Forbidden submatrices in 0-1 matrices

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Combinatorial Challenges, 2006

Introduction

Definition

A **0-1 matrix** (or **pattern**) is a matrix with just 1's and 0's (blanks) at its entries.

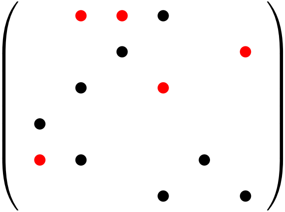
The pattern P is **contained** in the 0-1 matrix A if it can be obtained from a submatrix of it by deleting (changing to 0) extra 1 entries.
Note that permuting rows or columns is not allowed!

Introduction

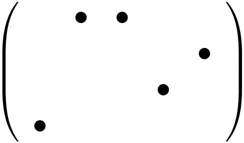
Example

The 0-1 matrix A **contains** the pattern P:
(dot for 1 entry, blank space for 0)

$A =$



$P =$



Introduction

Definition

The **extremal function** of P **ex(n,P)** is the maximum number of 1 entries in an n by n matrix not containing P.

Our aim is to determine this function for some patterns P.

This question is a variant of the Turán-type extremal graph theory.

Main papers

Z. Füredi (1990)

D. Bienstock, E. Győri (1991)

Mainly determining the extremal function of pattern Q:

$$Q = \begin{pmatrix} & \bullet & \bullet \\ \bullet & & \bullet \end{pmatrix}$$

Z. Füredi, P. Hajnal (1992)

Examining the extremal function of all patterns with four 1 entries and determine it for many cases.

A. Marcus, G. Tardos (2004)

Determining the extremal function of permutation patterns.

G. Tardos (2005)

Simple Bounds

Proposition

The extremal function of the 1 by 1 pattern with a single 1 entry is 0*, for any other pattern* $ex(n, P) \geq n$*.*

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Proposition

If a pattern P *contains a pattern* Q, *then* $ex(n, Q) \leq ex(n, P)$.

Proof.

If a matrix avoids Q, then avoids P too. □

Adding a column with one 1 entry on the boundary

Theorem

(Füredi, Hajnal) If P' can be obtained from P by attaching an extra column to the boundary of P and placing a single 1 entry in the new column next to an existing one in P, then $ex(n, P) \leq ex(n, P') \leq ex(n, P) + n$.

Example

$$P = \begin{pmatrix} \bullet & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix} \qquad P' = \begin{pmatrix} & \bullet & & \bullet \\ \color{red}{\bullet} & \bullet & & \\ & & \bullet & \end{pmatrix}$$

Adding a column with one 1 entry inside the pattern

Theorem

(Tardos) If P' is obtained from the pattern P by adding an extra column between two columns of P, containing a single 1 entry and the newly introduced 1 entry has 1 next to them on both sides, then $ex(n, P) \leq ex(n, P') \leq 2ex(n, P)$.

Example

$$P = \begin{pmatrix} \bullet & & & \bullet \\ & \bullet & \bullet & \end{pmatrix} \qquad P' = \begin{pmatrix} \bullet & & & & \bullet \\ & \bullet & {\color{red}\bullet} & \bullet & \end{pmatrix}$$

Permutation Matrices

Theorem

(Marcus, Tardos) For all permutation matrices P we have $ex(n, P) = O(n)$ (A **permutation matrix** *is a matrix with exactly one 1 entry in each column and row).*

Davenport-Schinzel sequences

Example

The sequence *deedecadedabbe* **contains** the sequence *abab*.

Definition

Davenport-Schinzel sequences are the ones not containing *ababa* type sequences.

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Definition

Similarly to 0-1 matrices, $ex(n, u)$ is the maximum length of a string on n symbols not containing the string u.

Davenport-Schinzel sequences

Theorem

(Hart, Sharir) For DS-sequences we have $ex(n, ababa) = \Theta(n\alpha(n))$, *where* $\alpha(n)$ *is the inverse Ackermann function.*

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Theorem

(Füredi, Hajnal) For the extremal function of the pattern S_1 we have $ex(n, S_1) = \Theta(n\alpha(n))$.

$$S_1 = \begin{pmatrix} & \bullet & & \bullet \\ \bullet & & \bullet & \end{pmatrix}$$

Davenport-Schinzel sequences

Theorem

(Klazar, Valtr) The string $a_1 a_2 \dots a_{k-1} a_k a_{k-1} \dots a_2 a_1 a_2 \dots a_{k-1} a_k$ *has linear extremal function for all* $k \geq 1$.

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Corollary

For every $k \geq 1$ the pattern P_k has extremal function $ex(n, P_k) = O(n)$, where ex.

$$P_4 = \begin{pmatrix} & & & \bullet & \bullet & & & \\ & & \bullet & & & \bullet & & \\ & \bullet & & & & & \bullet & \\ \bullet & & & & & & & \bullet \end{pmatrix}.$$

Adding two 1's between and left to other two

Theorem

(Keszegh, Tardos) Let A be a pattern which has two 1 entries in its first column in row i and $i+1$ for a given i. Let A' be the pattern obtained from A by adding two new rows between the ith and the $(i+1)$th row and a new column before the first column with exactly two 1 entries in the intersection of the new column and rows. Then $ex(n, A') = O(ex(n, A))$.

Example

$$A = \begin{pmatrix} \bullet & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix} \qquad A' = \begin{pmatrix} & \bullet & & \bullet \\ \bullet & & & \\ \bullet & & & \\ & \bullet & & \\ & & \bullet & \end{pmatrix}$$

Adding two 1's between and left to other two

Theorem

(Tardos) $ex(n, L_1) = O(n)$.

$$L_1 = \begin{pmatrix} \bullet & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix}$$

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Corollary

(Keszegh, Tardos) $ex(n, L_2) = O(n)$.

$$L_2 = \begin{pmatrix} & \bullet & & \bullet \\ \bullet & & & \\ \bullet & & & \\ & & \bullet & \end{pmatrix}$$

Minimal non-linear patterns

Theorem

(Keszegh, Tardos) We have $ex(n, H_0) = \Theta(n \log n)$*, where*

$$H_0 = \begin{pmatrix} & \bullet & \bullet & \\ & & & \bullet \\ & & & \bullet \\ \bullet & & & \end{pmatrix}.$$

Definition

It is easy to see that by deleting any 1 entry from it we obtain a pattern with four 1 entries and with linear extremal function.

We call these type of patterns **minimal non-linear** patterns. So far, this is the only pattern with more than four 1 entries, known to be the member of this class of patterns.

Minimal non-linear patterns

Definition

Similarly, we call a pattern **minimal non-quasilinear** if by deleting any 1 entry we get an almost linear pattern (linear except for $\alpha(n)$ terms).

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Conjecture

There are infinitely many minimal non-linear patterns.

Remark

There are some patterns H_k which are prime candidates for being such.

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- Everything is at least linear

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- L_1, and as corollaries of the rules above: P_k, L_2, etc.
- Other linear patterns? Other rules to build new linear patterns?

Other linear patterns?

Conjecture

$ex(n, G) = O(n)$.

$$G = \begin{pmatrix} & \bullet & \bullet & & \\ & & & & \bullet \\ & & & \bullet & \\ \bullet & & & & \end{pmatrix}$$

Remark

Solving this would help to decide whether the patterns H_k are really minimal non-linear patterns.

Other linear patterns?

Conjecture

1. *For any permutation pattern by doubling the column containing the 1 entry in its first row we obtain a pattern with linear extremal function.*
   (Note that this would prove that $ex(n, G) = O(n)$*.)*

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1. *For any permutation pattern by doubling the column containing the 1 entry in its first row we obtain a pattern with linear extremal function.*
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2. *By doubling one column of a permutation pattern we obtain a pattern with linear extremal function.*

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1. *For any permutation pattern by doubling the column containing the 1 entry in its first row we obtain a pattern with linear extremal function.*
   (Note that this would prove that $ex(n, G) = O(n)$.)
2. *By doubling one column of a permutation pattern we obtain a pattern with linear extremal function.*
3. *By doubling every column of a permutation pattern we obtain a pattern with linear extremal function.*