Profile vectors in the poset of subspaces

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Motivation

For a family $\mathcal{F}$ of subsets of $[n] = \{1, 2, ..., n\}$ P.L. Erdős, P. Frankl, G.O.H. Katona (1985) defined

$$p(\mathcal{F}) \in \mathbf{R}^{n+1}, p(\mathcal{F})_i = |\{F \in \mathcal{F} : |F| = i\}|.$$

For a set $\mathbf{A}$ of families they started to investigate $\langle \mu(\mathbf{A}) \rangle$, the convex hull of the profiles of the families in $\mathbf{A}$.

In the same way for a family $\mathcal{U}$ of subspaces of an n-dimensional vectorspace V over $GF(q)$ we define

$$p(\mathcal{U}) \in \mathbf{R}^{n+1}, p(\mathcal{U})_i = |\{U \in \mathcal{U} : \dim U = i\}|.$$

The reduction method

The method provides a superset of the profile polytope if we know the profile polytope of the families reduced to some simpler structure.

In the Boolean poset, reduction to

- maximal chain ($\{C_0 \subseteq C_1 \subseteq ... \subseteq C_n\}$ with $|C_i| = i$),
- pair of complement chains (union of two maximal chains $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $C \in \mathcal{C}_1$ iff $[n] \setminus C \in \mathcal{C}_2$),
- circle (the intervals (sets of consecutive elements) for a fixed cyclic permutation of $[n]$).

In the poset of subspaces, reduction to

- chain,
- pair of orthogonal chains.

Easy consequences I

Using the reduction method one can determine the profile polytope of the following sets of families (among others):

- k-Sperner families,
- orthogonal complement-free families,
- families without pairs of subspaces U_1, U_2 with $U_1 \cap U_2 = \{\underline{0}\}$, $\langle U_1 \cup U_2 \rangle = V$ (property *).

Theorem. The extreme points of the profile polytope of k-Sperner families are the following:

for all $0 \leq z \leq k$ and $\beta = \{\beta_1, ..., \beta_z\}$ with $0 \leq \beta_1 < ... < \beta_z \leq n$ the vectors v_β

$$(v_\beta)_i = \begin{cases} \begin{bmatrix} n \\ i \end{bmatrix} & \text{if } i \in \beta \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Easy consequences II

Theorem. The extreme points of the convex hull of the profile vectors of orthogonal complement-free families are the vectors corresponding to the families consisting of

a, a set I of levels with the property that the ith and the $n-i$th levels cannot be both in I, if n is odd,

b, a set I of levels with the property that the ith and the $n-i$th levels cannot be both in I and possibly half of the subspaces with dimension $n/2$ one from each pair of orthogonal complementary subspaces, if n is even.

Theorem. The extreme points of the profile polytope of families with property * are the vectors corresponding to the families consisting of

a, a set I of levels with the property that the ith and the $n-i$th levels cannot be both in I, if n is odd,

b , a set I of levels with the property that the ith and the $n-i$th levels cannot be both in I and possibly a maximal star of subspaces (subspaces containig a fixed vector) with dimension $n/2$, if n is even.

The method of inequalities

To do:

- finding (linear) inequalities valid for the profiles of all families in the set examined
- determining the extreme points of the polytope determined by the above inequalities
- finding a family for any extreme point that has this vector as profile

Intersecting families

Theorem. (Erdős - Ko - Rado for vectorspaces) If $\mathcal{U} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ is a t-intersecting family and $n \geq 2k - t$, then

$$|\mathcal{U}| \leq \max\{\begin{bmatrix} n-t \\ k-t \end{bmatrix}, \begin{bmatrix} 2k-t \\ k \end{bmatrix}\}.$$

- Hsieh ‘77 for large enough ns (including $t = 1$, $n \geq 2k + 1$)
- Greene - Kleitman ‘78 for $t = 1$, $k|n$ (so for $n = 2k$ as well)
- Frankl - Wilson ‘86 for all cases

A generalization of Hsieh's theorem

Theorem. The following generalization of Hsieh's theorem holds:

a, if $2k \leq n \leq 2k+2$ and $d = 0$ or $d = n - k$

or

b, if $n \geq 2k+3$ and $k + d \leq n$

then for any intersecting family $\mathcal{U}$ of k-dimensional subspaces of an n-dimensional vectorspace V with all members disjoint from a fixed d-dimensional subspace W of V we have

$$|\mathcal{U}| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}^{*(d)}$ denotes the number of k-dimensional subspaces of an n-dimensional vectorspace V that are disjoint from a fixed d-dimensional subspace W of V.

'Sketch' of the proof

Proof.

- calculating the eigenvalues of the apropriate incidency matrix (as Frankl and Wilson did) is not straightforward
- if $k|d|n$ or $k|n, d = 0$ (so in particular when $n = 2k, d = 0$) the trick of Greene and Kleitman (partitioning $V \setminus W$ into isomorphic copies of $V_k \setminus \{\underline{0}\}$) works
- for the remaining cases the 'lengthy computations' of Hsieh go through, since he used 'mostly' the quick growth of the q-nomial coefficients, and the ratio of growth needed there is valid for the growth of

$$\frac{{n \brack k}^{*(d)}}{{n-1 \brack k-1}^{*(d)}}$$

(for the $q = 2$ case we needed some further computations)

The inequalities

Lemma. For the profile vector f of any family $\mathcal{U}$ of intersecting subspaces of an n-dimensional vectorspace, and for any $k < n/2$ and $n/2 < d \leq n-k$, the following holds

$$c_{k,d} f_k + f_d \leq \begin{bmatrix} n \\ d \end{bmatrix},$$

where $c_{k,d} = q^d \frac{\left[{n-k \atop d}\right]}{\left[{n-d-1 \atop k-1}\right]}$, and eqaulity holds if $f_k = 0, f_d = \left[{n \atop d}\right]$ or if $f_k = \left[{n-1 \atop k-1}\right], f_d = \left[{n-1 \atop d-1}\right]$.

Proof. doublecount the disjoint pairs formed from $\mathcal{U}_k$ and $\mathcal{U}'_d = \left[{V \atop d}\right] \setminus \mathcal{U}_d$ giving

$$\begin{bmatrix} n \\ d \end{bmatrix}^{*(k)} f_k \leq (\begin{bmatrix} n \\ d \end{bmatrix} - f_d) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^{*(d)}$$

The profile polytope

Theorem. The essential extreme points of the profile polytope of the set of intersecting families are the vectors v_i $1 \leq i \leq n/2$ for even n and there is an additional essential extreme point v^+ for odd n, where

$$(v_i)_j = \begin{cases} 0 & \text{if } 0 \leq j < i \\ \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} & \text{if } i \leq j \leq n-i \\ \begin{bmatrix} n \\ j \end{bmatrix} & \text{if } j > n-i. \end{cases} \tag{2}$$

and

$$(v^+)_j = \begin{cases} 0 & \text{if } 0 \leq j < n/2 \\ \begin{bmatrix} n \\ j \end{bmatrix} & \text{if } j > n/2. \end{cases} \tag{3}$$

Remark. This is the 'same' as in the Boolean case, just using q-nomial coefficients instead of binomial coefficients.

t-intersecting families

In the Boolean case the complete intersection theorem of Ahlswede and Khatchatrian (1997) settled the problem of the largest t-intersecting families when n is small compared to k and t. The optimal families vary 'widely' when n ranges from $2k-t$ to infinity.

By the theorem of Frankl and Wilson we know that in tha case of subspaces there are only three possibilities for the optimal family.

'Conjecture'. Determining the profile polytope of t-intersecting families is much easyer in the poset of subspaces than in the Boolean case.