Group valued edge-colorings of cubic graphs

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The problem

- cubic graph G, Abelian group A
- an edge coloring $c : E(G) \to A \setminus \{0\}$ such that:
  $c(e) \neq c(f)$ for any e and f sharing a vertex
  $c(e_1) + c(e_2) + c(e_3) = 0$ for any e_1, e_2 and e_3 sharing a vertex
- a proper coloring and a flow-like condition
- a cubic graph G is $\mathbb{Z}_2 \times \mathbb{Z}_2$-edge-colorable iff it has a nowhere-zero 4-flow

- Is there a group A such that any cubic bridgeless graph is A-edge-colorable?
- What are the necessary conditions for A-edge-colorability?

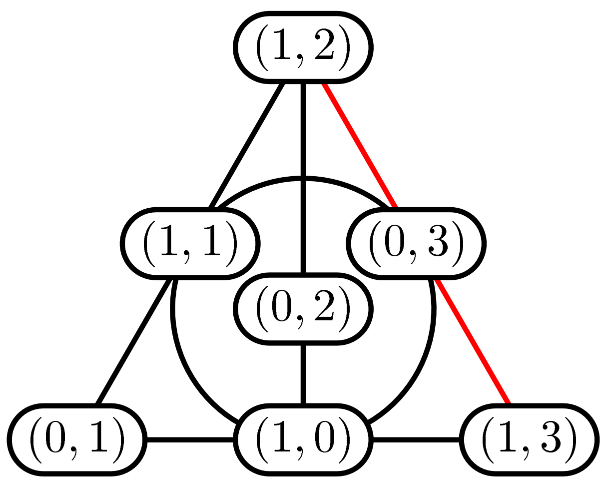
Classification of the groups

Never	3-edge-colorable	Open	Always
$\mathbb{Z}_3$	$\mathbb{Z}_2 \times Z_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_{12}$
$\mathbb{Z}_5$	$\mathbb{Z}_7$	$\mathbb{Z}_{10}$	$\dots$
	$\mathbb{Z}_8$	$\mathbb{Z}_{11}$	

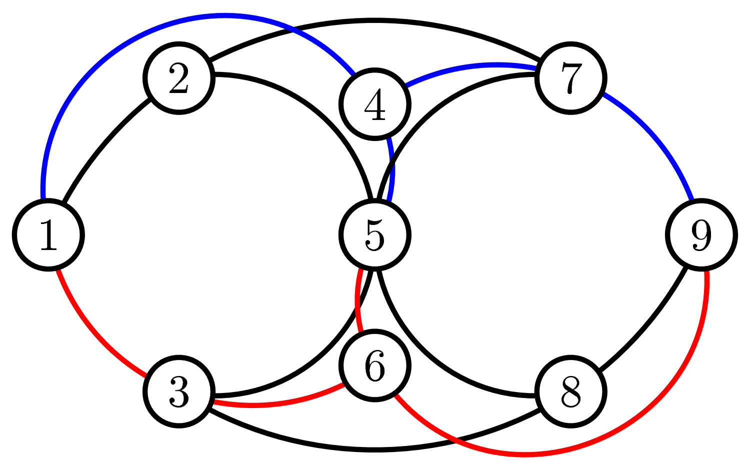
- Note that the property of being A-edge-colorable depends on the structure not only on the order of A.

A DIFFERENT VIEW

- a coloring by triple systems such that the colors of the edges incident with the same vertex should form a triple (hypergraph homomorphism point of view)



$Z_2 \times Z_4$

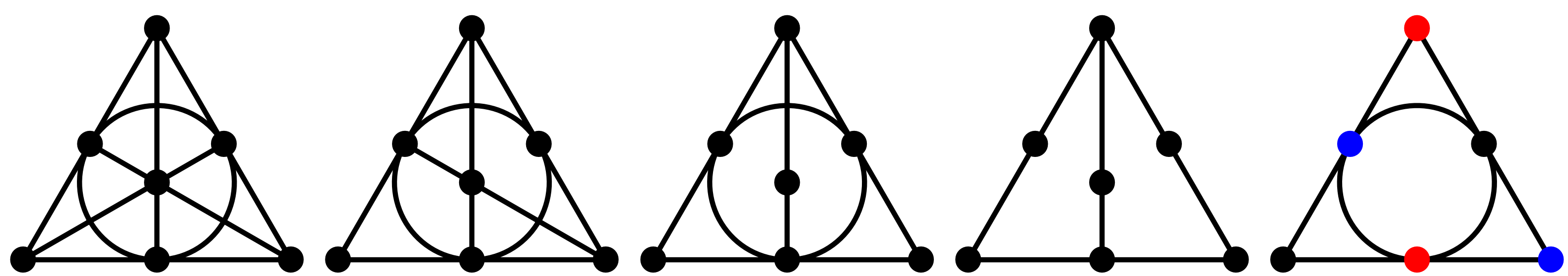


Z_{10}

- There exists a Steiner triple system that edge-colors every cubic graph[Griggs, Knor and Škoviera, 2004].

Colorings by the Fano plane

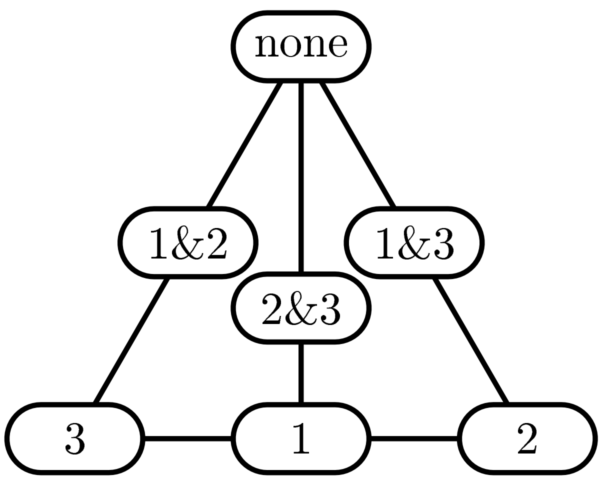
- every cubic bridgeless graph has a coloring using six lines of the Fano plane [Máčajová, Škoviera 2005]

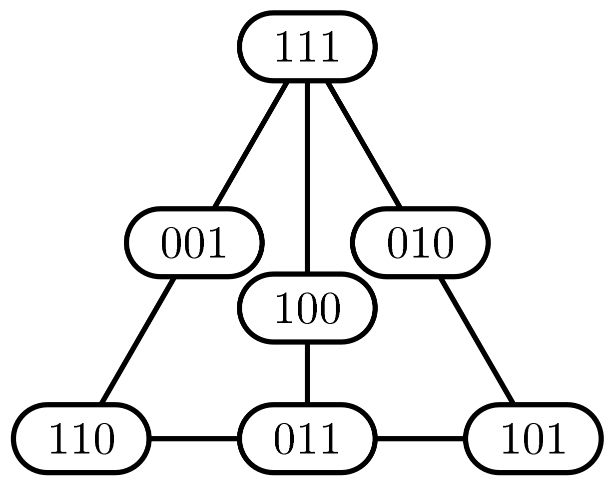


- Does every cubic bridgeless graph have an edge-coloring using five lines of the Fano plane?
- Does every cubic bridgeless graph have an edge-coloring using four lines of the Fano plane?

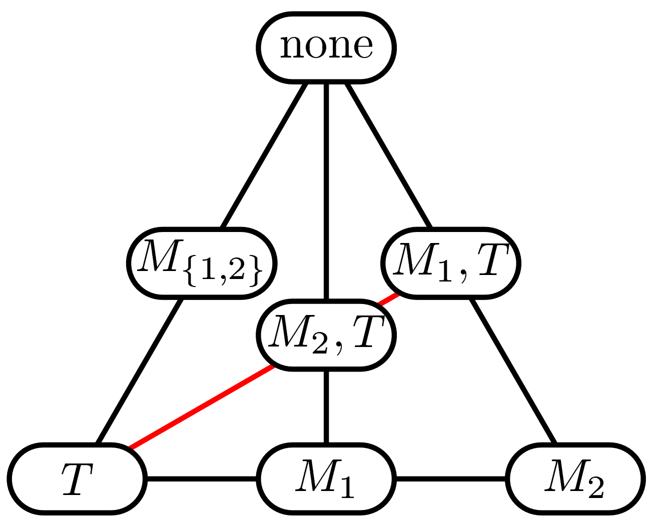
Coverings by perfect matchings

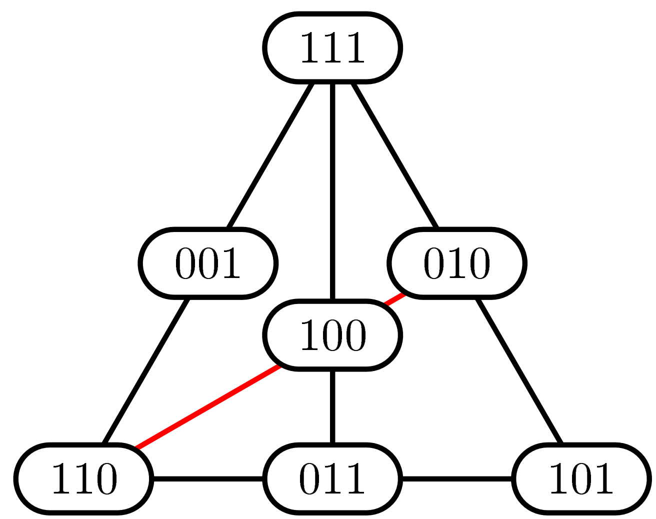
- Conjecture [Fan and Raspaud]:
  Every cubic bridgeless graph has three perfect matchings with empty intersection.
- Conjecture [Berge and Fulkerson]:
  Every cubic bridgeless graph has six perfect matchings covering each edge twice.
- The conjecture of Fan and Raspaud is equivalent to the Four Line Conjecture.





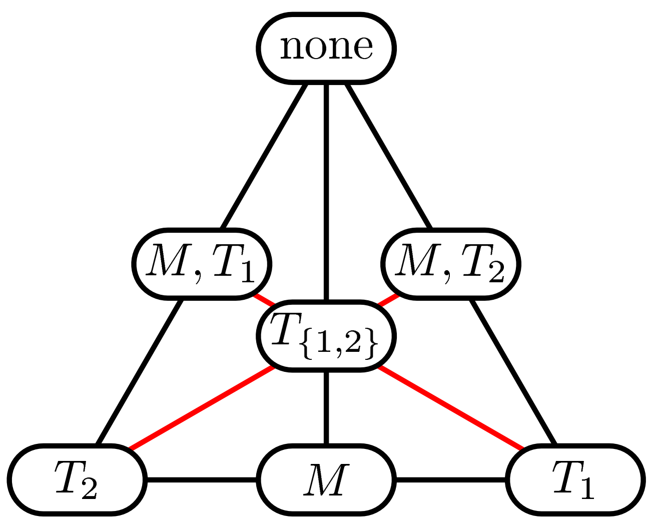
Coverings by parity subgraphs

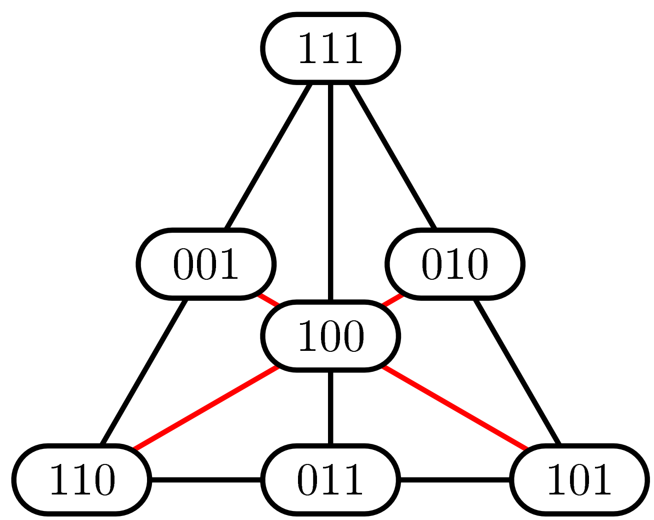




- a parity subgraph of G is a subgraph H such that the parities of degrees of vertices in G and H are the same
- a perfect matching is a parity subgraph of a cubic graph
- The existence of two perfect matchings and a parity subgraph with an empty intersection is equivalent to the Five Line Conjecture (and thus to $\mathbb{Z}_2 \times \mathbb{Z}_4$-edge-colorings).

Six Line Theorem





- The existence of one perfect matching and two parity subgraphs with an empty intersection is equivalent to the Six Line Theorem.

Some results...

- a short proof of the Six Line Theorem
- Rainbow Lemma
  Each cubic bridgeless graph contains a perfect matching M such that its edges can be colored with three colors (red, green and blue) in the following way: each cycle of the complementary 2-factor is incident with the same number modulo two of the red/green/blue edges.
- consider a matching M that avoids 3-cuts of the graph G
- contract the complementary 2-factor and find a nowhere-zero 4-flow
- parity conditions on red/green/blue edges
- Each $\mathbb{Z}_2 \times \mathbb{Z}_4$-edge-colorable cubic graph is also $\mathbb{Z}_3 \times \mathbb{Z}_3$-, $\mathbb{Z}_{10}$- and $\mathbb{Z}_{11}$-edge-colorable.
- Every cubic bridgeless graph contains two sets of disjoint even cycles such that each vertex is contained in at least one of the cycles.

SUMMARY

Six Line Theorem, edge colorings with groups of order twelve and more

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$\mathbb{Z}_3 \times \mathbb{Z}_3$-, $\mathbb{Z}_{10}$- and $\mathbb{Z}_{11}$-edge-colorings

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Five Line Conjecture $\Longleftrightarrow$ $\mathbb{Z}_2 \times \mathbb{Z}_4$-colorings

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two matchings and parity subgraph with empty intersection $\Longleftrightarrow$
Conjecture of Fan and Raspaud $\Longleftrightarrow$ Four Line Conjecture

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Conjecture of Berge and Fulkerson

- integer edge-colorings $(\pm 1, \ldots, \pm 7)$