Distinguishing Chromatic Number of Cartesian Products of Graphs

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Cartesian Product of Graphs

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs.

$G \square H$ denotes the Cartesian product of G and H.

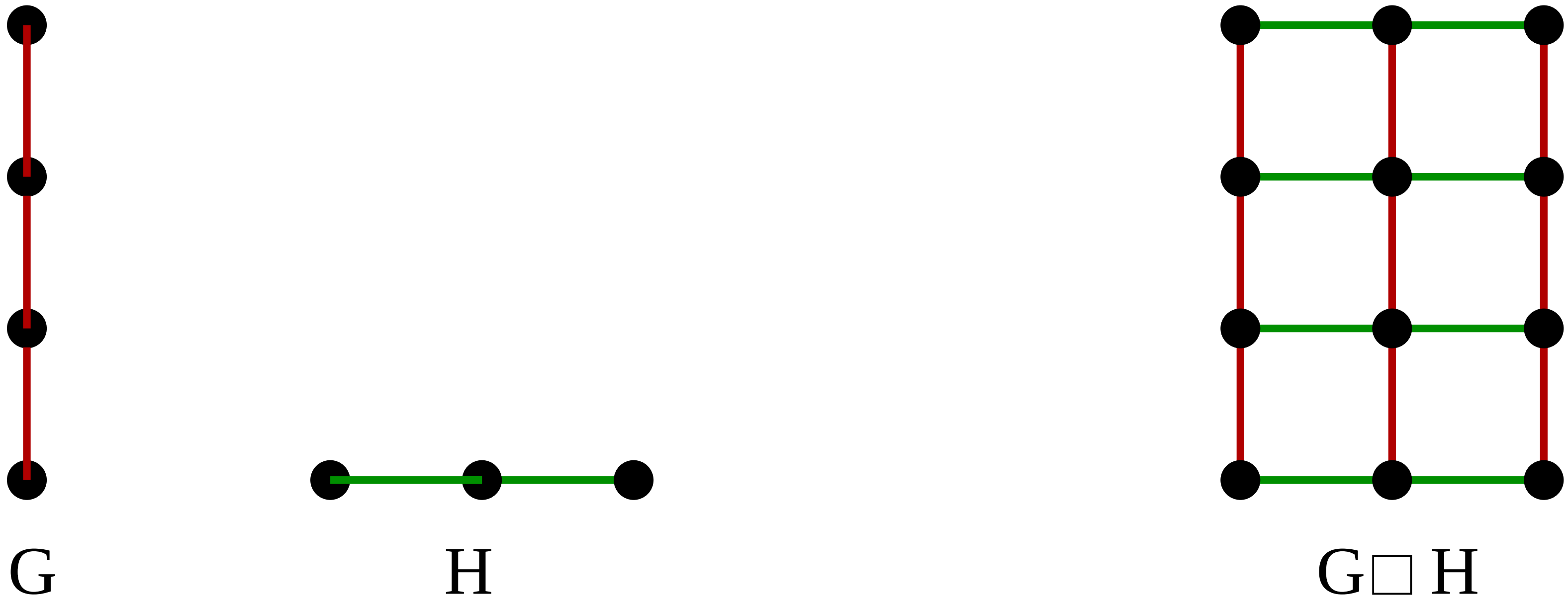
$V(G \square H) = \{(u, v) | u \in V(G), v \in V(H)\}$.

vertex (u, v) is adjacent to vertex (w, z) if either $u = w$ and $vz \in E(H)$ or $v = z$ and $uw \in E(G)$.

Extend this definition to $G_1 \square G_2 \square \ldots \square G_d$.

Denote $G^d = \square_{i=1}^{d} G$.

Cartesian Product of Graphs



Cartesian Product of Graphs

A graph G is said to be a prime graph if whenever $G = G_1 \Box G_2$, then either G_1 or G_2 is a singleton vertex.

Prime Decomposition Theorem [Sabidussi(1960) and Vizing(1963)] Let G be a connected graph, then $G \cong G_1^{p_1} \Box G_2^{p_2} \Box \ldots \Box G_d^{p_d}$, where G_i and G_j are distinct prime graphs for $i \neq j$, and p_i are constants.

Theorem [Imrich(1969) and Miller(1970)]
All automorphisms of a cartesian product of graphs are induced by the automorphisms of the factors and by transpositions of isomorphic factors.

Chromatic Number

Let $G = (V(G), E(G))$ be a graph.

Denote $n(G) = |V(G)|$, number of vertices in G.

A proper k-coloring of G is a labeling of $V(G)$ with k labels such that adjacent vertices get distinct labels.

Chromatic Number, $\chi(G)$, is the least k such that G has a proper k-coloring.

Chromatic Number

Fact: Let $G = \Box_{i=1}^{d} G_i$. Then $\chi(G) = \max_{i=1,\ldots,d} \{\chi(G_i)\}$

Let f_i be an optimal proper coloring of G_i, $i = 1, \ldots, d$.

Canonical Coloring $f^d : V(G) \to \{0, 1, \ldots, t-1\}$ as

$$f^d(v_1, v_2, \ldots, v_d) = \sum_{i=1}^{d} f_i(v_i) \bmod t\,, \qquad t = \max_i \{\chi(G_i)\}$$

Distinguishing Number

A distinguishing k-labeling of G is a labeling of $V(G)$ with k labels such that the only color-preserving automorphism of G is the identity.

Distinguishing Number, $D(G)$, is the least k such that G has a distinguishing k-labeling.

Introduced by Albertson and Collins in 1996.

Since then, especially in the last five years, a whole class of research literature combining graphs and group actions has arisen around this topic.

Distinguishing Number

Some motivating results :

Theorem [Bogstad + Cowen, 2004]
$D(Q_d) = 2$, for $d \geq 4$,
where Q_d is the d-dimensional hypercube.

Theorem [Albertson, 2004]
$D(G^4) = 2$, if G is a prime graph.

Theorem [Klavzar + Zhu , 2005+]
$D(G^d) = 2$, for $d \geq 3$.

Follows from $D(K_t^d) = 2$, for $d \geq 3$.

Distinguishing Chromatic Number

A distinguishing proper k-coloring of G is a proper k-coloring of G such that the only color-preserving automorphism of G is the identity.

Distinguishing Chromatic Number, $\chi_D(G)$, is the least k such that G has a distinguishing proper k-coloring.

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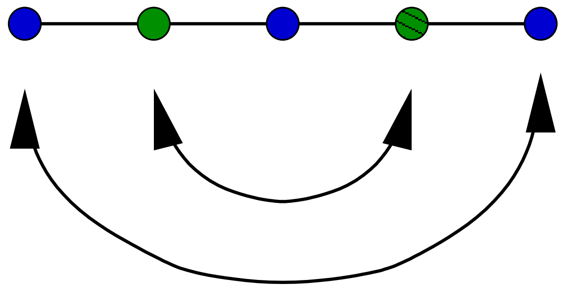
A distinguishing proper k-coloring of G is a proper k-coloring of G such that the only color-preserving automorphism of G is the identity.

Distinguishing Chromatic Number, $\chi_D(G)$, is the least k such that G has a distinguishing proper k-coloring.

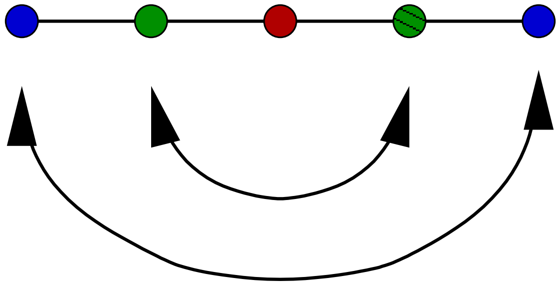
- A proper coloring of G that breaks all its symmetries.
- A proper coloring of G that uniquely determines the vertices.

Examples



Not Distinguishing

Examples



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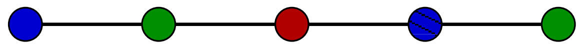

Distinguishing

$\chi_D(P_{2n+1}) = 3$ and $\chi_D(P_{2n}) = 2$

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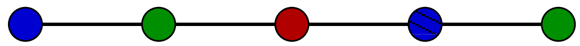

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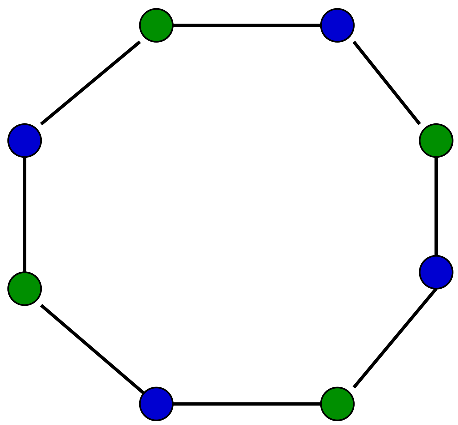

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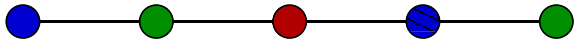

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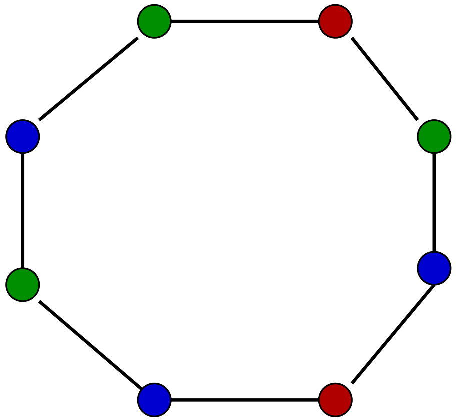

Distinguishing

$\chi_D(C_n) = 3$ except $\chi_D(C_4) = \chi_D(C_6) = 4$

Motivation

Distinguishing Chromatic Number, $\chi_D(G)$, is the least k such that G has a distinguishing proper k-coloring.

The chromatic number, $\chi(G)$, is an immediate lower bound for $\chi_D(G)$.

How many more colors than $\chi(G)$ does $\chi_D(G)$ need?

Theorem [Collins + Trenk, 2006]
$\chi_D(G) = n(G) \Leftrightarrow G$ is a complete multipartite graph.

$\chi_D(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t} n_i$ while $\chi(K_{n_1,n_2,\ldots,n_t}) = t$

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Theorem [Collins + Trenk, 2006]

$\chi_D(G) \leq 2\Delta(G)$, with equality iff $G = K_{\Delta,\Delta}$ or C_6.

Main Theorem

Theorem 1 [Choi + Hartke + K., 2005+]
Let G be a graph. Then there exists an integer d_G such that for all $d \geq d_G$, $\chi_D(G^d) \leq \chi(G) + 1$.

By the Prime Decomposition Theorem for Graphs, $G = G_1^{p_1} \square G_2^{p_2} \square \ldots \square G_k^{p_k}$, where G_i are distinct prime graphs.

Then, $$d_G = \max_{i=1,\ldots,k} \{\frac{\lg n(G_i)}{p_i}\} + 5$$

Note, $n(G) = (n(G_1))^{p_1} * (n(G_2))^{p_2} * \cdots * (n(G_k))^{p_k}$

At worst, $$d_G = \lg n(G) + 5$$

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$$d_G = \max_{i=1,\ldots,k} \{\frac{\lg n(G_i)}{p_i}\} + 5$$

when, $n(G) = (n(G_1))^{p_1} * (n(G_2))^{p_2} * \cdots * (n(G_k))^{p_k}$

d_G is unlikely to be a constant, as the example of Complete Multipartite Graphs indicates –

pushing $\chi_{D}(K_{n_1,n_2,\ldots,n_t})$ down from $n(G)$ to $t+1$!

Proof Idea for Theorem 1

Fix an optimal proper coloring of G.

Embed G in a complete multipartite graph H.

Form H by adding all the missing edges between the color classes of G.

Now work with H.

BUT $G \subseteq H \nRightarrow \chi_D(G) \leq \chi_D(H)$!

Proof Idea for Theorem 1

Fix an optimal proper coloring of G.

Embed G in a complete multipartite graph H.

Form H by adding all the missing edges between the color classes of G.

Then construct a distinguishing proper coloring of H^d that is also a distinguishing proper coloring of G^d.

Study Distinguishing Chromatic Number of Cartesian Products of Complete Multipartite Graphs.

Hamming Graphs and Hypercubes

Theorem 2 [Choi + Hartke + K., 2005+]
Given $t_i \geq 2$, $\chi_D(\square_{i=1}^d K_{t_i}) \leq \max_i\{t_i\} + 1$,
for $d \geq 5$.

Hamming Graphs and Hypercubes

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Corollary : Given $t \geq 2$, $\chi_D(K_t^d) \leq t + 1$, for $d \geq 5$.

Both these upper bounds are 1 more than their respective lower bounds.

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Corollary : Given $t \geq 2$, $\chi_D(K_t^d) \leq t + 1$, for $d \geq 5$.

Both these upper bounds are 1 more than their respective lower bounds.

Corollary : $\chi_D(Q_d) = 3$, for $d \geq 5$.

Complete Multipartite Graphs

Theorem 3 [Choi + Hartke + K., 2005+]
Let H be a complete multipartite graph. Then
$\chi_D(H^d) \leq \chi(H) + 1$, for $d \geq \lg n(H) + 5$.

This is already enough to prove Theorem 1 for prime graphs.

Complete Multipartite Graphs

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This is already enough to prove Theorem 1 for prime graphs.

Theorem 4 [Choi + Hartke + K., 2005+]
Let $H = \square_{i=1}^{k} H_i^{p_i}$, where H_i are distinct complete multipartite graphs. Then

$$\chi_D(H^d) \leq \chi(H) + 1,$$

for $d \geq \max_{i=1,\ldots,k} \{\frac{\lg n_i}{p_i}\} + 5$, where $n_i = n(H_i)$.

Main Theorem

Theorem 1 [Choi + Hartke + K., 2005+]
Let G be a graph. Then there exists an integer d_G such that for all $d \geq d_G$, $\chi_D(G^d) \leq \chi(G) + 1$.

By the Prime Decomposition Theorem for Graphs, $G = G_1^{p_1} \square G_2^{p_2} \square \ldots \square G_k^{p_k}$, where G_i are distinct prime graphs.

Then, $$d_G = \max_{i=1,\ldots,k} \{\frac{\lg n(G_i)}{p_i}\} + 6$$

Outline of the Proof for Hamming Graphs

Start with the canonical proper coloring f^d of cartesian products of graphs, $f^d : V(K_t^d) \to \{0, 1, \ldots, t-1\}$ with

$$f^d(v) = \sum_{i=1}^{d} f(v_i) \mod t,$$

where $f(v_i) = i$ is an optimal proper coloring of K_t.

Outline of the Proof for Hamming Graphs

Derive f^* from f^d by changing the color of the following vertices from $f^d(v)$ to $*$:

Origin : $0000\ldots000$.

Group 1 : $A = \bigcup_{i=1}^{\lfloor \frac{d}{2} \rfloor} A_i$, where $A_i = \{e^1_{i,j} \mid 1+i \leq j \leq d+1-i\}$

v^* : the vertex with all coordinates equal to 1 except for the ith coordinate which equals 0.

$e^1_{i,j}$ is the vertex with all coordinates equal to 0 except for the ith and jth coordinates which equal 1.

Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of K_t^d by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

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Step 1. Distinguish v^* from the Origin and the Group 1 by counting their distance two neighbors in the color class $*$.

Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of K_t^d by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

Step 2. Distinguish the Origin by counting the distance two neighbors in color class $*$.

Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of K_t^d by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

Step 3. Assign the vector representations of weight one, with 1 as the non-zero coordinate, to the correct vertices.

Assign the vector $e_1^1 = 100\ldots000$ to the vertex, neighboring the Origin, with most neighbors in Group 1.

Assign the vector e_i^1 to the vertex, neighboring the Origin, with most neighbors in Group 1 other than the vertices assigned the labels e_j^1, $1 \leq j \leq i-1$.
Distance to v^* breaks ties.

Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of K_t^d by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

Step 4. Assign the vector representations of weight one, with $k > 1$ as the non-zero coordinate, to the correct vertices, by recovering the original canonical colors of all the vertices.

Outline of the Proof for Hamming Graphs

Uniquely identify each vertex of K_t^d by reconstructing its original vector representation by using only the colors of the vertices and the structure of the graph.

Step 5. Assign the vector representations of weight greater than one to the correct vertices.

Let x be a vertex with weight $\omega \geq 2$. Then x is the unique neighbor of the vertices, $y_1, y_2, \ldots, y_\omega$, formed by changing exactly one non-zero coordinate of x to zero that is not the Origin.