Circular chromatic number of hexagonal chains with orientations

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Joint work with

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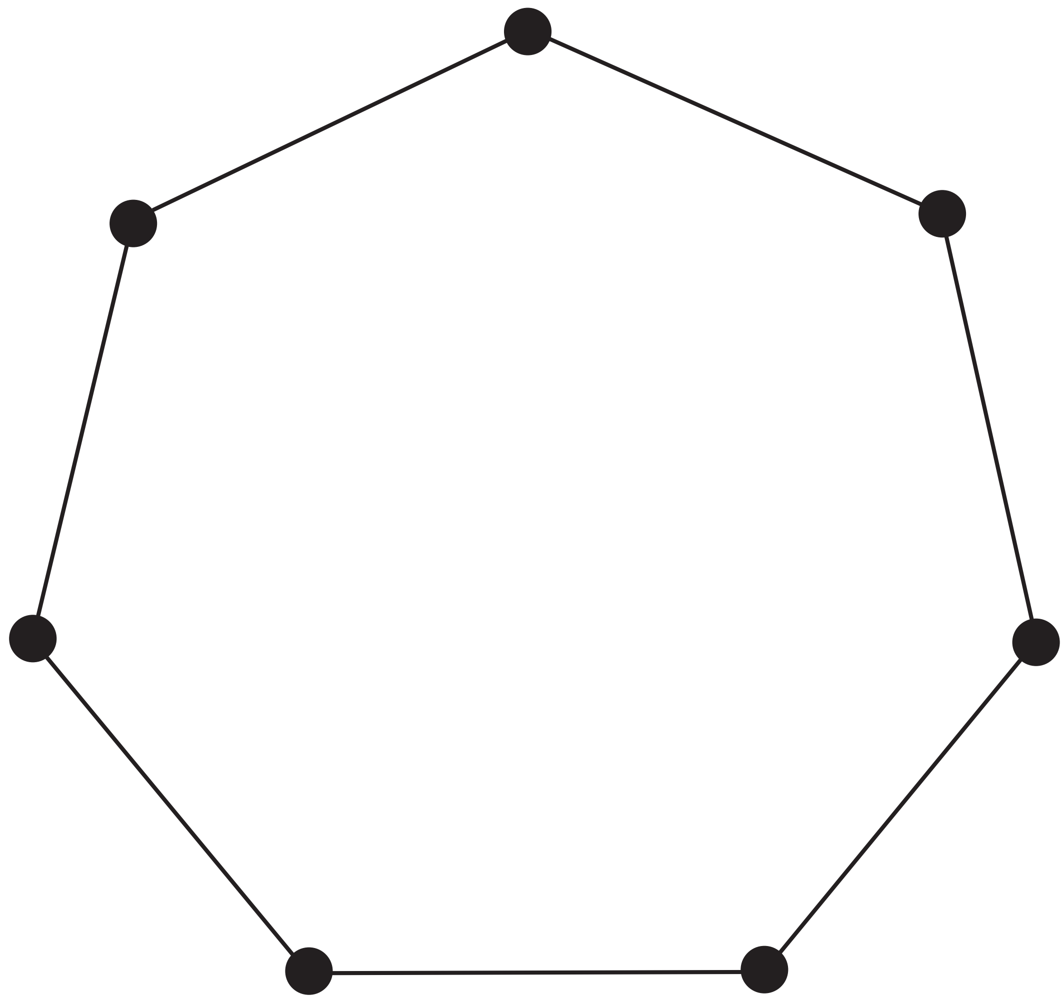
DIMACS/DIMATIA/Renyi Combinatorial Challenges Meeting,

DIMACS Center, April 2006

What is a graph coloring?



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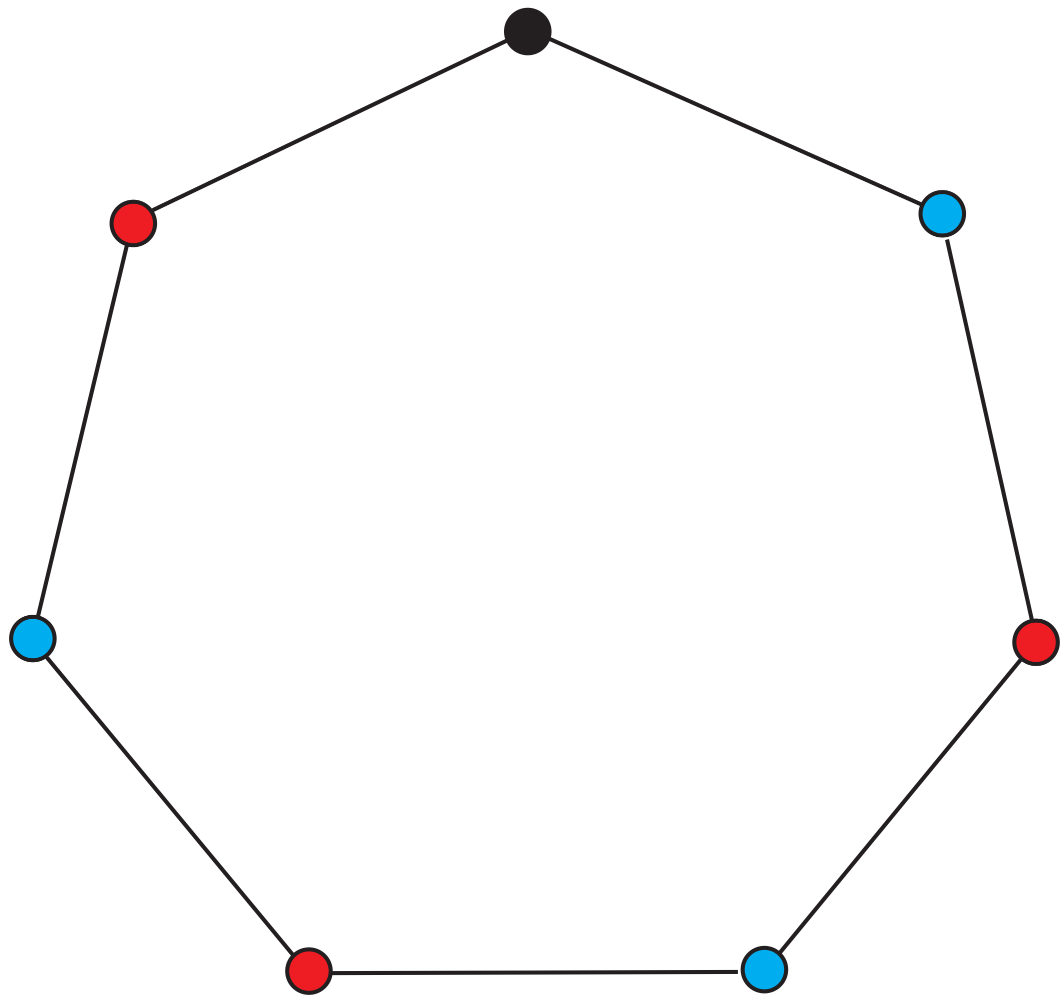

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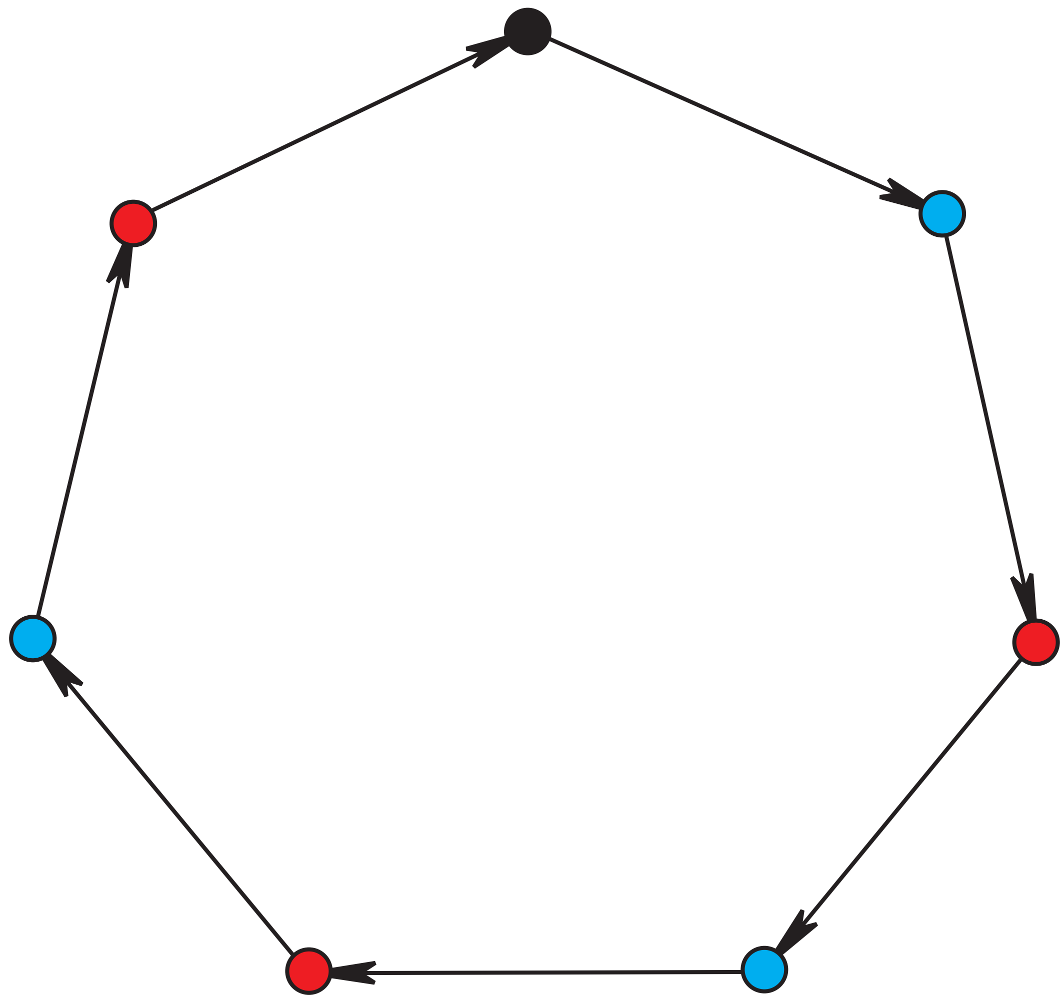

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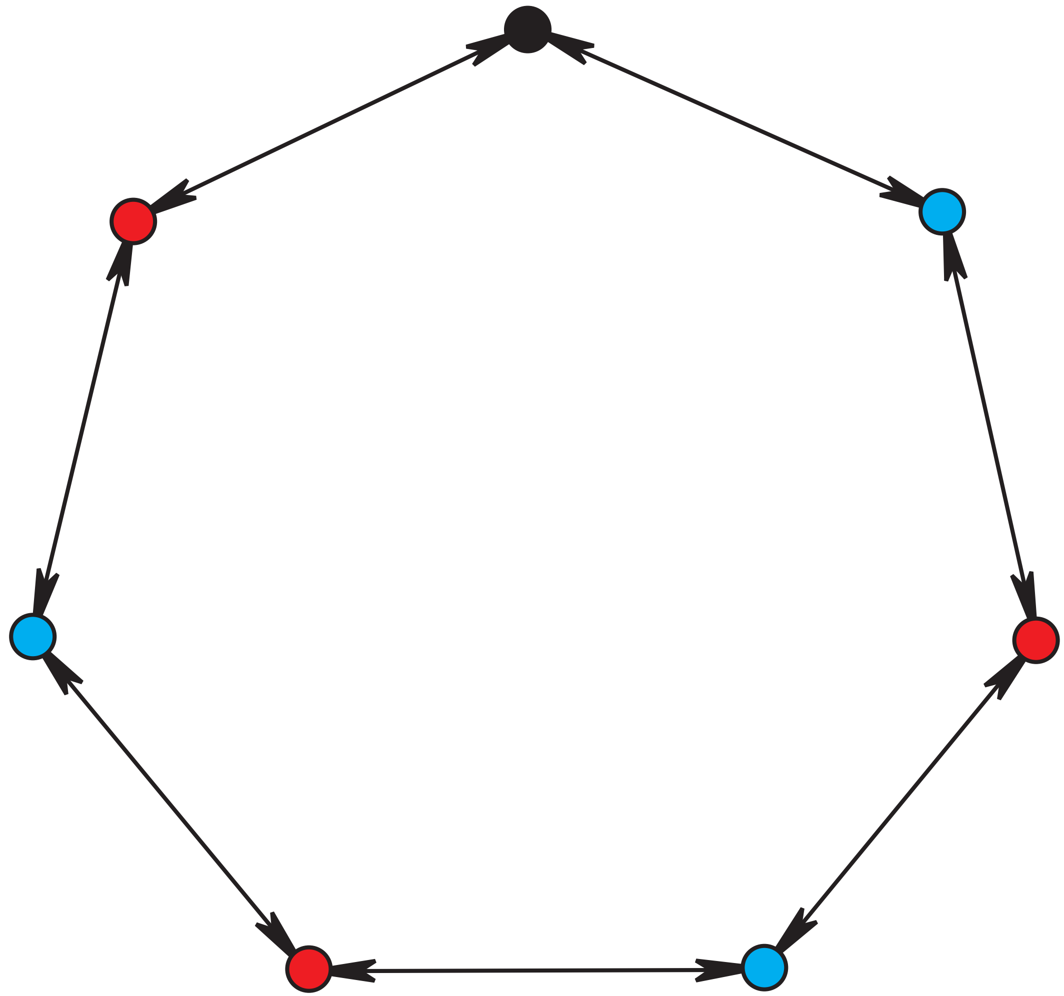

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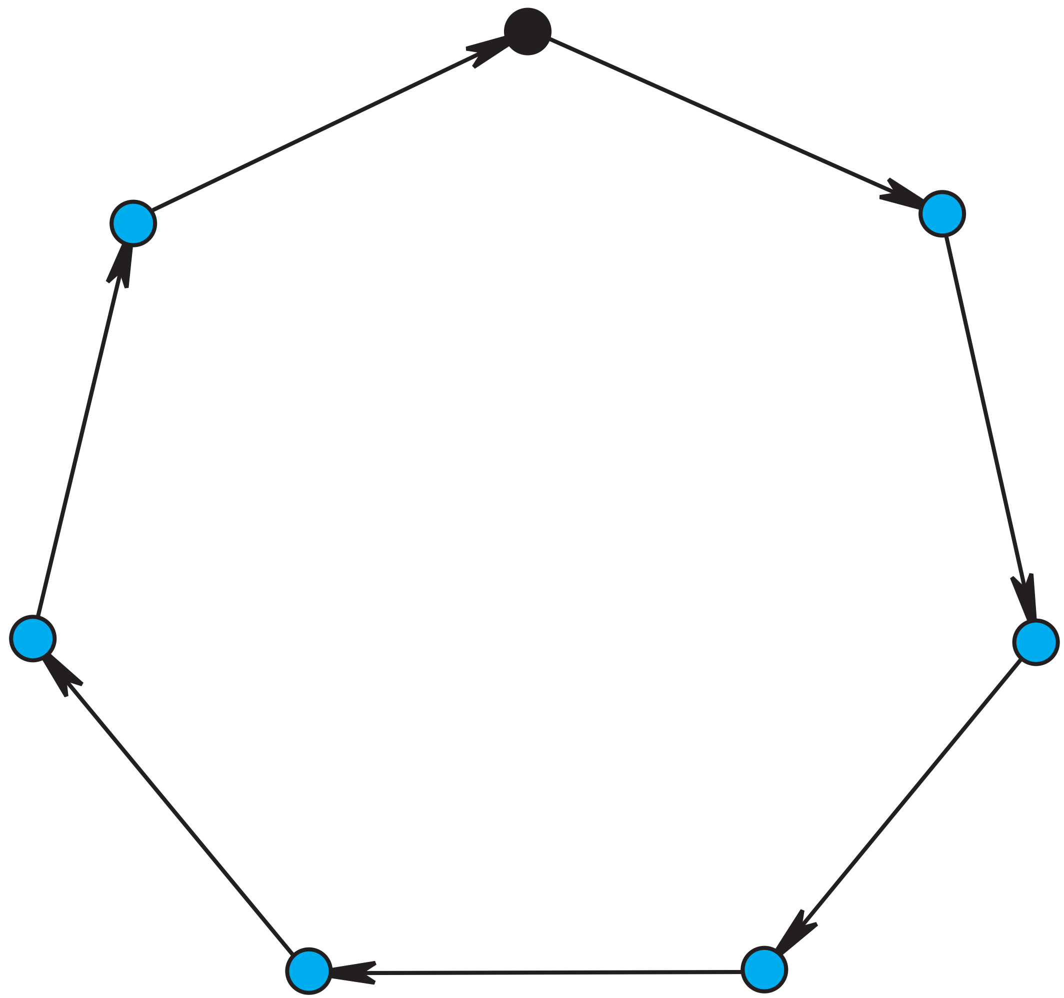

Focus

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- An example of computation.

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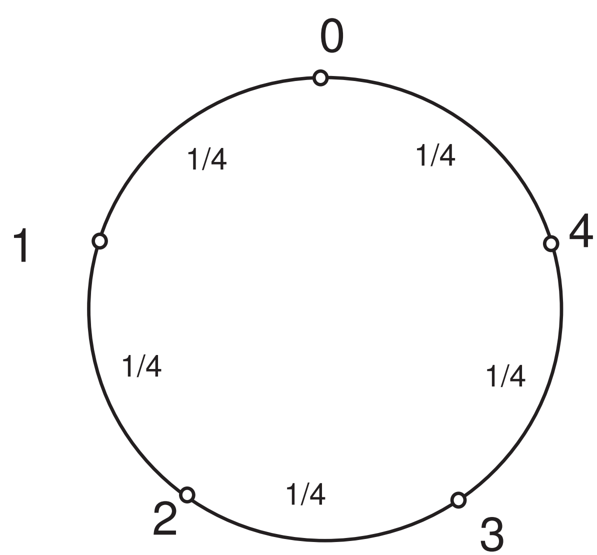

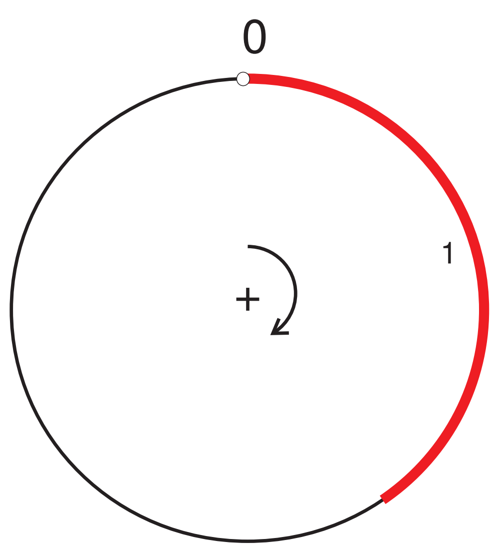

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  - $\chi_c(D) = \inf \{p \mid \text{exists circ. } p\text{-coloring of } D\}$.

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- Conjecture (Škrekovski): $\chi(D) \leq 2$ for planar D.

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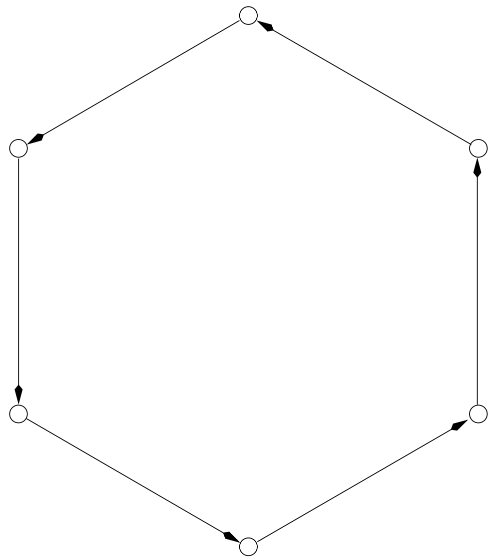

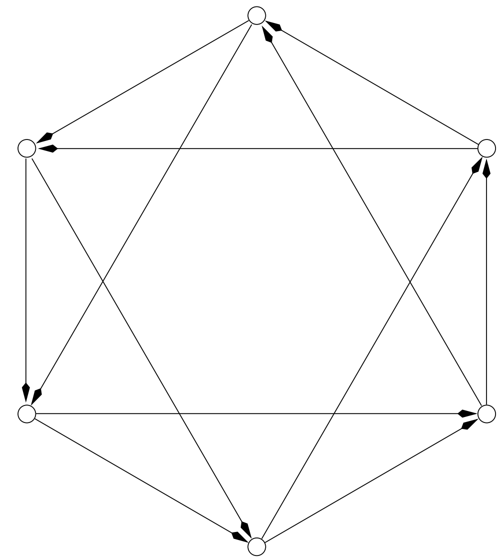

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- For $k,d \in \mathbb{Z}$, $k \ge d$:

$$\chi_c(\vec{G}(k,d)) = \frac{k}{d}.$$

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- D digraph.

  $\chi_c(D) \leq \frac{k}{d}$ if and only if there exists an acyclic homomorphism $f : D \to \vec{G}(k, d)$.

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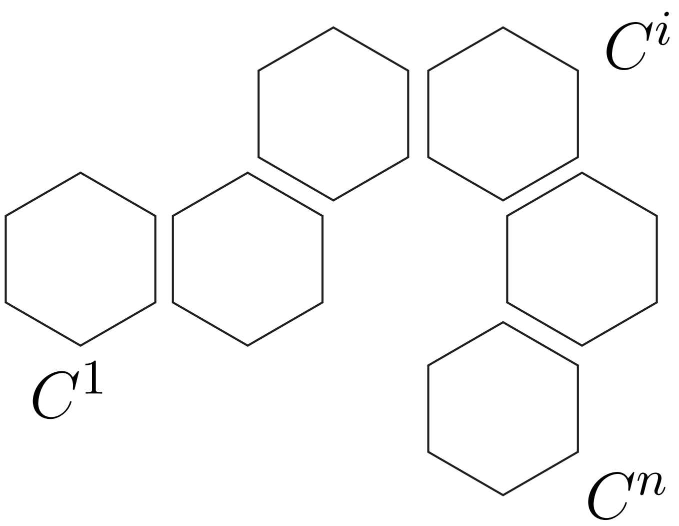

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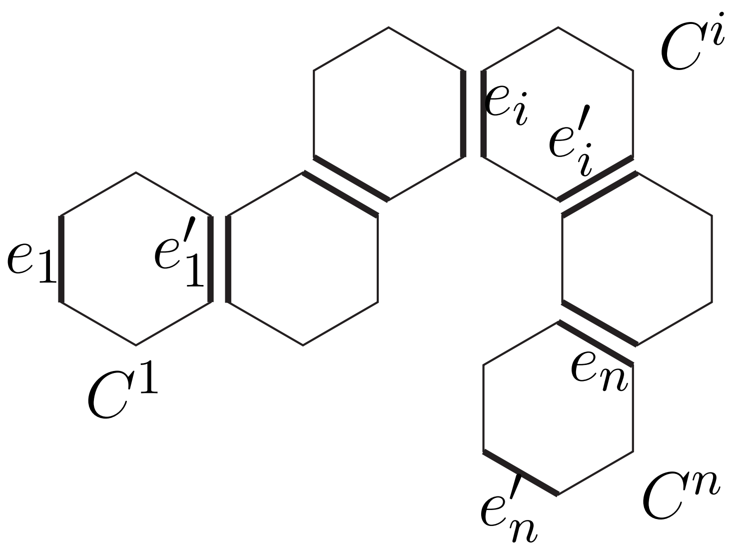

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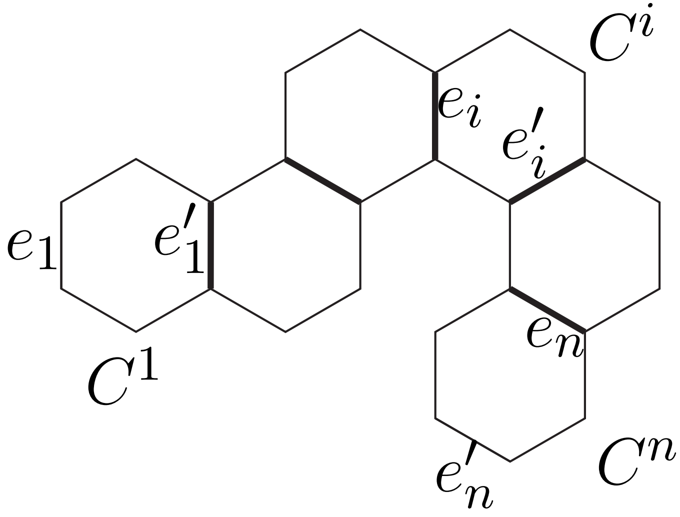

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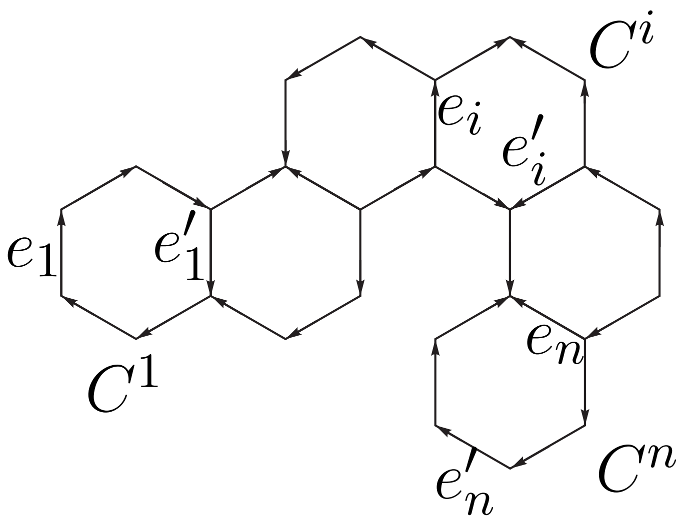
- Orientation of edges.



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Problem: Given an oriented hexagonal system S, find its circular chromatic number $\chi_c(S)$.

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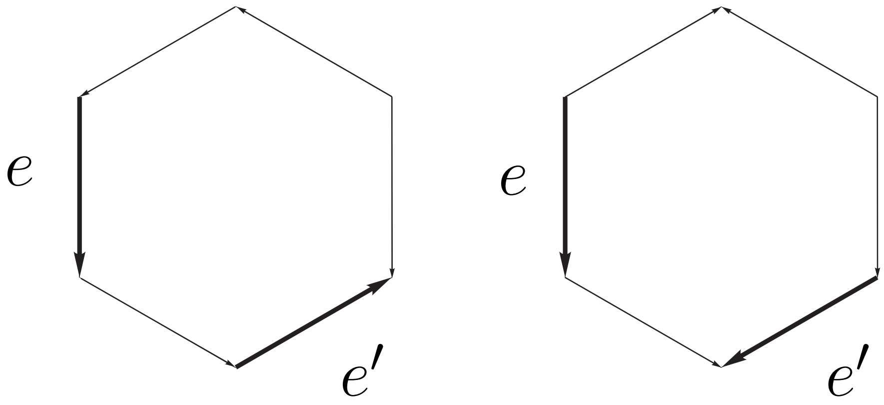
Theorem: Let $\lambda(S)$ be the characteristics of S and let $d \in \mathbb{N}$ be largest with $w \in \mathcal{O}^d$ being a subword of $\lambda(S)$. Then $\chi_c(S) = \frac{5d+1}{4d+1}$.

Properties of a single oriented C_6

- Are e and e' oriented coherently?



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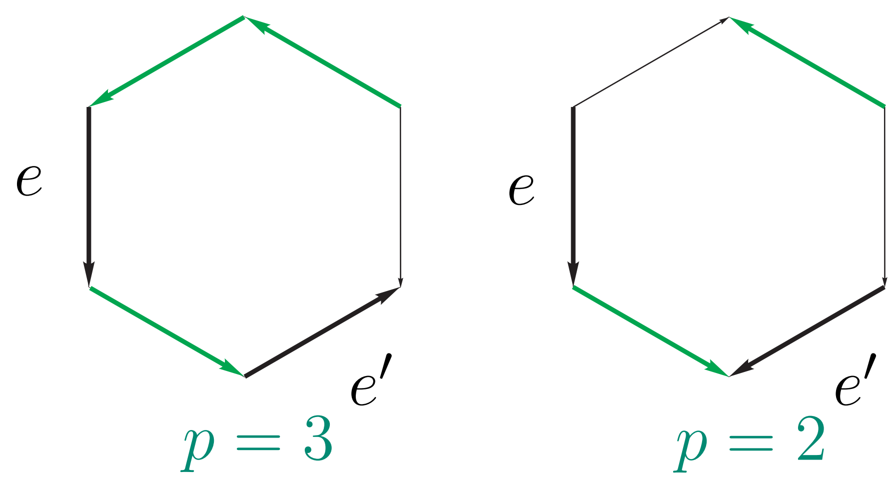

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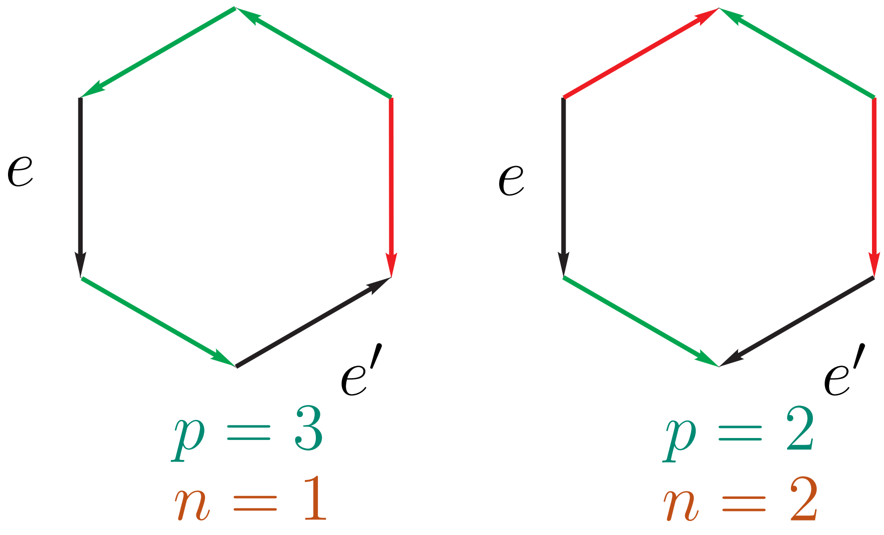

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- Does there exist $c : S \to \vec{G}(k, k-d)$, such that
  - $c(v) - c(u) = d_1$,
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- Yes: S admits the distance pair (d_1, d_2) for $\vec{G}(k, k-d)$.

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- Given: Orientation S of C_6, edges $e = uv, e' = u'v' \in E(C_6)$, integers d_1, d_2, k, d.
- e and e' are oriented coherently. S admits (d_1, d_2) if and only if $n = 0, p = 4, k \leq d_1 + d_2 + 4d$ or $n = 1, p = 3, d_1 + d_2 \leq d$ or $n \geq 2$.

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- e and e' are oriented incoherently.
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Outline of the approach

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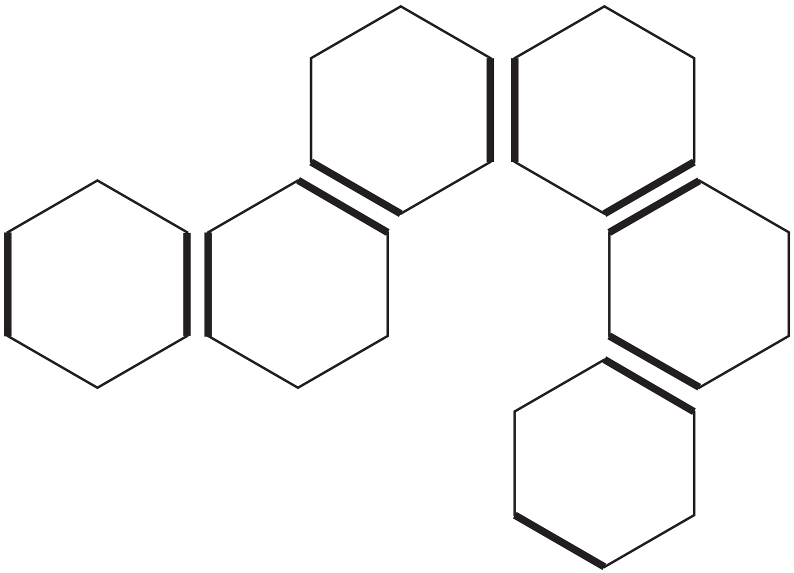

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- Propagating permissible distances through the chain.
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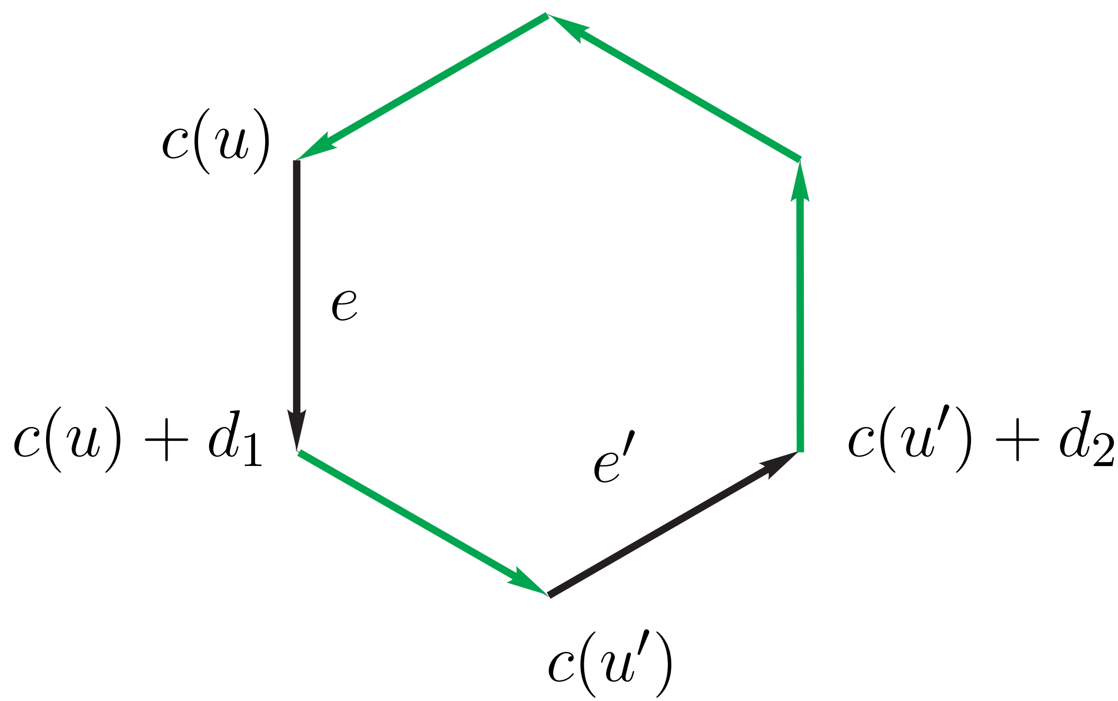
- attribute characteristic words to oriented hexagonal systems,
- establish lower bound on χ_c for certain subwords,
- find standard coloring with the minimum number of colors.

Classification of orientations I.

Directed cycle ($\diamond$): e, e' coherent, $p = 4$.



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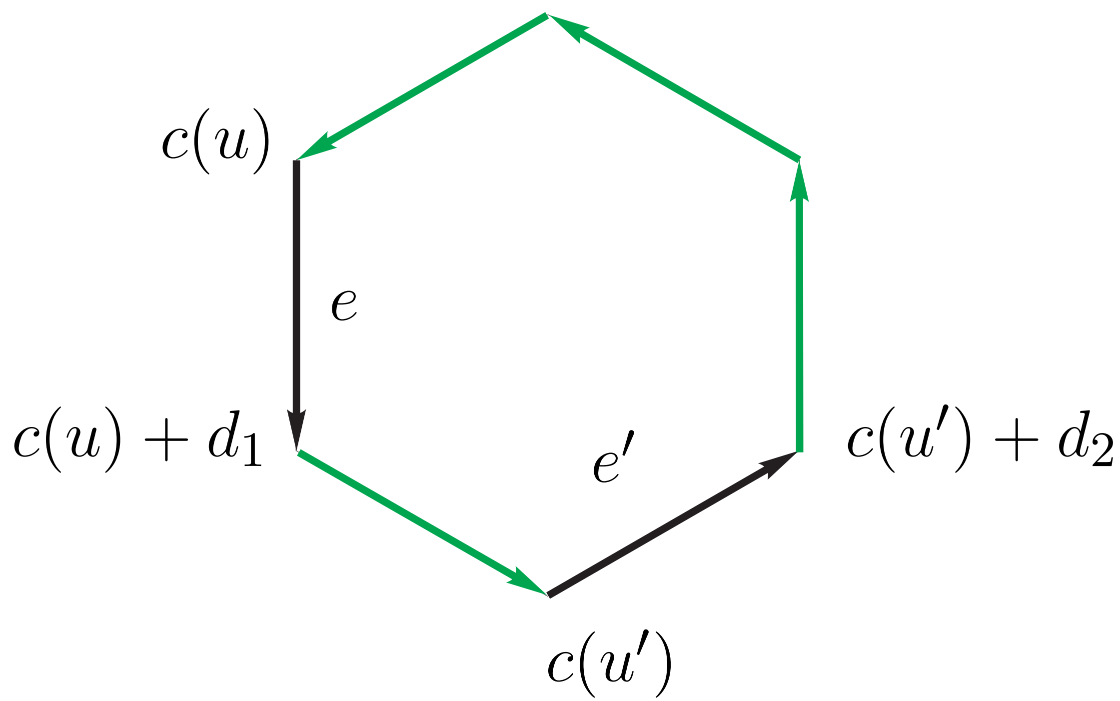
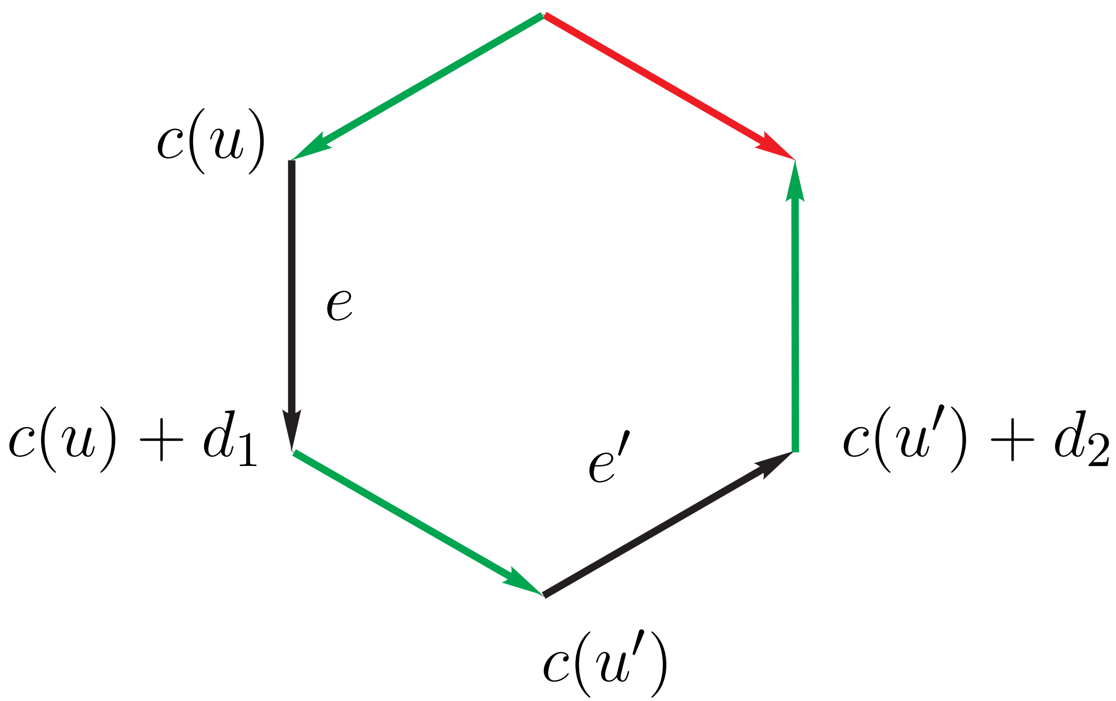


$k - 4d - d_1 \leq d_2 \leq d$. Small $\rightsquigarrow$ large.

Classification of orientations II.

Obstacle (∘): e, e' coherent, $p = 3$, $n = 1$.



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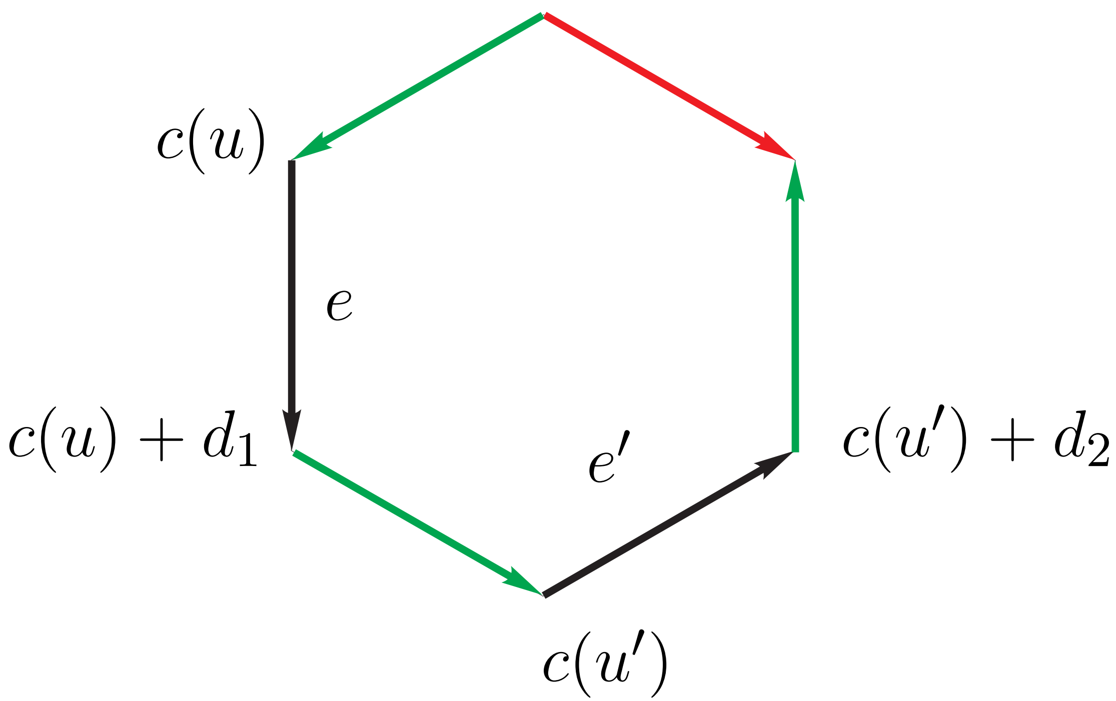
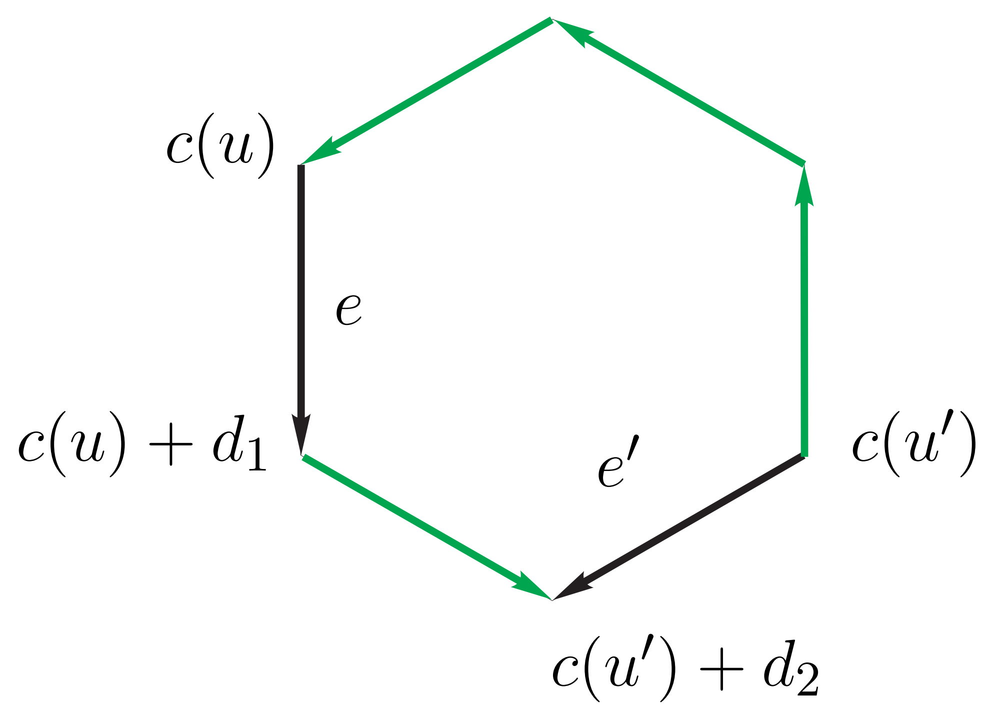


$0 \leq d_2 \leq d - d_1$. Large $\leadsto$ small.

Classification of orientations III.

Extension ($\triangleright$): e, e' incoherent, $p = 4$.



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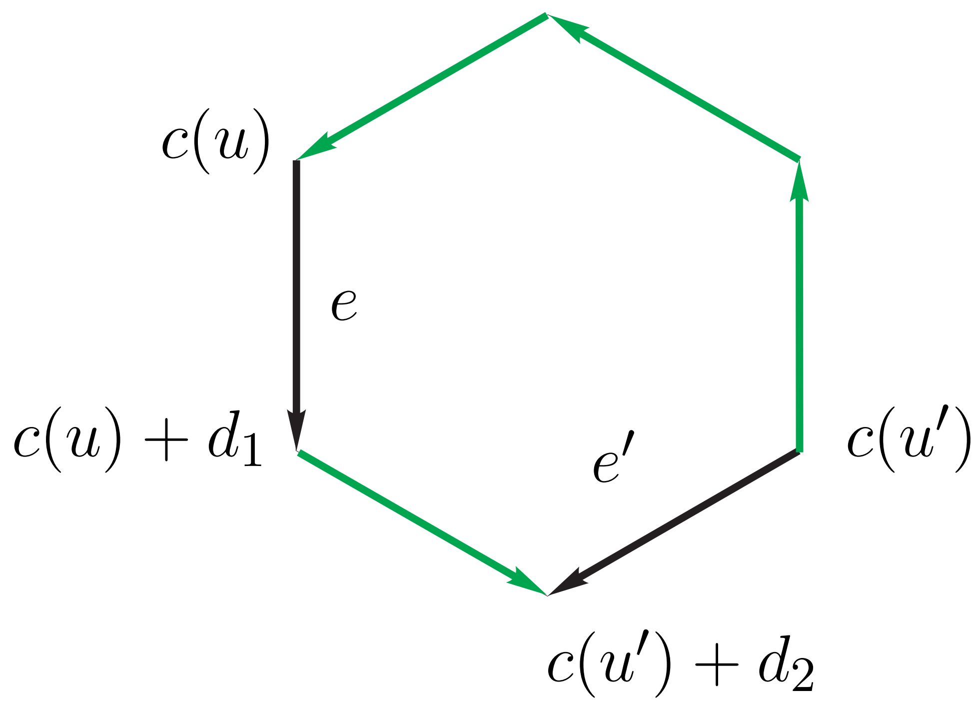
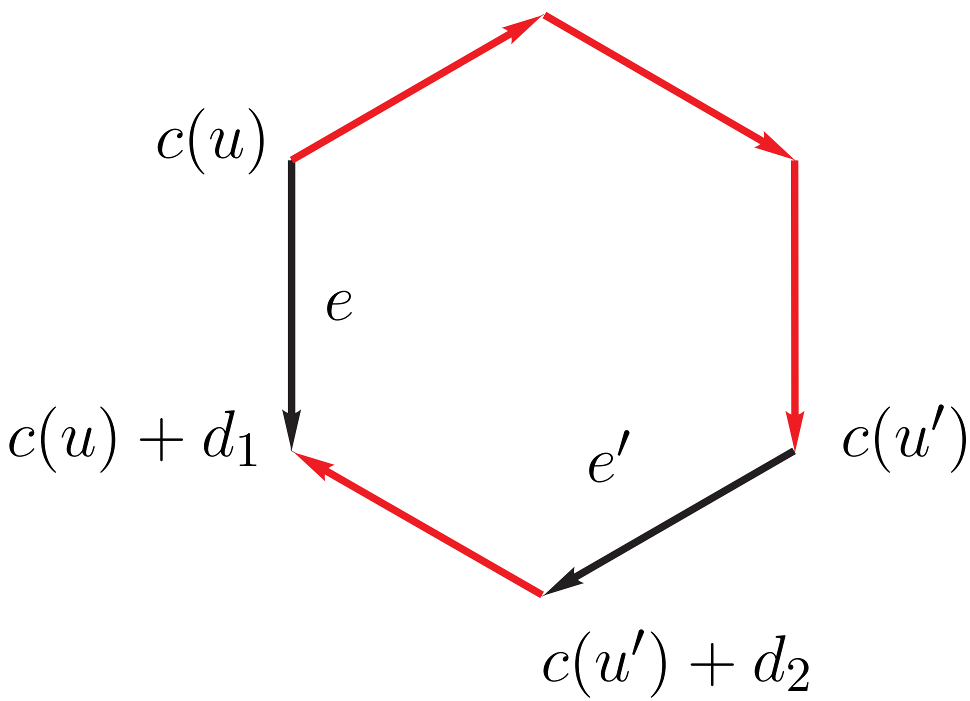


$d_1 \leq d_2 \leq d$. Large $\rightsquigarrow$ large.

Classification of orientations IV.

Antiextension ($\triangleleft$): e, e' incoherent, $n = 4$.



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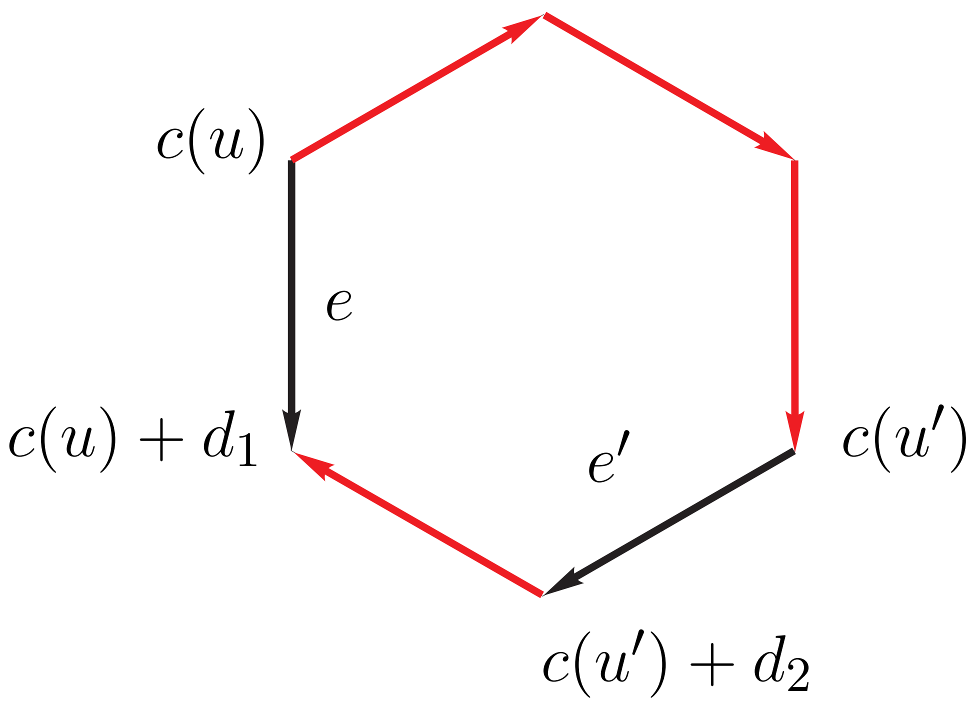

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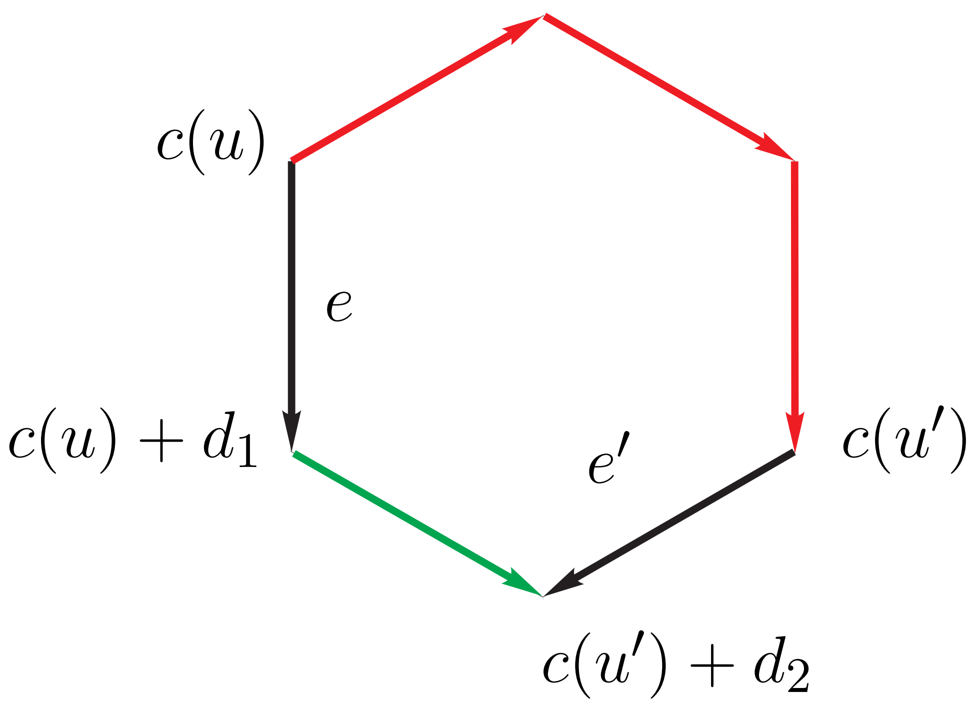
Other ($*$): at least two edges in every direction.



Classification of orientations V.

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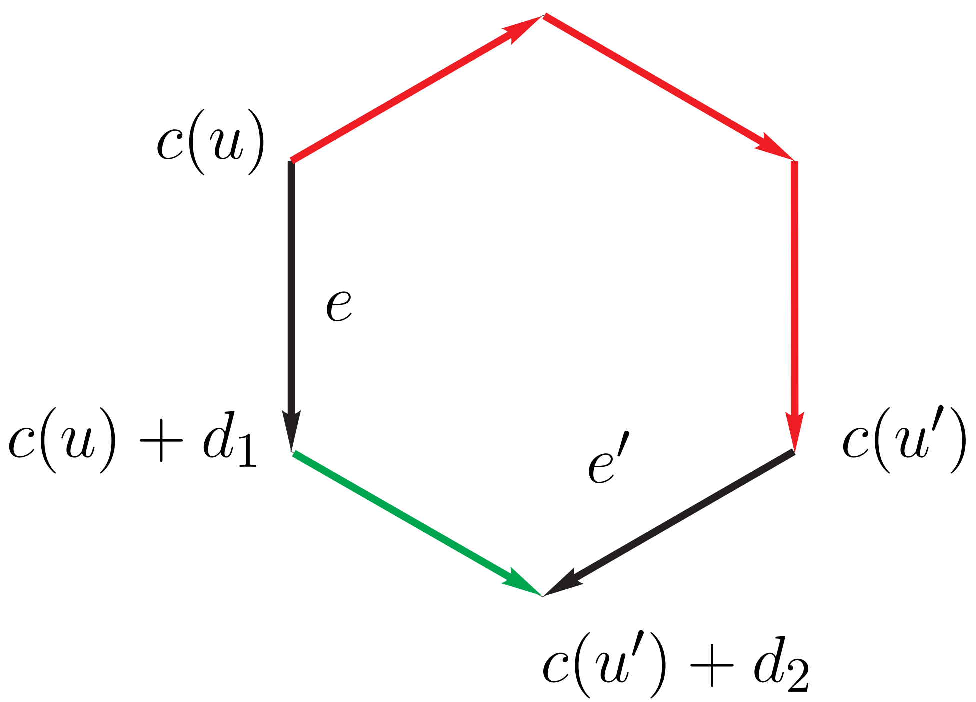
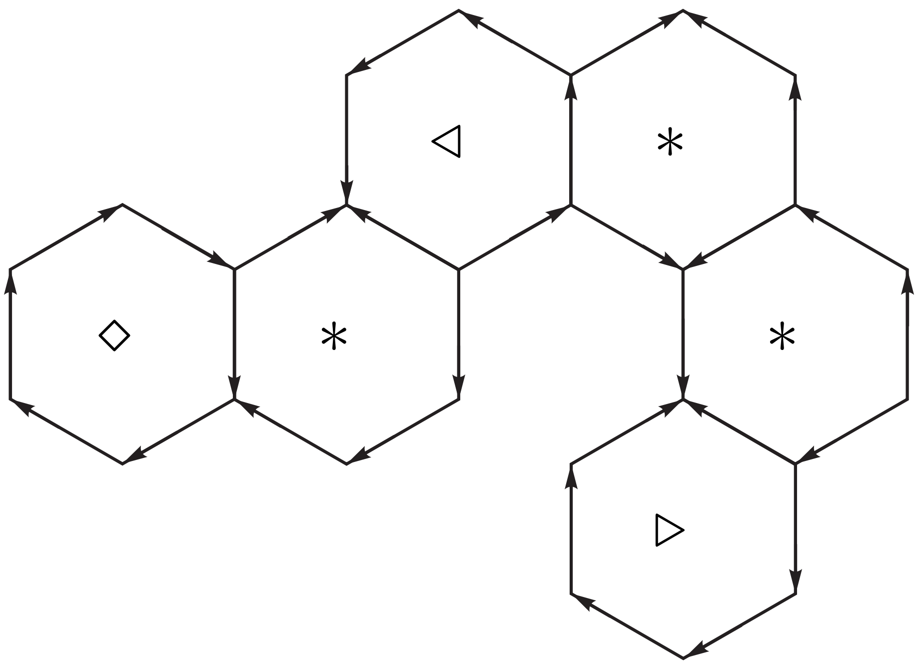


$0 \leq d_1, d_2 \leq d$. Freedom.

Hexagonal systems and words



Characteristics of orientation S:
$\lambda(S) = \diamond * \lhd * * \lhd$.

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  - Proof: Lemma, coloring algorithm.

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- S an oriented hexagonal system. Then: $\exists D \in \mathcal{H}_d \ni: S \to D \Leftrightarrow S \in \mathcal{S}_d \Leftrightarrow \nexists D \in \mathcal{F}_d \ni: D \to S$.