Subsums of a finite sum and extreme vertices of the hypercube

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Question 1 *How to give a set of n real numbers x_1, x*
them being equal to 0, such that they maximize the
$\sum_{i \in B} x_i$ *equal to 0, where the B's are subsets of* $\{1,$
are subsets of $\{1, 2, \ldots, n\}$ *of cardinality k or at most*

Question 2 *Let $x_1, x_2, \ldots, x_n$ be given numbers suc*
0*. What is maximum number of negative subsums (*
number of positive subsums) of exactly k of these nu

Question How many vertices (of a certain prope weight) of the n-dimensional cube can be picked such that is the subspace spanned by them, (or convex or the cone spanned by them) does not contain or do certain configurations of the hyperplane (elements, v weight(s), hyperplanes)?

Let C_n denote the set of vertices of the n dimens
let $M \subset C_n$ be a subset of it, span(M) denote the s
spanned by M and cone(M) denote the cone spanned

How big M can be such that $(1, 1, \ldots, 1) \notin$ span(M) o
span(M)?

Proposition 1 *If for an $M \subset C_n$ the size of $M > 2^{n-1}$ t*
span(M), and therefore $(1, 1, \ldots, 1) \in$ span(M).

Proposition 2 *If for an $M \subset C_n$ the size of $M > 2^{n-1}$*
C_n.

Proposition 3 *If for an $M \subset C_n$ the size of $M > 2^{n-1}$ t*
cone(M).

All the above bounds are sharp.

Theorem 1 *The maximum size of a subset M of th* *n-dimensional hypercube such that none of the vert* *$(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ are in span (M) is $\binom{n}{\lfloor n/2 \rfloor}$*

Theorem 2 *(Miller at al. 1991, Griggs, 1997)*
Let $x_1, x_2, \ldots, x_n$ be given numbers. The maximum *subsums $\sum_{i=1}^{k} x_{j_i}$ which can be given without the* *values x_i is $\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{i}\binom{\lfloor n/2 \rfloor}{i} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{i}\binom{\lfloor n/2}{\lfloor n/2 \rfloor}$*

Theorem 3 *Given a set of n real numbers $x_1, x_2, \ldots,$* *being equal to 0, the maximum number of sums $\sum_{i}$* *where the B's are subsets of $\{1, 2, \ldots, n\}$ is*

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{i}\binom{\lfloor n/2 \rfloor}{i} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{i}\binom{\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor - i} =$$

Problem 1 *(Littlewood-Offord) How do we select — distinct — complex numbers* $x_1, x_2, \ldots x_n$*, with* $|x_i| \geq$ *unit diameter ball* B*, such that they maximize the nu* *2^n sums* $\sum_{i \in I} a_i$*,* $I \subseteq [n]$*, lying inside* B*?*

It was first solved by Erdős, assuming the numbers x_i gument can be used to prove Theorem 3, more precis theorem:

Theorem 4 *Given a set of n real numbers $x_1, x_2, \ldots,$ being equal to 0, the maximum number of sums $\sum_{}$ fixed t (where the B's are subsets of $\{1, 2, \ldots, n\}$) is (*

Proof (Griggs, Erdős) Observe that replacing any of t not change the maximum number of subsums, only t will be changed to $t - x_i$. Therefore, we can chan x_i such that finally we will have sum of positive numb to maximize the subsums of them equal to a certair Surely there might be no two index sets containing ea the same subset sum value, so - by Sperner - there than $\binom{n}{\lfloor n/2 \rfloor}$ of them.

Summary: the maximum size of $M \subset C_n$ such that

	does not contain a specific $(0, 0, \ldots, 0, 1, 0, \ldots 0)$	does not contain any $(0, 0, \ldots, 0, 1, 0, \ldots 0)$'
span M	2^{n-1}	$\binom{n}{\lfloor n/2 \rfloor}$
cone M	—	—

The case when vertices of weight k are cons

Let C_n denote the vertices of the n dimensional hy
$M_k \subset C_n$ be a subset of it consisting of vertices o
(vertices with exactly k 1 coordinates).

How big M_k can be such that $(1, 1, \ldots, 1) \notin \text{span}(M_k)$ o
$\text{span}(M_k)$?

Consider all vertices of weight k with first coordinate
of them is $\binom{n-1}{k}$ and their span (cone) will definitely
$(1,0,0,\ldots,0)$ nor $(1,1,\ldots,1)$. However, for $(1,0,0,$
other constructions as well, like fixing the first coordin
number of them is $\binom{n-1}{k-1}$) or having either (but exa
first two coordinates 1, and the remaining $k-1$ 1 coor
remaining $n-2$ positions (the number of them is $2\binom{n}{k}$

Proposition 4 *If for an* $M_k \subset C_n$ *the size of* $M_k > \binom{n-1}{k}$
span(M_k)*.*

Proposition 5 *If for an* $M_k \subset C_n$ *the size of* $M_k > \max$
then $(1, 0, \ldots, 0) \in$ *span*(M_k)*, and therefore span*(M_k)

Remark 1

$$\max\{\binom{n-1}{k}, \binom{n-1}{k-1}, 2\binom{n-2}{k-1}\} = \begin{cases} \binom{n-1}{k-1} & \text{for} \\ 2\binom{n-2}{k-1} & \text{for} \\ \binom{n-1}{k} & \text{for} \end{cases}$$

Question 3 *Is it true that for any $M_k \subset C_n$ with size*
$(1, 1, \ldots, 1) \in$ *cone*(M_k)*?*

NOT TRUE IN GENERAL

Let $n = 3k + 1$ and consider all of those vertices of
which have at least one of the first three coordinates

Claim: the cone spanned by them will not contain (1

In this case there are $\binom{3k-2}{k}$ vertices of weight k havi
coordinates 0, which is less than $\binom{3k}{k-1}$, and therefor
vertices having at least one of the first three entri
$\binom{3k+1}{k} - \binom{3k-2}{k}$, which is more than $\binom{3k+1}{k} - \binom{3k}{k-1} =$

An alternative form of the same question:

Question 4 *Let* $x_1, x_2, \ldots, x_n$ *be given numbers such*
0. *What is maximum number of negative subsums (*
number of positive subsums) of exactly k *of these nu*

Answer: Obviously at least $\binom{n-1}{k}$ shown by the e
small (absolute value) negative numbers and one big
positive number, e.g. $\{-1, -1, -1, \ldots, -1, n\}$. Therefo
number of positive subsums is at most $\binom{n-1}{k-1}$ and in
bound is at least as big as $\binom{n-1}{k}$, but maybe bigger s

Consider $3k + 1$ numbers: $\{2 - 3k, 2$
$3, 3, \ldots, 3\}$ whose sum is 1. In this case there are $\binom{3k}{k}$
sums, which is less than $\binom{3k}{k-1}$, and therefore $\binom{3k+1}{k}$ -
subsums, which is more than $\binom{3k+1}{k} - \binom{3k}{k-1} = \binom{3k}{k} =$

Theorem 5 *(Manickam, Miklós, 1989) Let* $x_1, x_2, \ldots,$
bers such that $\sum_{i=1}^{n} x_i > 0$*. The minimum number of*
of exactly k *of these numbers is* $\binom{n-1}{k-1}$ *if* $n > n_1(k)$ *or*

Remark 2 *There are counterexamples for small* k*'s*
which does not divide n *either.*

Corollary 1 *If for an* $M_k \subset C_n$ *the size of* $|M_k| >$
$n > n_1(k)$ *or* k *divides* n*, then* $(1, 1, \ldots, 1) \in$ *cone*(M_k)

Question 5 *What is the maximum size of a subset* M
of the n*-dimensional hypercube (all of weight* k *or w*
such that none of the vertices o
$(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ *are in span* (M_k)*?*

OR

Question 6 *Let* $x_1, x_2, \ldots, x_n$ *be given numbers. W*
mum number of the subsums $\sum_{i=1}^{k} x_{j_i}$ *(of exactly* k
of these numbers) which can be given without the
values x_i*?*

OR

Question 7 *Let* $x_1, x_2, \ldots, x_n$ *be given numbers. W*
mum number of the subsums $\sum_{i=1}^{k} x_{j_i}$ *(of exactly* k *o*
these numbers) being equal to 0?

Let the answer to these 3 questions be defined by $M(n$
all of weight k), and $N(n,k)$ (with vectors of weight a

Definitely any number of the form $m_1(n_1,n_2,k_1,k_2)$
$n_1+n_2=n$ and $k_1+k_2=k$, or

$$m_2(n,k) = \sum_{i=1}^{\frac{k}{2}} \binom{\lceil \frac{n}{2} \rceil}{i} \binom{\lfloor \frac{n}{2} \rfloor}{i}.$$

is a lower bound for $N(n,k)$, and $m_1(n_1,n_2,k_1,k_2)$ is a
$M(n,k)$.

What is the maximum of $\binom{n_1}{k_1}\binom{n_2}{k_2}$?

Which is bigger, $\binom{2n}{2k}\binom{2n}{2k}$ or $\binom{3n}{3k}\binom{n}{k}$?

Theorem 6 *(Demetrovics, Katona, Miklós, 2004) Su*
n*. The maximum size of a subset* M *of vertices of t*
hypercube of weight k *(or of weight at most* k*) such*
vertices of the form $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ *are in span*

$$N(n,k) = \binom{\left\lfloor \frac{(n+1)(k-1)}{k} \right\rfloor}{k-1} \left(n - \left\lfloor \frac{(n+1)(k-}{k} \right.\right.$$

which answer is the maximum of $\binom{n_1}{k_1}\binom{n_2}{k_2}$ assuming
$k_1 + k_2 = k$ for fixed n and k with $n_1 = \left\lfloor \frac{(n+1)(k}{k} \right.$
$\left\lfloor \frac{(n+1)(k-1)}{k} \right\rfloor$, $k_1 = k - 1$, $k_2 = 1$, therefore also is
$M(n,k)$, the case with vectors all of weight exactly e

However, the above theorem is really only true if $n_1($

E.g., as Theorems 1–3 shows,

$$N(n,n) = \binom{n}{\lfloor n/2 \rfloor} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{i} \binom{\lfloor n/2 \rfloor}{i} = r$$

and not $m_1(n_1, n_2, k_1, k_2)$ for ceratin values of the par

It is expected that the same construction remains the
much larger than k, that is (assuming for convenienc

$$N(n,k) = \sum_{i=1}^{\frac{k}{2}} \binom{\lceil \frac{n}{2} \rceil}{i} \binom{\lfloor \frac{n}{2} \rfloor}{i}.$$

For example, it is known that $N(12,6) = \binom{6}{3}\binom{6}{3} + ($
$m_2(12,6)$, however, $N(20,6) = M(20,6) = \binom{17}{5} \cdot 3$, lik

Summary: the maximum size of $M_k \subset C_n$ such that

	does not contain a specific $(0, 0, \ldots, 0, 1, 0, \ldots 0)$	does not contain an $(0, 0, \ldots, 0, 1, 0, \ldots$
span M_k	$\binom{n-1}{k}^{*}$	$\binom{\lfloor \frac{(n+1)(k-1)}{k} \rfloor}{k-1} \left(n - \left\lfloor \frac{(n+1)(\right.\right.$
cone M_k	—	—

* provided $n > n_1(k)$.

Further questions

Find the maximum size of M such that span M avoi
weight $n-1$, or, in other words, given n real numb
find the maximum number of subset sums $\sum_{i \in B} x_i$ e
the B's are subsets of $\{1, 2, \ldots, n\}$, with the condition
sums $\left(\sum_{i=1}^{n} x_i\right) - x_j$ are equal to 0.

In general, find the maximum number of subset sums
0, for any set of n real numbers $x_1, x_2, \ldots, x_n$, with th
none of the sums $\sum_{j=1}^{r} x_{i_j}$ are equal to 0 (for a given

For $r = n$ the answer is 2^{n-1} by Proposition 1 and for
by Theorem 3.

For $r = n - 1$ the best construction is $\{1, 1, 0, 0, 0, \ldots$
ways).

The next question is to find the maximum size of M s
avoids all vertices of weight 2, or, given n real numb
($\neq 0$), find the maximum number of subsums $\sum_{i \in B} x$
B's are subsets of $\{1, 2, \ldots, n\}$, with none of the sums

Problem 2 *(Erdős-Moser) How do we select* **distin**
numbers $x_1, x_2, \ldots x_n$ *and a target sum* t *to maximiz*
subset sums $= t$*?*

that is,

Problem 3 *How do we select nonzero real number*
maximize the number of subset sums $= 0$*, with all* x_i

Select the n distinct integers closest to 0 and the ta
is the best construction (Stanley)

Our problem is:

Problem 4 *How do we select (nonzero?) real numbe*
maximize the number of subset sums $= 0$*, with all* x_i

(Strong) conjecture:

It will be maximized by the choice of

$$\{-1, -1, \ldots, -1, 2, \ldots 2\}$$

with $2n/3$ copies of -1 and $n/3$ copies of 2.