cap (c + (T \cup -T))| \leq |S_2| - (|S_2| - k)/k$$

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- Gives estimate $\mathrm{ch}_T(G) \leq (\Delta(k-1)+3)k^{\Delta}$.

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so complete disorder is not possible if $\mathrm{ch}_T(K_2)$ is infinite.
- It is also easy to see that if T contains an infinite arithmetic progression, $\mathrm{ch}_T(K_2)$ is not finite.
- Is it true that if $\mathrm{ch}_T(K_2)$ is infinite, then T contains an infinite arithmetic progression?
- Unfortunately, there are sets T such that $\mathrm{ch}_T(K_2)$ is infinite, but they do not contain any arithmetic progression of length greater than two.

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  - One is polynomial in Δ (for fixed $\mathrm{ch}_T(K_2)$).
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- Is it $c \cdot \Delta \cdot \mathrm{ch}_T(K_2)$ for some constant c?
  - If not, can we find a good lower bound?

Thank you for your attention!