Edge Choosability of Planar Graphs with no Two Adjacent Triangles

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April 29, 2006

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$\chi'_l(G)$: minimum k such that G has an L-edge-coloring whenever $|L(e)| \geq k$ for all $e \in E(G)$

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Theorem [Cranston 2006]

If G is planar, G does not contain a kite as a subgraph, and $\Delta(G) \geq 9$, then $\chi'_l(G) = \chi'(G) = \Delta(G)$.

Vizing's Theorem [1964]

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Theorem [Cranston 2005]

If G is planar, G does not contain a kite as a subgraph, and $\Delta(G) \geq \not{6}\ 7$, then $\chi'_l(G) \leq \Delta(G) + 1$.

Lemma:

If G is planar, G does not contain a kite as a subgraph, and $\Delta(G) \geq 7$, then G contains an edge uv with $d(u) + d(v) \leq \Delta(G) + 2$.

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If we can order the edges of G such that for each edge e at most k edges adjacent to edge e precede it in the ordering, then $\chi'_l(G) \leq k + 1$.

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This is called the Discharging Method

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Fix a face f. Show that $\mu^*(f) \geq 0$.

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so $\mu^*(v) \geq d(v) - 4 - d(v)/2(1/2) = 3d(v)/4 - 4 > 0$

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$$\mu^*(v) \geq d(v) - 4 - t/2 - (d(v) - t)/3$$

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