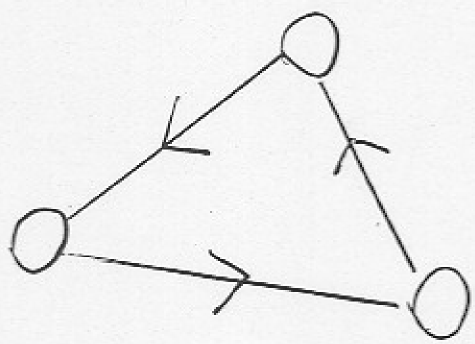
TOURNAMENTS

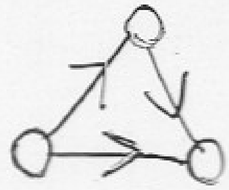


N. ALON

TEL AVIV + IAS

RANKING A TOURNAMENT

A TOURNAMENT IS AN ORIENTED COMPLETE GRAPH



$g(n)$ = MAX k SUCH THAT $\forall$ TOURNAMENT ON n VERTICES CONTAINS AN ACYCLIC SUBGRAPH WITH $\geq \frac{1}{2}\binom{n}{2} + \frac{1}{2}k$ EDGES.

ERDŐS + MOON (65): $\Omega(n) \leq g(n) \leq O(n^{3/2}(\log n)^{1/2})$

$$\Omega(n^{3/2}) \leq g(n) \leq O(n^{3/2})$$

SPENCER (70) SPENCER (80)

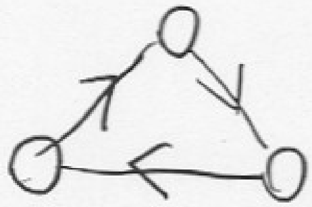
W. F. DE LA VEGA (83)

THUS, $\exists$ TOURNAMENTS THAT DO NOT ADMIT A "GOOD" RANKING.

PROBLEM: EXPLICIT CONSTRUCTIONS ?
(ERDÖS + MOON (65), SPENCER (85))

For the mathematician, even combinatorialist, not familiar with probabilistic methods a much weaker method can be quite impressive: There exist tournaments which cannot be ranked so that more than 51% of the games are in order. Construction of specific tournaments with this property appears to be quite difficult and quite possibly, in this author's opinion, impossible.

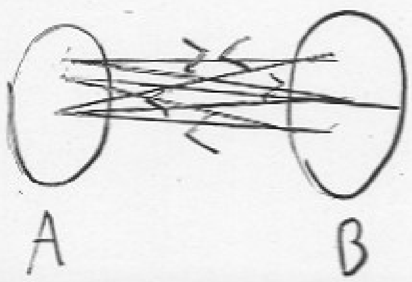
<u>DEF:</u> FOR A PRIME $p \equiv 3 \pmod 4$, THE QUADRATIC RESIDUE TOURNAMENT T_p HAS $V(T_p) = \mathbb{Z}_p$ WITH $i \to j \iff$ $i - j = \square$ (= IS A QUADRATIC RESIDUE)



<u>LEMMA:</u> IF $T_p = (V, E)$, $A, B \subseteq V$, $A \cap B = \emptyset$ THEN

$$\left| e(A,B) - e(B,A) \right| \leq |A|^{1/2} |B|^{1/2} p^{1/2}$$

$$e(A,B) = \left| \{ i \to j \in E : i \in A, j \in B \} \right|$$



PF (SKETCH): IF $C = (C_{u,v})_{u,v \in V}$

$$C_{u,v} = \begin{cases} 1 & \text{IF } u \to v \\ -1 & \text{IF } v \to u \\ 0 & \text{IF } v = u \end{cases}$$

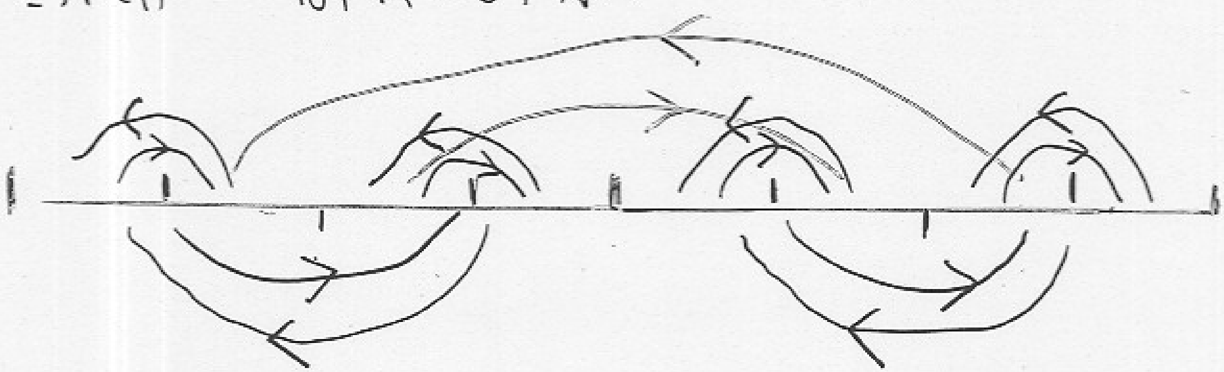
THEN $C^T C = p I - J$.

HENCE, THE EIGENVALUES OF $C^T C$ ARE 0 AND p AND THUS:

$$|e(A,B) - e(B,A)| = |x_A^t C x_B|$$
$$\leq \|x_A\|_2 \|C x_B\|_2 = |A|^{1/2} (x_B^t C^T C x_B)^{1/2}$$
$$\leq |A|^{1/2} p^{1/2} |B|^{1/2}.$$

$\square$

IF $T_p = (V, E)$ IS THE QUADRATIC RESIDUE TOURNAMENT, AND $v_1, v_2, \ldots, v_p$ IS A PERMUTATION OF ITS VERTICES, THEN ROUGHLY HALF THE EDGES ARE DIRECTED IN EACH DIRECTION



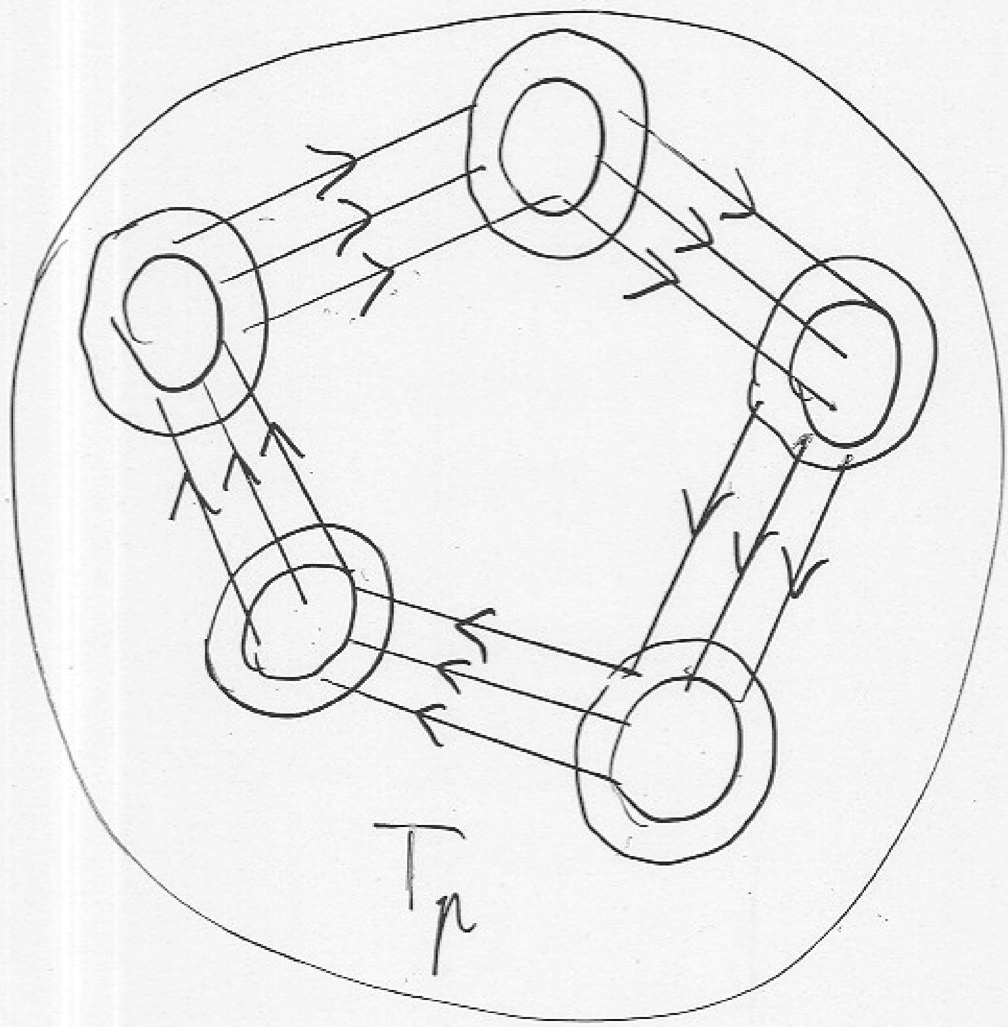
$\Rightarrow$ T_p CONTAINS NO ACYCLIC SUBGRAPH WITH MORE THAN $\frac{1}{2}\binom{p}{2} + cp^{3/2}\log p$ EDGES.

A RECENT APPLICATION [A(05), CHARBIT, THOMASSE, YEO (06)]:

THE PROBLEM OF COMPUTING THE MAX. NO. OF EDGES IN AN ACYCLIC SUBGRAPH OF A GIVEN INPUT TOURNAMENT IS NP-HARD.

- SETTLES A CONJECTURE OF BANG-JENSEN + THOMASSEN (92)
- NP HARDNESS UNDER RANDOMIZED REDUCTION PROVED BY AILON, CHARIKAR, NEWMAN (05).
- EVEN APPROXIMATING THE MAXIMUM TO WITHIN AN ADDITIVE ERROR OF $n^{2-\varepsilon}$ IS NP-HARD.

REDUCTION FROM MAXIMUM ACYCLIC SUBGRAPH IN A GENERAL DIGRAPH F:
A "BLOW UP" OF F WITH "T_p ON TOP" GIVES RESULT.



TOURNAMENTS, BOXES AND VOTING PARADOXES

N. ALON, G. BRIGHTWELL, H. KIERSTEAD, A. KOSTOCHKA & P. WINKLER

FELLOWSHIPS

EACH MEMBER OF A COMMITTEE OF $2k-1$ HAS A LINEAR ORDER ON CANDIDATES

X IS BETTER THAN Y IF $\geq k$ MEMBERS RANK X HIGHER.

UNDESIRABLE: $\forall$ CHOICE OF WINNERS $\exists$ A NON-WINNER WHICH IS BETTER THAN ALL WINNERS.

MAJORITY TOURNAMENTS

FOR $2k-1$ LINEAR ORDERS ON $[n]=\{1,2,\ldots,n\}$ THE k-MAJORITY TOURNAMENT IS THE TOURNAMENT ON $[n]$ IN WHICH

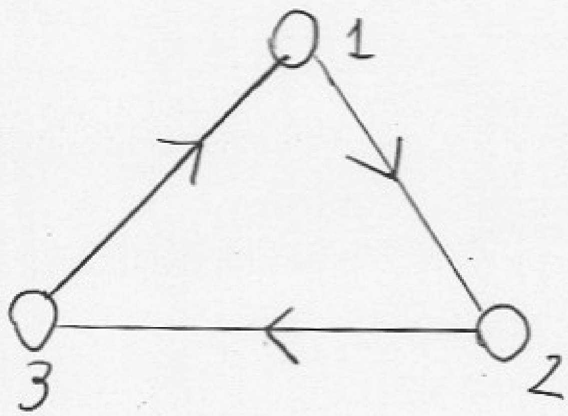
$i \to j \iff i$ IS ABOVE j IN $\geq k$ ORDERS.

EXAMPLE:

L_1; $1 \to 2 \to 3$

L_2; $2 \to 3 \to 1$

L_3; $3 \to 1 \to 2$



WHICH TOURNAMENTS ARE MAJORITY TOURNAMENTS

McGARVEY (53): EVERY TOURNAMENT IS A k-MAJORITY TOURNAMENT FOR SOME k, $k \leq n^2$. [n = NO. OF VERTICES].

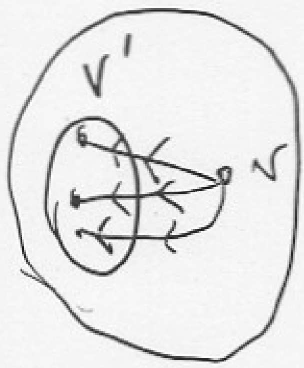
STEARNS (59): $k \leq O(n)$ SUFFICES

ERDŐS & MOSER (64): $O(n/\log n)$ ORDERS SUFFICE (AND THIS IS TIGHT).

SCHÜTTE, ERDÖS (63):

$\forall t$ $\exists$ A TOURNAMENT $T=(V,E)$ WITH NO DOMINATING SET OF SIZE $\leq t$.

THAT IS, $\forall V' \subseteq V$, $|V'|=t$ $\exists v \in V$ THAT BEATS EACH $v \in V'$.



HENCE, LARGE COMMITTEES MAY HAVE PROBLEMS EVEN WITH MANY FELLOWSHIPS.

WHAT ABOUT SMALL COMMITTEES ?

CONJECTURE (KIERSTEAD & TROTTER):

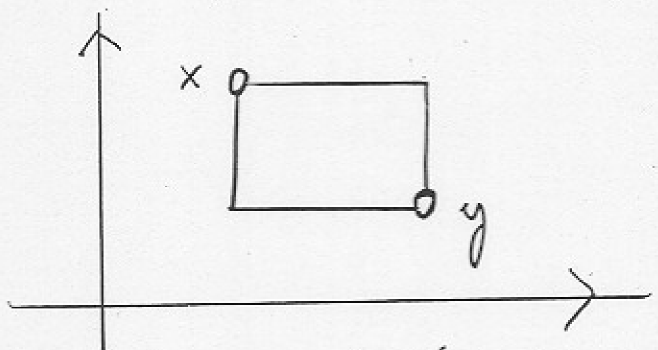
$\forall k\ \exists F(k)\ (<\infty)$ SUCH THAT EVERY k-MAJORITY TOURNAMENT HAS A DOMINATING SET OF SIZE $\leq F(k)$.

[THAT IS; $F(k)$ FELLOWSHIPS SUFFICE FOR A COMMITTEE OF SIZE $2k-1$].

DEF: $F(k)$ = SMALLEST F SUCH THAT EVERY k-MAJORITY TOURNAMENT HAS A DOMINATING SET OF SIZE $\leq F$.

A GEOMETRIC RESULT: FOR $x, y \in \mathbb{R}^d$

$\mathrm{Box}(x,y)$ = SMALLEST BOX WITH FACES PARALLEL TO THE COORDINATES HYPERPLANES, THAT CONTAINS x AND y.
[$\mathrm{Box}(x,y) = \{ z \in \mathbb{R}^d : z_i$ IS BETWEEN x_i AND y_i $\forall i \}$.]

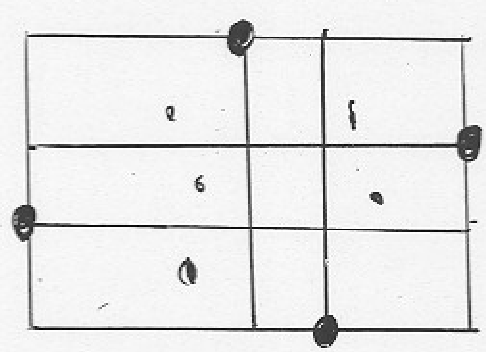


THM (BÁRÁNY + LEHEL 87): $\forall d$ $\exists c = c(d)$ $\forall$ FINITE $V \subseteq \mathbb{R}^d$
$\exists V' \subseteq V$, $|V'| \leq c(d)$ SUCH THAT

$$V \subseteq \bigcup_{x,y \in V'} \mathrm{Box}(x,y).$$

MOREOVER, $c(d) \leq (2d^2 + 1)^{d \cdot 2^d}$

E.G., $c(2) = 4$



<u>PROP:</u> $F(k) \leq C(2k-1)$

<u>PF:</u> GIVEN $2k-1$ LINEAR ORDERS $L_1, \dots, L_{2k-1}$ ON $[n]$, DEFINE $V = \{p^{(1)}, p^{(2)}, \dots, p^{(n)}\} \subseteq R^{2k-1}$:

$p^{(i)}_j$ = RANK OF i IN L_j

TAKE $V' \subseteq V$, $|V'| \leq C(2k-1)$ WITH

$$V \subseteq \bigcup_{x,y \in V'} \mathrm{Box}(x,y).$$

$\{i \mid p^{(i)} \in V'\}$ DOMINATES V:

INDEED, IF $p^{(r)} \in \mathrm{Box}(p^{(i)}, p^{(j)})$, THEN EITHER i OR j IS ABOVE r IN $\geq k$ ORDERS.

COROLLARY:

FOR A COMMITTEE OF 3

$F(2) \leq C(3) \leq \left(2 \cdot 3^{2^3} + 1\right)^{3 \cdot 2^3} = 13{,}123^{24}$

FELLOWSHIPS SUFFICE!

CAN WE DO BETTER?

CAN ACTUALLY SHOW:

- $\Omega(k/\log k) \leq F(k) \leq O(k \log k)$
- $F(2) = 3$

[I.E., FOR A COMMITTEE OF 3, 3 FELLOWSHIPS SUFFICE].

OPEN: $F(k) = ?$

$F(1) = 1$, $F(2) = 3$, $4 \leq F(3) < 360$

[HOW MANY FELLOWSHIPS DOES A COMMITTEE OF 5 NEED?]

$$\Omega(k/\log k) \leq F(k) \leq O(k \log k)$$

MORAL: BIGGER COMMITTEES NEED BIGGER BUDGET!