Coverings and packings for radius 1 adaptive block coding

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DIMACS/DIMATIA/Rényi Inst. Combinatorial Challenges 2006

Outline

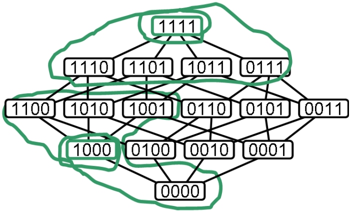
1. Background
   - Non-adaptive and adaptive radius 1 codes
   - Liar games
   - Previous work
2. New Contribution
   - Constructive bottom-up algorithm
   - Ingredients of the proof
   - Exact sizes of optimal codes
3. Open questions and concluding remarks

1-balls & non-adaptive radius 1 block codes defined

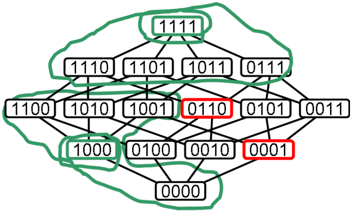
- Hypercube $Q_{n,t} := \{x_1 \cdots x_n \in \{0, \ldots, t-1\}^t\}$
- Hamming distance $d(x, y) = |\{i : x_i \neq y_i\}|$
- 1-ball $B_1(u) := \{u \in Q_{n,t} : d(u, v) \leq 1\}$
- 1-ball size $b_1(n, t) := 1 + n(t-1)$

1111
0111
1011
1101
1110

$B_1(1111)$



Packing code in $Q_{4,2}$



Covering code in $Q_{4,2}$

Optimal radius 1 block codes defined

- $F_t(n, 1) :=$ maximum size of packing of 1-balls in $Q_{n,t}$
- $K_t(n, 1) :=$ minimum size of covering of 1-balls in $Q_{n,t}$
- Sphere bound. $F_t(n, 1) \leq \frac{t^n}{1+n(t-1)} \leq K_t(n, 1)$

For $t = 2$:

- Hamming codes. $(n + 1)|2^n \Rightarrow F_2(n, 1) = K_2(n, 1)$
- Asymptotics (Kabatyanskii and Panchenko). $\lim_{n\to\infty} \frac{F_t(n,1)}{K_t(n,1)} = 1$

1-sets & radius 1 adaptive block codes defined

- a 1-set consists of

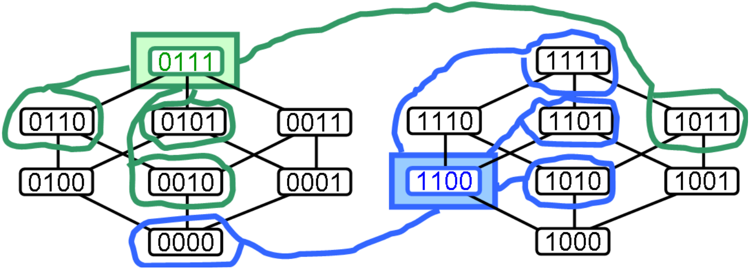
$$\begin{array}{rlr} \text{a stem} & x_1 \cdots x_{i-1} x_i \ \cdots x_n & \in Q_{n,t} \\ n(t-1) \text{ children} & x_1 \cdots x_{i-1} y_i * \cdots * & \in x_1 \cdots x_i y_i Q_{n-i,t}, \end{array}$$

  where $y_i \in [t] \setminus x_i$.

- Examples.

$$n=4, t=2: \quad \begin{array}{l} 1100 \\ \underline{0}010 \\ 1\underline{0}01 \\ 11\underline{1}1 \\ 110\underline{1} \end{array} \qquad n=4, t=3: \quad \begin{array}{l} 2021 \\ \underline{0}000 \\ \underline{1}102 \\ 2\underline{1}21 \\ 2\underline{2}00 \\ 20\underline{0}2 \\ 20\underline{1}1 \\ 202\underline{0} \\ 202\underline{2} \end{array}$$

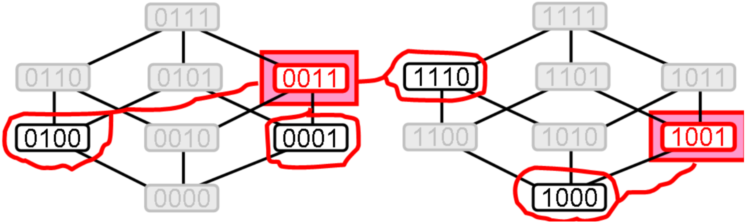
Example radius 1 adaptive packing



Adaptive packing code in $Q_{4,2}$

0111	1100
1011	0000
0010	1011
0101	1101
0110	1101

Example radius 1 adaptive covering



Adaptive covering code in $Q_{4,2}$

Previous packing, plus:

0011	1001
1110	· · · ·
0100	· · · ·
0001	· · · ·
· · · ·	1000

signature $= 5, 5, 4, 2$

Optimal radius 1 adaptive block codes defined

- $F'_t(n,1) :=$ maximum size of packing of 1-sets in $Q_{n,t}$
- $K'_t(n,1) :=$ minimum size of covering of 1-sets in $Q_{n,t}$
- Sphere bound$^+$.
  $F_t(n,1) \le F'_t(n,1) \le \frac{t^n}{1+n(t-1)} \le K'_t(n,1) \le K_t(n,1)$

Optimal radius 1 adaptive block codes defined

- $F_t'(n, 1) :=$ maximum size of packing of 1-sets in $Q_{n,t}$
- $K_t'(n, 1) :=$ minimum size of covering of 1-sets in $Q_{n,t}$
- Sphere bound$^+$.
  $F_t(n, 1) \le F_t'(n, 1) \le \frac{t^n}{1+n(t-1)} \le K_t'(n, 1) \le K_t(n, 1)$
- Binary case (EIS, CHLL; P, EPY)

n	1	2	3	4	5	6	7	8	9	10	11
$F_2(n, 1)$	1	1	2	2	4	8	16	20	40	72	144
$F_2'(n, 1)$	1	1	2	2	4	8	16	28	50	92	170
$K_2'(n, 1)$	1	2	2	4	6	10	16	30	52	94	172
$K_2(n, 1)$	1	2	2	4	7	12	16	32	≤ 57	≤ 105	≤ 180

Liar games defined

2-player perfect information game

- Players: Paul – partitioner/questioner
  Carole – chooser/responder
- q rounds of Game play:
  Paul partitions $[n] \to A_1 \dot\cup \cdots \dot\cup A_t$
  Carole selects a part, other parts get 1 lie
  Elements with $\leq k$ lies survive

Possible winning conditions for Paul

- Original. ≤ 1 element survives (Rényi, Ulam)
- Pathological. ≥ 1 element survives (Ellis+Yan)

Equivalence of liar games and packings/coverings

- Offline partitions by Paul
  - Winning strategy in original game $\leftrightarrow$ nonadaptive packing in hypercube
  - Winning strategy in pathological game $\leftrightarrow$ nonadaptive covering in hypercube

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- Online partitions by Paul
  - Winning strategy in original game $\leftrightarrow$ adaptive packing in hypercube
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Equivalence of liar games and packings/coverings

- Offline partitions by Paul
  - Winning strategy in original game $\leftrightarrow$ nonadaptive packing in hypercube
  - Winning strategy in pathological game $\leftrightarrow$ nonadaptive covering in hypercube
- Online partitions by Paul
  - Winning strategy in original game $\leftrightarrow$ adaptive packing in hypercube
  - Winning strategy in pathological game $\leftrightarrow$ adaptive covering in hypercube

Remarks. Parameters n, t, k must match!

Many generalizations:

attributions $\subseteq 2^{\{\text{Spencer, Yan, Dumitriu, Ellis, Ponomarenko,Nyman}\}}$

Sample of previous bounds on adaptive codes

Adaptive packing codes/liar games

- (Berlekamp '67) Fixed k, weight function
- (Spencer+Winkler '91) $k \sim q/3, q/4$ (balls off a cliff...)
- (Spencer '92) $F'_2(n,k) \pm C_k$ for fixed k
- (Pelc, Guzicki, Deppe) exact $F'_2(n,k)$ for $k = 1, 2, 3$, resp.
- (Cicalese+Mundici, Spencer$\oplus${Dumitriu,Yan}) half-lie: $k = 1$ and fixed k, resp.
- (Spencer+Dumitriu, Ellis+Nyman) fixed k; arbitrary channel, arbitrary channel**s**, resp.

Adaptive covering codes/pathological liar games

- (Ellis+Yan, Ellis+Ponomarenko+Yan) half-lie for $k = 1$, $K'_2(n,k) \pm C_k$ for fixed k

Example collaboration.

Example collaboration.

S. "The $\{x_2, x_1 x_3\}$ partitioning is clearly best."

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Ł. "Who are you playing for, Paul or Carole?"

Example collaboration.

S. “The $\{x_2, x_1 x_3\}$ partitioning is clearly best.”

Ł. “Who are you playing for, Paul or Carole?”

S. “I don’t remember, but the answer is $1/3$.”

Philosophy of approach

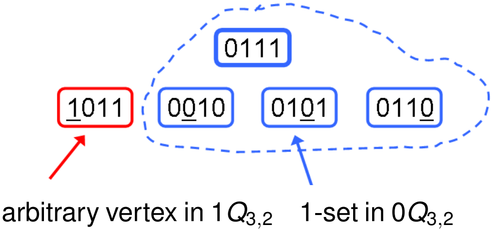
Previously, 4 proofs for any choice of k, t, channel

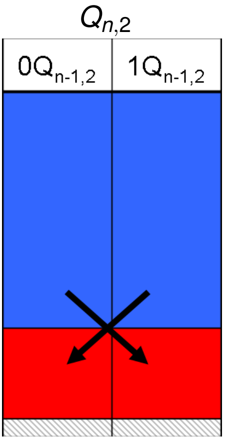
- Upper (sphere) bound for adaptive packing
- Exhibition of good adaptive packing
- Lower (sphere) bound for adaptive covering
- Exhibition of good adaptive covering

Goal: single unified proof (& fast algorithm)

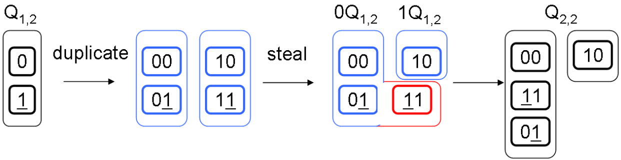
Decomposition structure of 1-sets in $Q_{n,2}$

Observation. $t = 2$

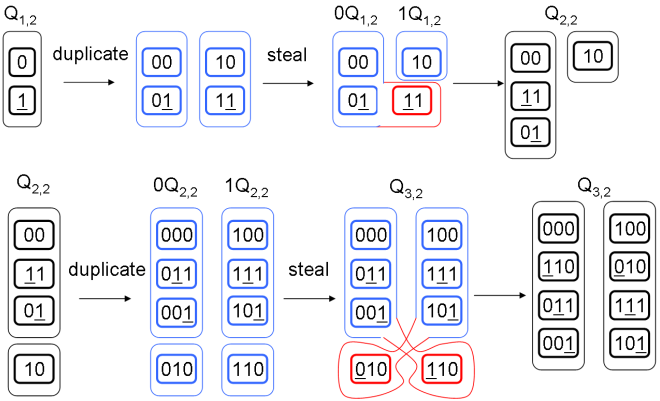




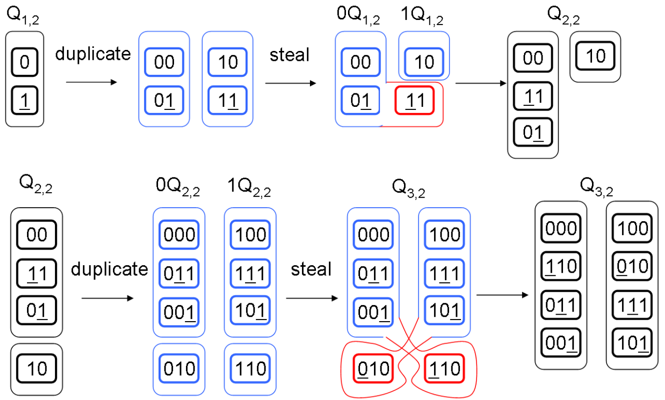
Packing within covering **duplication** algorithm



Packing within covering **duplication** algorithm



Packing within covering **duplication** algorithm



Signature encoding of $Q_{2,2} \to Q_{3,2}$

$Q_{2,2}$		$0Q_{2,2}$	$1Q_{2,2}$		$0Q_{2,2}$	$1Q_{2,2}$		$Q_{3,2}$
3	dup. $\to$	3	3	st. $\to$	4	4	$\leftrightarrow$	4
1		1	1					4

Dominant signature of a collection of 1-sets defined

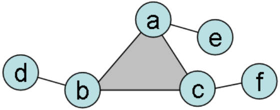
Definition

A dominant signature of a collection $\mathcal{F}$ of m 1-sets is an ordering $F_1, \ldots, F_m$ and a sequence $\alpha_1, \ldots, \alpha_m$ such that for all J,

$$|F_1 \cup F_2 \cup \cdots \cup F_J| = \alpha_1 + \alpha_2 + \cdots + \alpha_J,$$

and for each J, no J-subset of $\mathcal{F}$ is larger than $\sum_{i=1}^{J} \alpha_i$.

Remark. Always monotonic decreasing, but doesn't always exist:



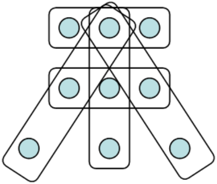
3, 4, **5**, 6 (triangle first)
2, 4, **6**, 6 (triangle last)

$\{abc, ae, bd, cf\}$

Dominant signature of a collection of 1-sets defined

More counterexamples.

(Vera Asodi) Minimum ground set 3-uniform counterexample



(Penny Haxell, Gábor Tardos) Whiteboard

Radius 1 adaptive codes with dominant signatures

Lemma

If the set of all 1-sets $\mathcal{F}$ in $Q_{n,t}$ has a dominant signature $(\alpha(n,t,i))_{i\geq 1}$, then

- *Packing:* $\alpha(n,t,1) = \cdots = \alpha(n,t,F_t'(n,1)) = 1 + t(n-1)$,
- *Covering:* $\alpha(n,t,K_t'(n,1)) > \alpha(n,t,K_t'(n,1)+1) = 0$, *and*
- *Partial covering: for all J, the most vertices which can be covered by J 1-sets is exactly*

$$\alpha(n,t,1) + \alpha(n,t,2) + \cdots + \alpha(n,t,J) \;=\; |F_1 \cup F_2 \cup \cdots \cup F_J|\,.$$

Remark. Dominant signature $\Rightarrow$ optimal packing within optimal covering.

More iterations of the duplication algorithm

$Q_{3,2}$
4
4

$\xrightarrow{\text{dup.}}$

$0Q_{3,2}$	$1Q_{3,2}$
4	4
4	4

$\xrightarrow{\text{st. (a)}}$

$0Q_{3,2}$	$1Q_{3,2}$
5	5
3	3

$\xrightarrow{\text{st. (b)}}$

$0Q_{2,2}$	$1Q_{2,2}$
5	5
4	2

$Q_{4,2}$
5
5
4
2

$\xrightarrow{\text{dup.}}$

$0Q_{4,2}$	$1Q_{4,2}$
5	5
5	5
4	4
2	2

$\xrightarrow{\text{st. (a)}}$

$0Q_{4,2}$	$1Q_{4,2}$
6	6
6	6
4	4

$\xrightarrow{\text{st. (b)}}$

$0Q_{4,2}$	$1Q_{4,2}$
6	6
6	6
5	3

$Q_{5,2}$
6
6
6
6
5
3

$\xrightarrow{\text{dup.}}$

$0Q_{5,2}$	$1Q_{5,2}$
6	6
6	6
6	6
6	6
5	5
3	3

$\xrightarrow{\text{st. (a)}}$

$0Q_{5,2}$	$1Q_{5,2}$
7	7
7	7
7	7
7	7
4	4

$\xrightarrow{\text{st. (b)}}$

$0Q_{5,2}$	$1Q_{5,2}$
7	7
7	7
7	7
7	7
5	3

Domination preserved by duplication algorithm

Lemma (Domination lemma)

If the input signature for the duplication algorithm is dominant for $Q_{n,t}$, then the output signature is dominant for $Q_{n+1,t}$.

Remark. i.e., optimal packing within covering for $n \Rightarrow$
optimal packing within covering for $n+1$

Lemma (Covering lemma)

Let $(\alpha(n,t,i))_{i=1}^{M}$ be a dominant signature for $Q_{n,t}$, and $\sum_{i=0}^{t-1} M_i = M$. Then the most vertices of $Q_{n+1,t}$ which can be covered by M_i 1-sets with stem in $iQ_{n,t}$ is $C(n+1,t,M_0,\ldots,M_{t-1}) :=$

$$\sum_{i=0}^{t-1}\left(\sum_{j=1}^{M_i}\alpha(n,t,j)+\min\left\{t^n-\sum_{j=1}^{M_i}\alpha(n,t,j), M-M_i\right\}\right).$$

Proving the domination lemma

Proof sketches of the covering and domination lemmas.

• The inner summation is concave; therefore $C(n+1, t, M_0, \ldots, M_{t-1})$ is maximized when $|M_i - M_j| \leq 1$ for all i, j.

$$\sum_{j=1}^{M_i} \alpha(n,t,j) + \min\left\{ t^n - \sum_{j=1}^{M_i} \alpha(n,t,j), M - M_i \right\}.$$

• Output signature β of duplication has $M_0 \geq \cdots \geq M_{t-1} \geq M_0 - 1$.

• Output signature $(\beta(n+1,t,j))_{j \geq 1}$ satisfies for all K (w/abuse),

$$\sum_{j=1}^{K} \beta(n+1,t,j) = C(n+1, t, \lceil K/t \rceil, \ldots, \lfloor K/t \rfloor).$$

□

Sample case computation for duplication output

Case: • last row which makes a full steal is r

- last nonzero β at index $M > t \cdot r$
- give back all stealing done by indices $> t \cdot r$
- $\beta = t^n - \sum_{i=1}^{r}(\alpha(n,t,i) + t - 1) + t - (M - t \cdot r)$ at indices $t \cdot r + 1, \ldots, M$. Then

$$\begin{aligned}
\sum_{j=1}^{M} \beta(n+1,t,j) &= \sum_{j=1}^{tr} (\alpha(n,t,\lceil j/t \rceil) + t - 1) \\
&\quad +(M - tr)\left(t^n - \sum_{j=1}^{r}(\alpha(n,t,j) + t - 1) + t - M + tr\right) \\
&= \sum_{i=1}^{t}\left(\sum_{j=1}^{r}\alpha(n,t,j) + rt - r\right) + (M - tr)(t - (M - tr)) \\
&\quad +(M - tr)\left(t^n - \sum_{j=1}^{r}\alpha(n,t,j) - rt + r\right) \\
&= \sum_{i=1}^{M-tr} t^n + \sum_{i=M-tr+1}^{t}\left(\sum_{j=1}^{r}\alpha(n,t,j) + M - r\right) \\
&\geq C(n+1, t, \lceil M/t \rceil, \ldots, \lceil M/t \rceil, \lfloor M/t \rfloor, \ldots, \lfloor M/t \rfloor).
\end{aligned}$$

Optimal adaptive packing and covering numbers

Theorem

Let $t \geq 2$ and $n \geq t+1$. There exists an optimal radius 1 adaptive packing contained in an optimal radius 1 adaptive covering of $Q_{n,t}$ with respective sizes $F'_t(n,1)$, $K'_t(n,1)$ given by

$t^{n-1} \mod b_{n,t}(1)$	$F'_t(n,1)$	$K'_t(n,1)$
$i,\ 0 \leq i < t$	tS	$tS+i$
$t, \ldots, b_{n-1,t}(1)-1$	tS	$tS+t$
$b_{n-1,t}(1)-1+i,\ 1 \leq i < t$	$tS+i$	$tS+t$

where $b_{n,t}(1) = 1 + n(t-1)$ and $S = \lfloor t^{n-1}/b_{n,t}(1) \rfloor$.

Remarks. For $1 \leq n \leq t+1$:

- $F'_t(n,1) = t^{n-2}$
- $K'_t(n,1)$ boundable but complicated

Open questions

Related to this technique:

- For which other communication channels does the domination lemma hold for radius 1?
- Can we extend the duplication algorithm to other communication channels, radius > 1?

General adaptive coding questions:

- Tight bounds when radius $= \omega(1)$?
- Can known methods improve coin-weighing, batch-testing, or liar games with restricted questions?

And of course, do fixed-radius nonadaptive codes approach the sphere bound?

SIAM Victoria Minisymposium: “Liar games and error-correcting codes”

Organizers: Ioana Dumitriu, Joel Spencer, New York University

- Berlekamp. History of Block Coding with Noiseless Feedback
- Dumitriu+Spencer. Liar Games with a Fixed Number of Constrained Lies
- Ahlswede+Cicalese+Deppe. Searching with Lies under Error Cost Constraints
- Ahlswede+Deppe. Non-Binary Error-Correcting Codes with Noiseless Feedback
- Ellis+Nyman. Multichannel Liar Games with a Fixed Number of Lies

Survey: A. Pelc. Searching games with errors–fifty years of coping with liars, Theoret. Comput. Sci. ‘02.