General Geographical Threshold Graphs

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Introduction

-A large class of real networks: Internet, WWW, social networks, biological, independent systems have short diameter [Bollobas and Riordan] high cluster coefficients, and asymptotically power-low behavior.

-There has been an extensive research towards establishing a mathematical model which emulates the networks with the above defined characteristics: Barrabasi and Albert proposed a preferential attachment which has the the power-low degree distribution, etc.

-We analyze the geographical threshold graph (GTG) model of the network. In GTG model edges are created according to a threshold function $\mathcal{T}$ which depends on the distances between vertices and the density distribution of weights $f(w)$ in the graph.

-The main contribution of this work is that for a given degree distribution $p_d(k)$ we derive a set of sufficient conditions on $\mathcal{T}$ which enables us to analytically calculate the density function of weights $f(w)$.

-Since our model is very general, the analysis which we derived can be applied in many different areas, e.g., the wireless communication systems, the financial markets, etc.

Outline

-Introduction of both a non-geographical and a geographical threshold model of the network,

-Framework for finding $f(w)$ for a given $p_d(k)$ and derive a set of sufficient conditions on a threshold function,

-Short discussion on the inverse problem,

-Connectivity problem,

-Conclusions and several directions for a future work.

Notations

We shall consider the graph $G = (V, E)$

- V a set of vertices

- E a set of edges.

- d_{min} will denote minimum degree

- d_{max} will denote maximum degree

- $p_d(k)$ a degree distribution

Non-Geographical Threshold Model

A non-negative weight w, taken randomly and independently from a probability distribution $f(w)$, is assigned to each vertex $v \in V$.

Two vertices v and v', with the weights w and w', are connected if and only if:

$$g(w, w') \geq \theta,$$

where $\theta \geq 0$ is a given threshold.

Specifically, for the "additive law" $g(w, w') = w + w'$ and a finite graph $|V| = n$:

$$\Pr[v \leftrightarrow v'|w] = \Pr[g(w, w') \geq \theta] = 1 - F(\theta - w)$$

and

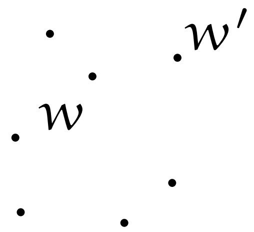
$$k(w) = (n-1)(1 - F(\theta - w)),$$

where F is the cumulative density function of f.

$$\begin{aligned} p_d(k) &= f(w)\left|\frac{dw}{dk}\right| \\ &= \frac{f(\theta - F^{-1}(1 - \frac{k}{n-1}))}{(n-1)f(F^{-1}(1 - \frac{k}{n-1}))}. \end{aligned}$$

Geographical threshold model

In the geographical case it is assumed that the weighted vertices $v \in V$ are uniformly and independently distributed in the d-dimensional Euclidian space.



Two vertices v, v', with weights w, w' respectively, within the Euclidian distance $r = d(v, v')$ are connected if and only if

$$\mathcal{T}(w, w', r) = g(w, w')h(r) \geq \theta.$$

Masuda et al. have shown that for

-additive model $\frac{w+w'}{r^\beta} \geq \theta$ or
-gravity model $\frac{e^w e^{w'}}{r^\beta} \geq \theta$

and for different distributions of the weight w

-exponential $f(w) = \lambda e^{-\lambda w}, w \geq 0$,
-logistic $f(w) = \beta e^{-\beta w}/(1 + e^{-\beta w})^2$,
-Pareto $f(w) = a/w_0(w_0/w)^{a+1}, w \geq w_0$, etc.

the graph G has the power-low degree distribution $p_d(k) \sim k^{-\alpha}$, when $n \to \infty$.

Q: For a given degree distribution $p_d(k)$ as an input; what is a set of sufficient conditions on a threshold function, which enables us to analytically calculate the density function of weights $f(w)$ (i.e., how to distribute resources over the net in order to get the desired degree distribution)?

The vertex degree

Now, we will make some assumptions for computing $k(w)$. First, $h(r)$ is a monotonically decreasing function; the further the points, the smaller the probability they are connected.

From the monotonicity of the function h, the inverse h^{-1} exists. It follows v, v' with weights w, w' are connected iff they lie within the ball of the radius

$$r_0 = h^{-1}(\theta / g(w, w')).$$

Then the degree $k(w)$ of the vertex v is given by:

$$k(w) \quad = \quad \int_{w' \geq 0} \int_{r \geq 0} f = \int_{w'} f(w') \# \mathcal{B}^d(v, r_0) dw'$$

Let the number of the points inside the ball $\#\mathcal{B}^d(v, r_0)$ be proportional to the volume of the ball, $\#\mathcal{B}^d(v, r_0) = \rho Vol$ (we will not deal with the floor function).

Finally,

$$k(w) = \frac{\pi}{2}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\int_{w'} f(w')\Big(h^{-1}(\frac{\theta}{g(w,w')})\Big)^d dw'.$$

In a general case this is not analytically solvable, especially if we are asked to calculate not only the degree $k(w)$ for a given weight w, but also the degree distribution $p_d(k)$ in the graph G, for $k \geq 0$.

First assumption: $h(r) = r^{-\beta}$, then $h^{-1}(t) = t^{-1/\beta}$.

$$k(w) = \frac{\pi}{2}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\int_{w'} f(w')\Big(\frac{g(w,w')}{\theta}\Big)^{\frac{d}{\beta}} dw'$$

$$\frac{dk}{dw} = \frac{\pi}{2}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\frac{d}{\beta}\theta^{-\frac{d}{\beta}}\int_{w'} f(w')g(w,w')^{\frac{d}{\beta}-1}\frac{\partial g(w,w')}{\partial w}dw',$$

(Leibnitz's criterion will be satisfied, under conditions on g, bellow)

Let as define the ratio between the dimension of the space d, and β-factor, $\nu = \frac{d}{\beta}$. We will analyze two cases, $\nu \neq 1$ and $\nu = 1$,

for $g(w, w')$ being additively

$$g(w,w') = \psi_1(w) + \psi_2(w'),$$

i.e., multiplicatively separable

$$g(w,w') = \psi_1(w)\psi_2(w')$$

Conditions on $g(w, w')$

Let the following be satisfied.

(0) $g(w, w')$ is a symmetric function; neither vertex is privileged.

(1) $g(w, w')$ is an increasing in both arguments; for fixed w, the higher the weight w', the higher the probability that vertices v and v' are connected.

(2) $\psi_1(w)$ and $\psi_2(w)$ are continuous and differentiable functions, with the continuous first derivatives on the interval of interest.

(3) A technical detail: for the unbounded degree distribution, $d_{max} \to \infty$, a bounded function $\psi(w)$ cannot be used.

Main Theorems

[Thm 01] Let the connection between two vertices in the network be defined by $g(w,w')/r^{\beta} \geq \theta$. Also, let $g(w,w')$ be multiplicatively separable; i.e., $g(w,w') = \psi_1(w)\psi_2(w')$, and let assumptions (0)-(3) hold. Then if the desired degree distribution of the network is $p_d(k)$, the weight distribution $f(w)$ can be computed as:

$$f(w) = p_d(C\psi_1^{\gamma}(w))\nu C\psi_1^{\gamma-1}(w)\psi_1'(w) \quad (1)$$

with the domain $\psi_1^{-1}((\frac{d_{min}}{C})^{1/\nu}) \leq w \leq \psi_1^{-1}((\frac{d_{max}}{C})^{1/\nu})$, where $\mathcal{E} = \int p_d(\mu)\mu d\mu$, $C_1 = \frac{\pi}{2}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\frac{d}{\beta}\theta^{-\frac{d}{\beta}}$ and $C = \sqrt{C_1\mathcal{E}K^{\nu}}$.

[Thm 02] Let the connection between two vertices in the network be defined by $g(w, w')/r^\beta \geq \theta$. While the parameter $\nu = 1$, let $g(w, w')$ be additively separable; i.e., $g(w, w') = \psi_1(w) + \psi_2(w')$, and let assumptions (0)-(3) hold. Then if the desired degree distribution of the network is $p_d(k)$, the weight distribution $f(w)$ can be computed as:

$$f(w) = p_d(C_1\psi_1(w) + D)\psi_1'(w). \quad (2)$$

with the domain $\psi_1^{-1}(\frac{d_{min}-D}{C_1}) \leq w \leq \psi_1^{-1}(\frac{d_{max}-D}{C_1})$, where $\mathcal{E} = \int p_d(\mu)\mu d\mu$, $C_1 = \frac{\pi}{2}\frac{\rho}{\Gamma(\frac{d}{2}+1)}\theta^{-1}$, and $D = (C_1 K + \mathcal{E})/2$.

Eg. Inverse problem

Claim For any given distribution of the weights $f(w)$, we can construct the gravity model $g(w, w') = \psi(w)\psi(w')/r^{\beta}$, i.e., we can find the function $\psi(w)$ such that the degree distribution of the network is the power law $p_d(k) = (\alpha - 1)k^{-\alpha}$.

Let $s = \nu(\alpha - 1) > 0$ and $B = C^{\alpha-1}/(\alpha - 1)$.

$$\psi(w) = [s(K - BF(w))]^{-1/s}.$$

Mean Value of Weights (Resources)

The next condition that we usually have to meet in practice, is that we do not have infinite resources. For PL it follows

$EW = \int w f(w) dw$ is finite if $\int \frac{dw}{\psi^s(w)}$ is finite.

Connectedness

Geographical Random Graphs
-n vertices are randomly distributed over the unit disk
-two vertices are connected iff they are within some, given, value $r(n)$

The graph is (not) connected with the high probability iff

$$r^2(n) = c\frac{\log n + \gamma_n}{n}$$

for γ_n tends to +(-) infinity.

2D GTG Model

n vertices are uniformly and randomly distributed over the unit disk. Every vertex is assigned to a random nonnegative weight w, according the weight distribution $f(w)$ (with cdf F).

Two vertices v, v', with the weights w, w', at the distance $d(v, v')$ are connected iff

$$\frac{w + w'}{d(v, v')^\alpha} > 1,$$

where $\alpha > 2$.

Corollary: Let the weight distribution be such that $F^{-1}(\frac{1}{n^{1+\varepsilon}}) \gtrsim (\frac{\log n}{n})^{\alpha/2}$, for some $0 < \varepsilon < \alpha/2 - 1$, then the structure is connected with the probability one, as $n \to \infty$.

Conclusions

We showed that for two specific classes of the threshold functions in the GTG model, for any given degree distribution on the network, the weight distribution of the vertices can be analytically computed. The threshold functions on weights which we considered were either additively or multiplicatively separable.

The advantage is that the model is very general, both weights of vertices and lengths of edges play role.

Many areas for the application: the wireless communication systems, social networks, the financial markets, etc.

It would be interesting to see:

-if our result can be generated to other types of threshold functions,

-strict conditions on connectivity,

-the amount of “randomness”, i.e., the similarity to other random structures that do not have embedded geometry.

Thank You

References

Naoki Masuda, Hiroyoshi Miwa, Norio Konno, “Geographical threshold graphs with small-world and scale-free properties”, *Phys. Rev. E 71, 036108 (2005)*