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A New Mathematical Approach for the Design of Digital Communication Systems[a]

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Content

I - Introduction

Purpose:

To show that the **topological structure** associated with each **metric space** (block diagram) should be considered in the design of a communication system.

What should be the approach?

The identification of the surface topology of each block diagram, starting with the graph associated with the DMC channel.

Consequences:

New mathematical concepts and approaches may be incorporated to the already known ones to achieve the goals of better performance and less complexity.

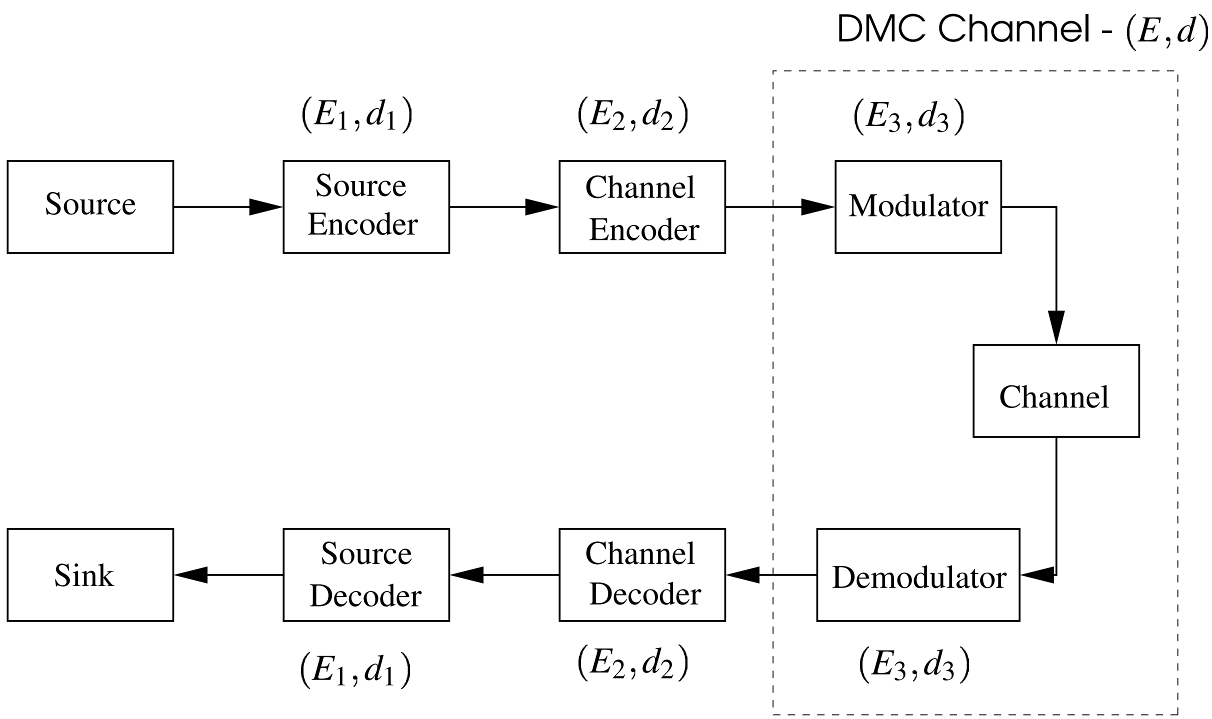


Figure 1: Communication system model

- **Current** design is based on **metric spaces** (vector space structure).
- **Proposal**: design should also consider the **topological structure** associated with each metric space.

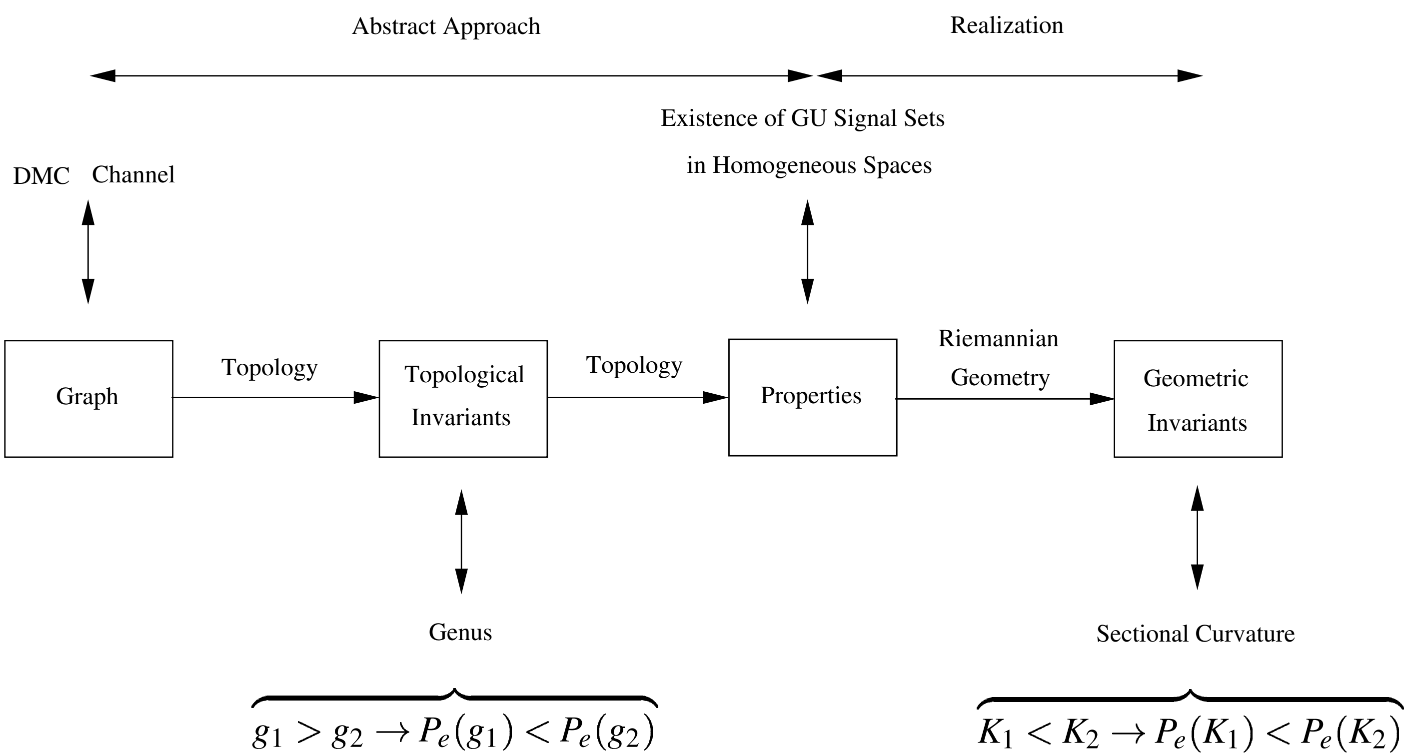


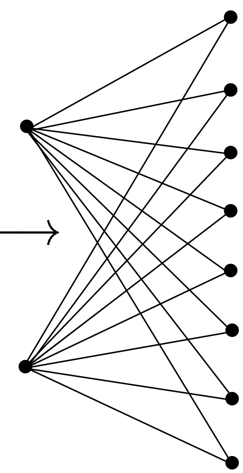
Figure 2: Proposal overview

Results:

- **To extend the concept** of geometrically uniform codes (Euclidean space) to **other spaces with constant curvature** (homogeneous spaces, in particular, to the hyperbolic space);
- To consider **the performance analysis** of a digital communication system **in n-dimensional manifolds**;
- **To show the best performance**, among the spaces with constant curvature, **is achieved when the curvature is negative**.

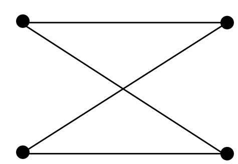
Motivation for the Proposal: First Case

$C_{2,8}[8,2] \equiv K_{2,8} \quad \longleftrightarrow$



$\hookrightarrow \quad \{\overbrace{S(8),T(6)},\underbrace{2T(4),3T(2)}\}$

$C_2[2] \equiv K_{2,2} \quad \longleftrightarrow$



$\hookrightarrow \quad \{\overbrace{S(2)}\}$

$K_{m,n}$ Embedding Compact surface

Figure 3: Embeddings

Motivation for the Proposal: Second Case

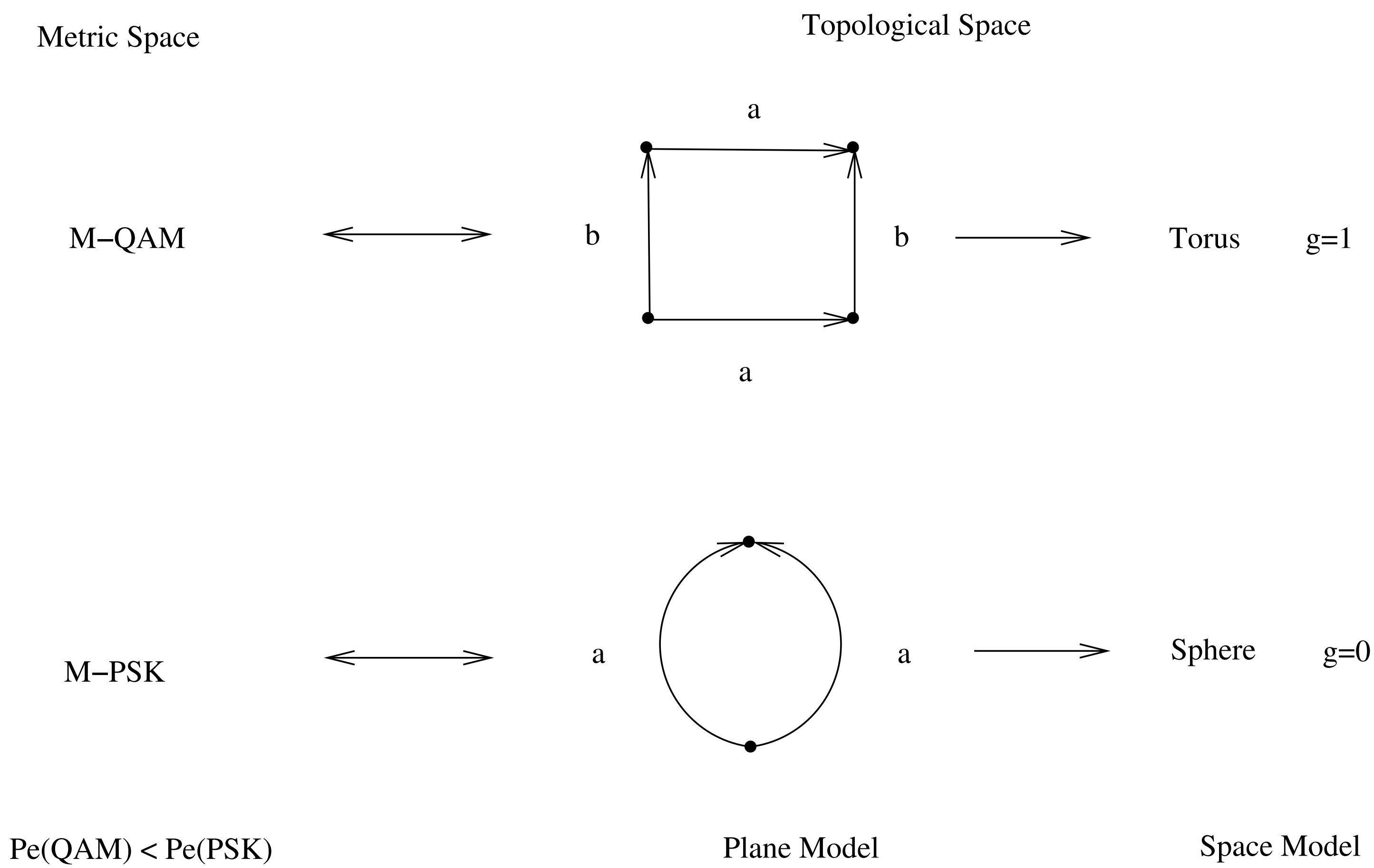


Figure 4: Metric and Topological spaces

Conform Homeomorphic

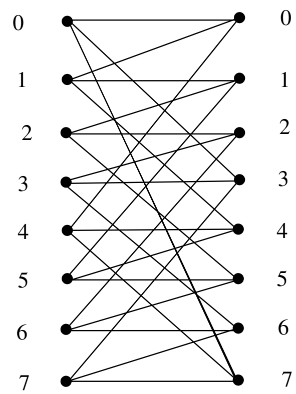
Transformations

$$\downarrow$$

Riemann Surfaces $\Longleftrightarrow$ **2-D manifolds**

(conformal geometry)		(Riemannian geometry)
$\Updownarrow$		$\Updownarrow$
Algebraic Combinatorial Topology		Differential Geometry
Euler charact, $\chi(\Omega) = 2 - 2g$	$\longleftrightarrow$	Sectional curvature, K
$g = 0 \leftrightarrow \chi(\Omega) > 0$	$\longleftrightarrow$	$K > 0$, Elliptic
$g = 1 \leftrightarrow \chi(\Omega) = 0$	$\longleftrightarrow$	$K = 0$, Euclidean
$g \geq 2 \leftrightarrow \chi(\Omega) < 0$	$\longleftrightarrow$	$K < 0$, Hyperbolic
Compact, Oriented Surfaces		Homogeneous Spaces

$C_8[3] \longleftrightarrow$



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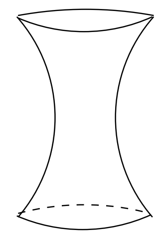
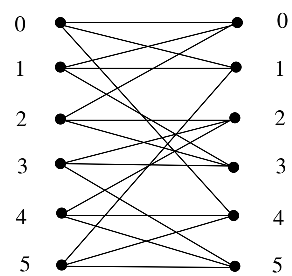


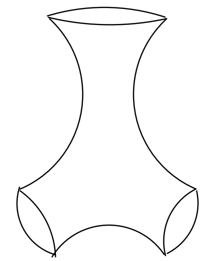
Catenoid

$\equiv$ sphere/2 endings

$C_6[3] \longleftrightarrow$



$\hookrightarrow$



Trinoid

$\equiv$ sphere/3 endings

Figure 5: Embeddings in Minimal Surfaces

Classification Theorem for Surfaces (with boundary):

Every compact, connected surface is topologically equivalent to a sphere, or a connected sum of tori, or a connected sum of projective planes (with some finite number of discs removed).

Example 1 $\Omega = 2T \to g = 2, \quad \chi(\Omega) = -2$

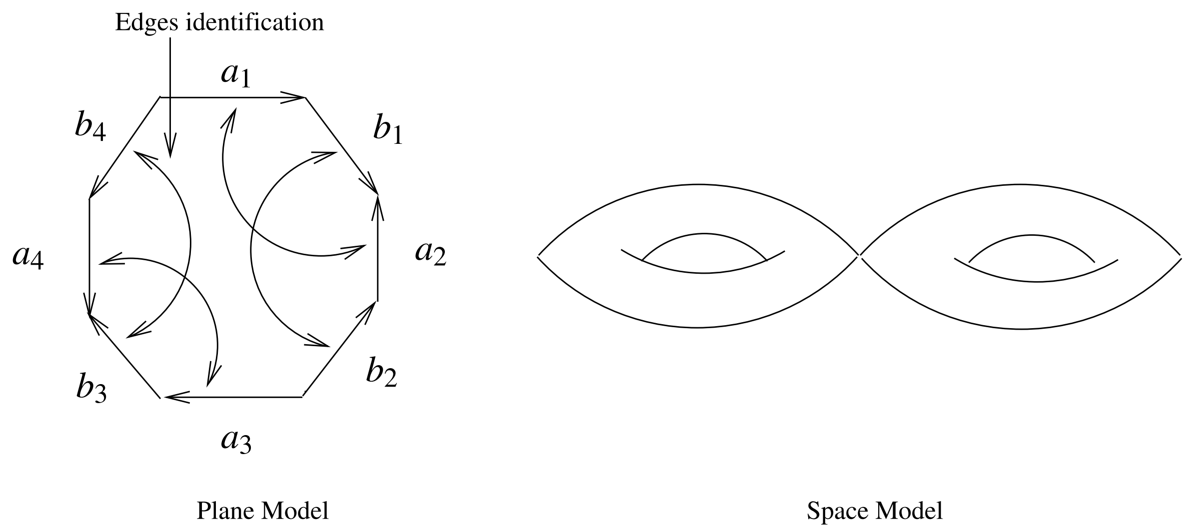


Figure 6: 2 Torus

The majority of DMC channels of practical interest are embedded in compact surfaces with genus $g = 1, 2, 3$.

This motivates us to **construct signal sets matched to groups**, equivalently, **GU signal sets**;

- The design of GU signal sets is strongly dependent on the existence of **regular tessellations** in homogeneous spaces.
- Homogeneous spaces are important for the **rich algebraic and geometric properties**, so far not fully explored in the context of communication and coding theory.
- The **algebraic structures** provide the means for systematic devices implementations whereas the **geometric properties** are relevant with respect to the efficiency of demodulation and decoding processes.

II - Embedding of Graphs in Surfaces

(i) **Minimum genus of an oriented surface** is, **(Ringel)**:

$$g_m(K_{m,n}) = \{(m-2)(n-2)/4\}, \text{ for } m,n \geq 2,$$

where $\{a\}$ denotes the least integer greater than or equal to the real number a.

(ii) **Maximum genus of an oriented surface** is, **(Ringeisen)**:

$$g_M(K_{m,n}) = [(m-1)(n-1)/2], \text{ for } m,n \geq 1,$$

where $[a]$ denotes the greatest integer less than or equal to the real number a.

(iii) **Minimum genus of a non-oriented surface** is, **(Ringel)**:

$$\tilde{g}(K_{m,n}) = [(m-2)(n-2)/2].$$

Theorem 1 (Ringeisen) *If a graph G has a 2-cell embedding in surfaces of genus g_m and g_M, then for every integer g, $g_m \leq g \leq g_M$, G has a 2-cell embedding in a surface of genus g.*

Assumption: All embeddings are 2-cell embeddings of $K_{m,n}$ preserving the Euler characteristic of Ω.

Needed Elements:

- A **model** $\mathfrak{F}_{mn} \equiv \Omega\,(\alpha) = \cup_{i=1}^{\alpha} R_i$ spanned by the minimum embedding of the graph $K_{m,n}$ in an oriented compact surface Ω.
- The **cardinality of the set of models** $\mathfrak{F}_{mn}$,

$$\mathfrak{M}_\alpha = \{\mathfrak{F}_{mn} : K_{m,n} \hookrightarrow \Omega(\alpha) \quad \text{and} \quad \mathfrak{F}_{mn} \equiv \Omega(\alpha)\}.$$

  The **number of regions** associated with $\mathfrak{F}_{mn}$ is constant and depends on $\chi(\Omega)$.

¿From this, and Theorem 1, we have

Proposition 1 *If $\Omega \equiv gT$ (a g-torus), then the number of regions of $\mathfrak{F}_{mn}$ is*

$$|\mathfrak{F}_{mn}| = 2 - 2g - m - n + mn, \; \forall \mathfrak{F}_{mn} \in \mathfrak{M}_\alpha.$$

Proposition 2 *Let $\mathfrak{F}_{mn} \equiv \Omega(\alpha)$ and $\mathfrak{F}_{mn} = \bigcup_{j=1}^{\alpha} R_{i_j}$, then i_j is always an even integer greater than or equal to 4.*

Lemma 1 *The cardinality of the set $\mathfrak{M}_\alpha$ is equal to the number of positive integer solutions of the following equations*

$$2mn = 4R_4 + 6R_6 + 8R_8 + \cdots \quad \text{and} \qquad \textstyle\sum_{i\geq 0} R_{4+2i} = \alpha,$$

where R_k denotes the number of regions with k edges.

Definition 1 *A **channel class** $C_{m,n}$ is the set consisting of all channels with m vertices in X and n vertices in Y.*

(i) Channel class $C_{m,n}[P,Q] = C_{m,n}[\{p_1,\cdots,p_m\},\{q_1,\cdots,q_n\}]$;

(ii) Channel class $C_{m,n}[p,q]$. **A type of soft-decision channel**;

(iii) Channel class $C_m[p]$. **A type of hard-decision channel**.

Let $\langle\Omega(\alpha)\rangle$ denote the set of surfaces corresponding to the embeddings of $C_{m,n}[P,Q] \hookrightarrow \Omega(\alpha)$, that is,

$$\langle\Omega(\alpha)\rangle = \{\Omega(\alpha), \Omega_1(\alpha-1), \cdots, \Omega_{\alpha-1}(1)\},$$

where $\Omega_i(j)$ denotes surface Ω with j regions and i discs removed.

Lemma 2 *If $C_m[p] \hookrightarrow g_m T(\alpha)$ and $C_m[p] \hookrightarrow \tilde{g}P(\beta)$ are minimum embeddings, then the* ***set of surfaces for the 2-cell embedding of the $C_m[p]$ channel*** *is*

$$\widehat{S}_{m,p} = \begin{cases} \left\{\langle(g_m+i)T(\alpha-2i)\rangle_{i=0}^{(\alpha-2)/2}\right\} \cup \left\{\langle(\tilde{g}+j)P(\beta-j)\rangle_{j=0}^{\beta-1}\right\} & \text{if } \alpha \text{ is even} \\ \left\{\langle(g_m+i)T(\alpha-2i)\rangle_{i=0}^{(\alpha-1)/2}\right\} \cup \left\{\langle(\tilde{g}+j)P(\beta-j)\rangle_{j=0}^{\beta-1}\right\} & \text{if } \alpha \text{ is odd.} \end{cases}$$

where the corresponding surfaces are denoted by T (torus) and P (projective plane).

Riemann Surfaces as Quotient Spaces for the Embeddings

Uniformization Theorem (Klein, Poincaré, Köbe): *Every simply connected Riemann surface is conformally equivalent to one of the three Riemann surfaces, universal covering,* $\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\} \equiv \mathbb{S}^2$*, Riemann sphere;* $\mathbb{C} \equiv \mathbb{R}^2$*, complex plane;* $\mathbb{H}^2$*, upper half-plane*.

A Riemann surface $\mathbb{H}^2/\Gamma$ may be constructed from $\mathbb{H}^2$ and a subgroup $\Gamma \subset Aut(\mathbb{H}^2)$. However, Γ has to satisfy: every element of Γ, except the identity, has no fixed points in $\mathbb{H}^2$, and acts properly discontinuously on $\mathbb{H}^2$. Γ is called a **Fuchsian group**.

The elements of a Fuchsian group are Möbius transformations:

- $Aut(\Delta)$ (Poincaré disc), $T(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$.
- $Aut(\mathbb{H}^2)$ (upper half-plane), $T(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$.

In order to obtain a **geometric image** of $\mathbb{H}^2/\Gamma$, we use a **fundamental domain** for Γ. An open set F of $\mathbb{H}^2$ is a **fundamental domain** for Γ if

- $T(F) \cap F = \emptyset, \forall T \in \Gamma,\ T \neq id$ (properly discontinuously);
- If $\widetilde{F}$ is the closure of F in $\mathbb{H}^2$, then $\mathbb{H}^2 = \bigcup_{T \in \Gamma} T(\widetilde{F})$.

Therefore, $\mathbb{H}^2/\Gamma \equiv \widetilde{F}$ with points in ∂F identified by the elements of Γ.

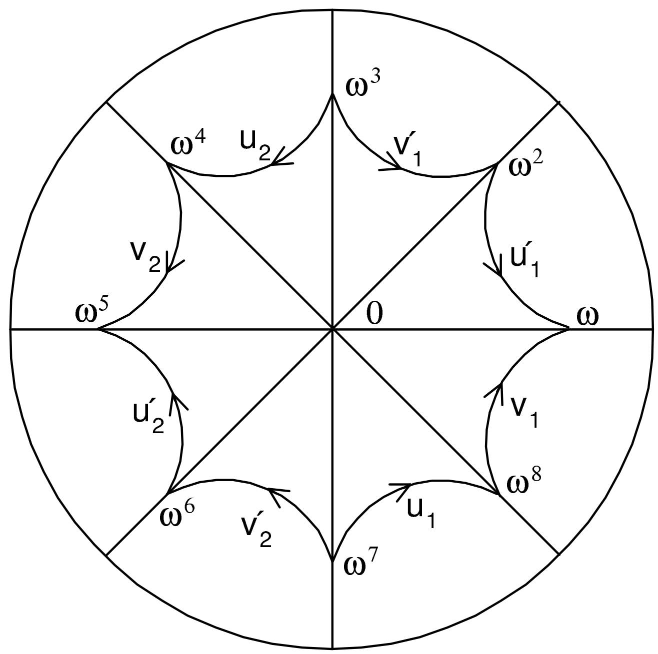


Figure 7: Fundamental domain of the 2-torus

Steps for the Embedding of DMC Channels in Surfaces

P1 - Identify the complete bipartite graph $K_{m,n}$ **having** $C_{m,n}[P,Q]$ **channel** as a subgraph;

P2 - Determine g_m **and** g_M **of the surfaces for the embedding of** $K_{m,n}$ **associated with the** $C_{m,n}[P,Q]$ **channel**. From Theorem 1, if $K_{m,n} \hookrightarrow \Omega$ and Ω has genus g, then $g_m \leq g \leq g_M$;

P3 - For each g use Proposition 1 and **identify a model** $\mathfrak{F}_{mn} \hookrightarrow \Omega(\alpha)$;

P4 - For each model in P3 **identify the set of surfaces generated by** $\Omega(\alpha)$, $\langle \Omega(\alpha) \rangle$, apply Lemma 2 to obtain the set of surfaces $\widehat{S}_{m,n}$ for the embeddings of the $C_{m,n}[P,Q]$ channel;

P5 - Use Lemma 1 and **identify in P4 the set of regular tessellations with** m **identical regions**.

Examples of Embedded DMC Channels

The surfaces are denoted by S (sphere), T (torus), P (projective plane) or K (Klein bottle).

BSC Channel

$$C_2[2] \equiv K_{2,2} \quad \hookrightarrow \quad \widehat{S}_{2,2} = \{S(2), S_1(1), P(2), P(1), P_1(1)\}.$$

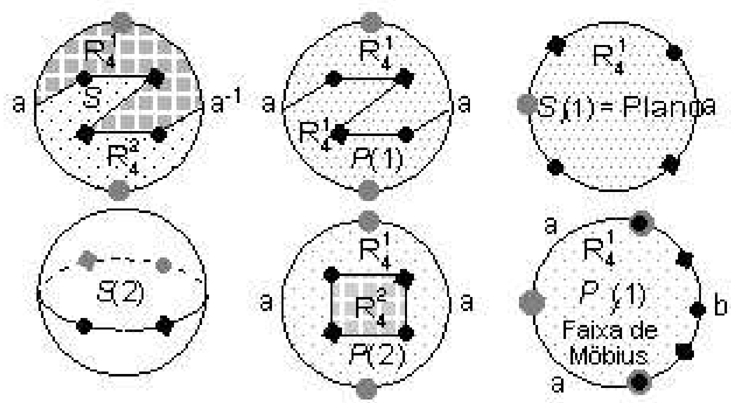


Figure 8: $C_2[2]$ channel

Ternary Channels with Degree 3

$$C_3[3] \equiv K_{3,3} \quad \hookrightarrow \quad \widehat{S}_{3,3} = \{\langle T(3)\rangle, 2T(1), \langle P(4)\rangle, \langle K(3)\rangle, \langle 3P(2)\rangle, 2K(1)\}.$$

$$\Xi_{3,3} = \{T[3R_6]\, P_1[3R_4], K[3R_6]\}.$$

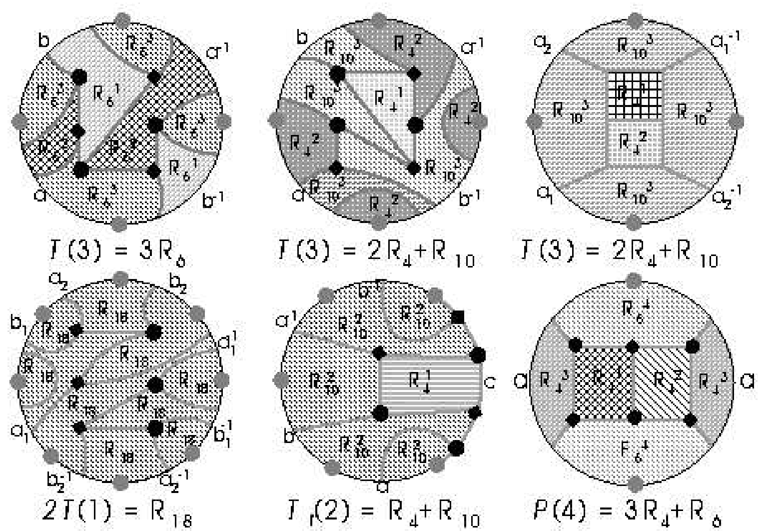


Figure 9: $C_3[3]$ channel

Minimal Surfaces - Embedding Channels in n-noid

Definition 2 *An α-totem channel is a tower of α $C_{m,n}[P,Q]$ channels, with $\alpha \geq 3$.*

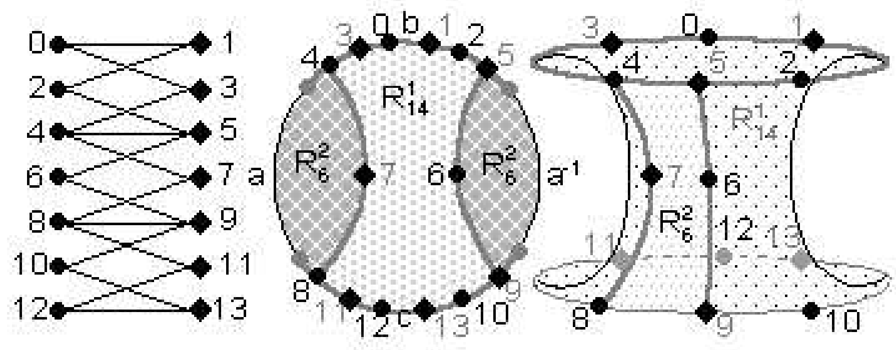


Figure 10: 3-totem channel embedded in a catenoid

$$\widehat{S}_3\text{-totem} = \{S(4), T(2), S_1(3), 2N(2), S_3(1), T_1(1)\}.$$

Embedding Channels in a Torus with 3 Boundary Components

Fig. 11 shows the embedding of the $C_6[4]$ channel in an oriented compact surface of genus one with three boundary components, denoted by T_3

$$C_6[4] \hookrightarrow T_3 = 9R_4.$$

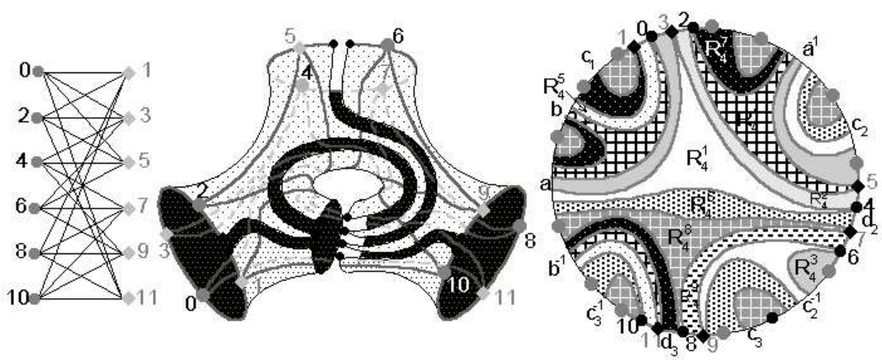


Figure 11: 8-totem channel embedded in a torus with $3N$

III - GU Signal Sets in Homogeneous Spaces

Among the important facts about Slepian signal sets, **”spherical codes”**, is the association of discrete groups of isometries to such signal sets. So is for Lattice signal sets, **”torus codes”**.

Forney´s introduction of **GU codes (isometry codes)** in 1991, intensified the search for signal sets associated with discrete groups of isometries in metric spaces and placed the previous two approaches in the same context.

Therefore, the next step is to look for signal sets as either subsets of regular tessellations in $\mathbb{H}^2$ or on compact (non-compact) surfaces with $g \geq 2$ obtained by quotient, **”g-torus codes”**.

Hyperbolic Signal Sets

Definition 3 *A **regular tessellation** in $\mathbb{H}^2$ is a partition of $\mathbb{H}^2$ by non-overlapping regular polygons with the same number of edges which intersect entirely on edges or vertices. A regular tessellation in which q regular p-gons meet at each vertex is denoted by $\{p,q\}$.*

Euclidean $\{p,q\}$ exists *iff* $(p-2)(q-2)=4$. Solutions: $\{4,4\}$; $\{6,3\}$; $\{3,6\}$.

Hyperbolic $\{p,q\}$ exists *iff* $(p-2)(q-2)>4$. Solutions: infinite.

Associated with each $\{p,q\}$ is the **complete symmetry group** $[p,q]$ of $\{p,q\}$. This is the isometry group of $\mathbb{H}^2$.

The group $[p,q]$ is generated by r_1, r_2 and r_3 in the sides of the hyperbolic triangle with angles $\frac{\pi}{2},\frac{\pi}{p},\frac{\pi}{q}$. The presentation of $[p,q]$ is

$$[p,q]=\left\langle r_1,r_2,r_3 : r_1^2=r_2^2=r_3^2=(r_2r_1)^p=(r_3r_2)^q=(r_1r_3)^2=e\right\rangle.$$

Tessellation $\{p,q\} \longrightarrow$ group $[p,q]$.

Dual tessellation $\{q,p\} \longrightarrow$ group $[q,p]$.

When $[p,q]=[q,p]$, it is called **self-dual**.

If $p=q=4g$, $g \geq 2$, then $\{4g,4g\}$ is such that the **identification of its sides** yields a **universal covering** for $\mathbb{H}^2$ as an **oriented compact surface of genus** g and its **group** $[4g,4g]$ has as a normal subgroup,

$$\pi_g = \left\langle a_1,\ldots,a_g,b_1,\ldots,b_g : \prod_{i=1}^{g}[a_i,b_i]=e \right\rangle \quad \textbf{(fundamental group)}$$

Since the fundamental region is a polygon with $4g$ sides, it follows that its symmetry group is $\mathbb{D}_{4g}$. Therefore, $[4g,4g]=\pi_g \ltimes \mathbb{D}_{4g}$.

Quotient Space Approach (Compact Surfaces)

The hyperbolic space is relevant for:

- Any g-torus, locally isometric to $\mathbb{H}^2$, can be obtained topologically by $\mathbb{H}^2/G \cong T_g$, where G is a Fuchsian group.
- For each genus g, this action occurs by the identification of edges in a regular polygon with $4g$ or $4g+2$ edges in $\mathbb{H}^2$ by $2g$ or $2g+1$ isometries which generate G.

This provides a systematic way of generating signal sets on T_g. Any Fuchsian group $\Gamma \in \mathbb{H}^2$ such that $G \subset \Gamma$ is a normal subgroup may be used to generate GU signal sets on T_g.

An important aspect when considering quotients E/G of spaces with constant curvature by discrete groups of isometries is that the geometry of the space E is induced to the quotient by the projection of orbits,

$$\begin{array}{rcl} \pi : E & \longrightarrow & E/G \\ x & \longrightarrow & \pi(x) = Gx \end{array}$$

that is, all local metric properties in E are preserved in E/G due to the existence for each $p \in E$ of a neighborhood $V_p \subset E$ such that $\pi_{|V_p}$ is injective due to the fact that G consists of isometries (therefore preserving metric properties).

This means that the study of signal sets in the quotient is related to the study of signal sets in E, however, locally.

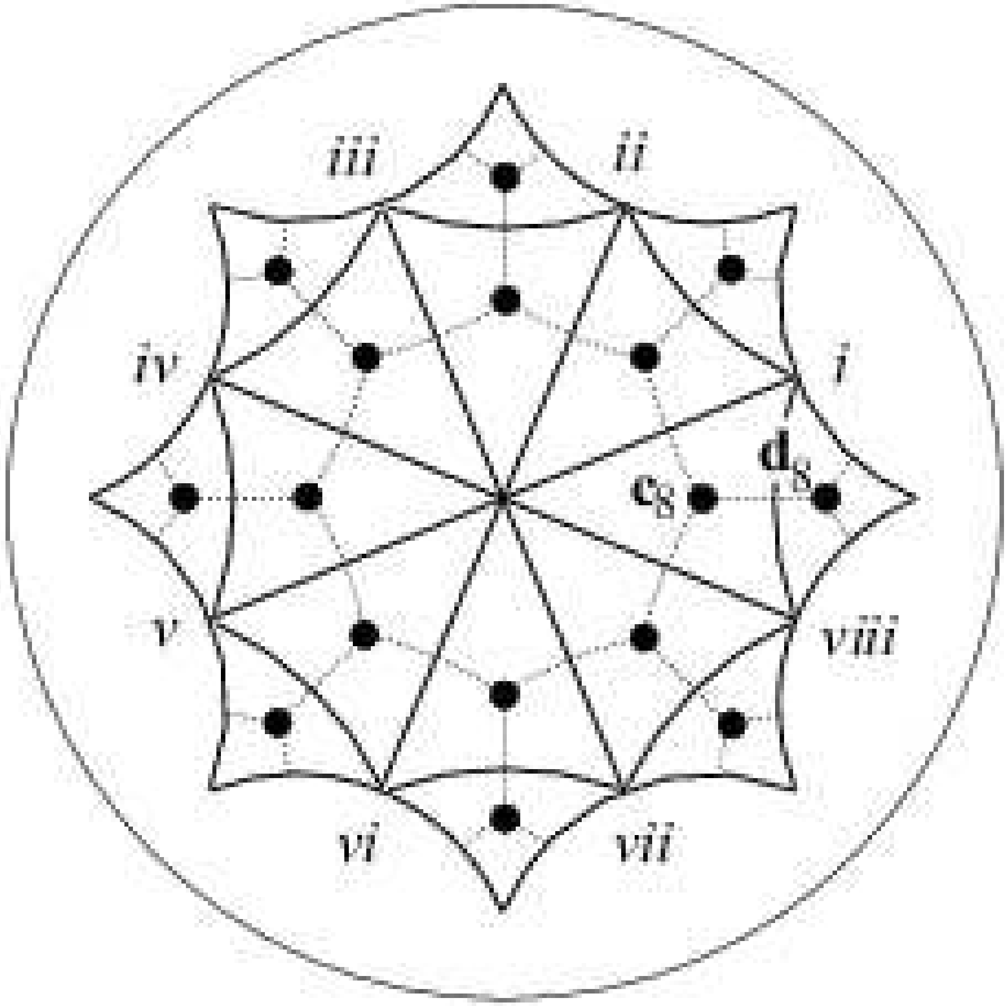


Figure 12: 2-torus.

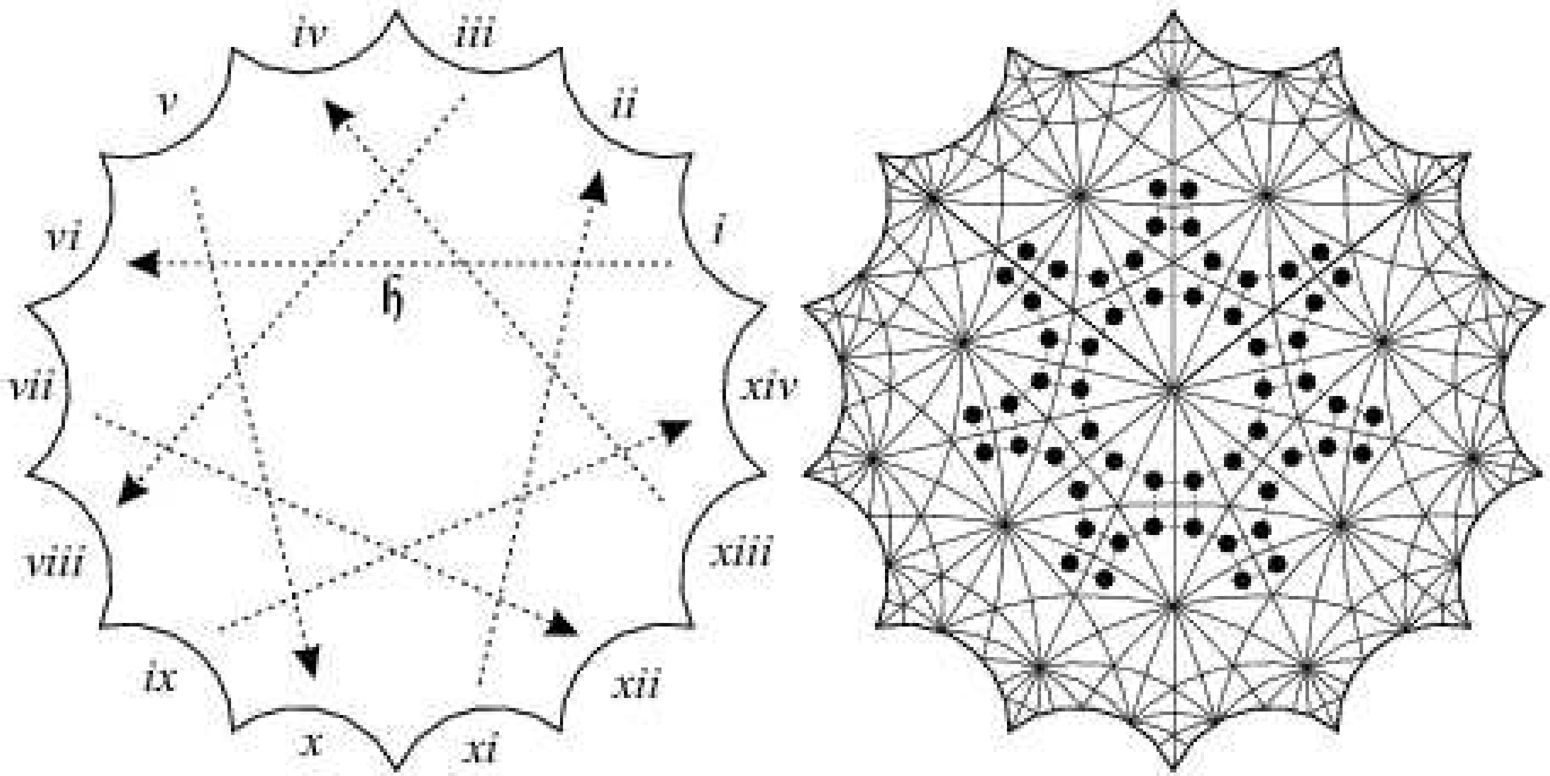


Figure 13: 3-torus.

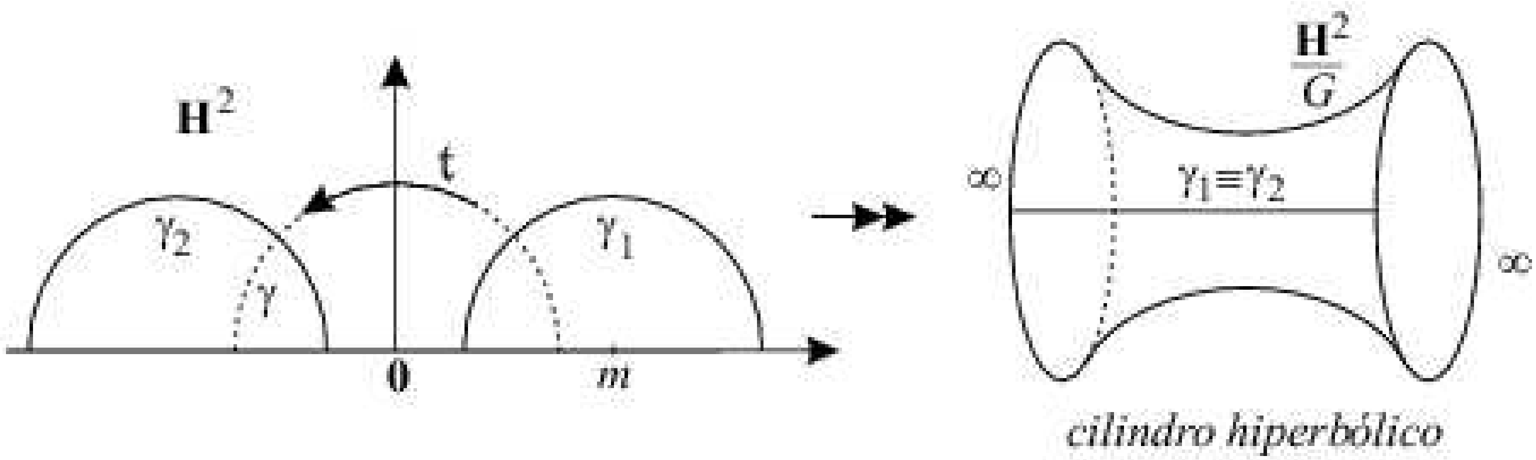


Figure 14: Hyperbolic Cylinder.

IV - Performance of Signal Sets in Riemannian Manifolds

Elements of Riemannian Geometry

Definition 4 ***A differentiable manifold of dimension*** n *consists of a set M and a family of bijective mappings* $\mathbf{x}_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ *of open sets* U_α *of* $\mathbb{R}^n$ *in* M *such that*

1. $\cup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$.
2. *For every pair* α *and* β*, with* $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \phi$*, the sets* $\mathbf{x}_\alpha^{-1}(W)$ *and* $\mathbf{x}_\beta^{-1}(W)$ *are open sets in* $\mathbb{R}^n$ *and the mappings* $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ *are differentiable.*
3. *The family* $\{(U_\alpha, \mathbf{x}_\alpha)\}$ *is maximal with respect to (1) e (2).*

Definition 5 ***A Riemannian metric*** *in a differentiable manifold* M ***is a correspondence associating*** *with each point* p *of* M ***a dot product*** $\langle\ ,\ \rangle_p$ *(that is, a positive definite, symmetric, bilinear form)* ***in the tangent space*** T_pM*, varying differentiably.*

We use a Riemannian metric to determine the length of a curve $c : I \subset \mathbb{R} \to M$ constrained to the closed interval $[a,b] \subset I$ as follows

$$s = \int_a^b \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{1/2} dt \ ,$$

where $\frac{dc(t)}{dt}$ denotes a vector field originating from $c(t)$.

A **parameterized curve** $\gamma : I \to M$ is a **geodesic** $\gamma(t) = (\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t))$ in a system of coordinates $(U, \mathbf{x})$ if and only if it satisfies

$$\frac{d^2\mathbf{x}_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{d\mathbf{x}_i}{dt} \frac{d\mathbf{x}_j}{dt} = 0, \quad k = 1, \ldots, n \ , \tag{1}$$

where Γ^m_{ij} are the Christoffel symbols of a Riemannian connection M, given by

$$\Gamma^m_{ij} = \frac{1}{2} \sum_k \left\{ \frac{\partial}{\partial \mathbf{x}_i} g_{jk} + \frac{\partial}{\partial \mathbf{x}_j} g_{ki} - \frac{\partial}{\partial \mathbf{x}_k} g_{ij} \right\} g^{km} \ ,$$

where g^{km} is an element of the matrix G^{km}, whose inverse is G_{km}.

Definition 6 (Carmo) *Given a point $p \in M$ and a 2-D subspace $\Sigma \subset T_pM$, with $\{x, y\}$ any basis for Σ,* ***the real number*** *$K(x,y) = K(\Sigma)$ is called* ***sectional curvature*** *of Σ in M, defined by*

$$K(x,y) = \frac{\langle R(x,y)x, y \rangle}{|x|^2 + |y|^2 - \langle x, y \rangle^2} \ ,$$

where $R(x,y)$ denotes the Riemann curvature tensor (depends only on the metric), and the denominator denotes the area of a 2-D parallelogram.

Signal Sets in Riemannian Manifolds

In designing signal sets in a Riemannian manifold we may consider different Riemannian metrics for the modulator, channel and demodulator. However, the composition has to be matched to the metric in the metric space (E,d) originated from the embedding of the DMC channel.

Definition 7 *A signal set* $\mathcal{X} = \{x_1, \ldots, x_m\}$ *in an* n*-dimensional Riemannian manifold* M *with a coordinate system* $(U, \mathbf{x})$ *is a set of* n*-dimensional points.*

$$\mathcal{X} = \{x_1 = (\mathbf{x}_{11}, \ldots, \mathbf{x}_{n1}), \ldots, x_m = (\mathbf{x}_{1m}, \ldots, \mathbf{x}_{nm})\} \subset U.$$

The distance between any two given points in M *will be the least geodesic distance.*

The noise action is a transformation that takes $x_m \in M$ to the received signal $y \in M$. This transformation is given by

$$y = \exp_{x_m}(v) \ , \quad v \in T_{x_m}M \ , \tag{2}$$

where $\exp_{x_m} : T_{x_m}M \to M$ is called **exponential map**.

Note that the exponential map takes the noise in the tangent space to the manifold M.

Knowing the pdf of the n-D random vector $v = (v_1, \ldots, v_n)$, $v \in T_{x_m}M$. Then

$$p_{Y/X=x_m}(y/x_m) = p_{\mathcal{V}}(v = \exp_{x_m}^{-1}(y))|J| \ , \tag{3}$$

where $|J|$ is the Jacobian of the transformation.

If each v_j, $j = 1, \ldots, n$, is gaussian then y is also gaussian, since $T_{x_m}M$ is a n-D vector space.

The pdf of y is **well defined** if the Riemannian manifold M is **complete**, that is, for every $x_m \in M$ the map $\exp_{x_m}(v)$ is defined for every $v \in T_{x_m}M$.

Let the v_js, $j = 1, \ldots, n$, be gaussian r.v. with zero mean and equal variances. Then, the pdf of y given x_m, is

$$p(y/x_m) = k_1 e^{-k_2 d^2(y,x_m)} \sqrt{det(G)} \, , \tag{4}$$

where $d^2(y, x_m)$ is the squared geodesic distance, $\sqrt{det(G)}$ is the volume element of the manifold M, and k_1, k_2 are constants satisfying the condition

$$\int_M k_1 e^{-k_2 d^2(y,x_m)} \sqrt{det(G)} \, d\mathbf{x}_1 \ldots d\mathbf{x}_n = 1 \, . \tag{5}$$

Let R_m be the decision region of x_m. Then

$$P_{e,m} = 1 - \int_{R_m} p(y/x_m)\, d\mathbf{x}_1 \ldots d\mathbf{x}_n \ . \tag{6}$$

$P_{e,m}$ does not depend on a particular coordinate system $(U, \mathbf{x})$.

$P_e(\mathcal{X})$ in a Riemannian manifold may be written as $P_e = \sum_{\mathcal{X}} P(x_m) P_{e,m}$.

The average energy of $\mathcal{X}$ is $E_t = \sum_{\mathcal{X}} P(x_m) d^2(x_m, \bar{x})$, where $d^2(x_m, \bar{x})$ is the squared geodesic distance and $\bar{x}$ is the center of mass of the signal set.

It is known that the center of mass minimizes the average energy. Therefore, $\bar{x}$ is the unique solution to

$$\left.\frac{\partial E_t}{\partial \mathbf{x}_j}\right|_{x=\bar{x}} = \sum_{\mathcal{X}} P(x_m) d(x_m, \bar{x}) \left.\frac{\partial d(x_m, x)}{\partial \mathbf{x}_j}\right|_{x=\bar{x}} = 0 \ , \qquad j = 1, \ldots, n \ . \tag{7}$$

The noise power is defined as

$$\sigma^2 = \int_M d^2(y, x_m) p(y/x_m)\, d\mathbf{x}_1 \ldots d\mathbf{x}_n. \tag{8}$$

Determining the PDF in Spaces with K Constant

We consider an example of an M-PSK signal set in a 2-D Riemannian manifold. For this example, we use the geodesic polar coordinate system, (ρ,θ).

In this system, the coefficients $g_{11}(\rho,\theta)$, $g_{21}(\rho,\theta) = g_{12}(\rho,\theta)$ and $g_{22}(\rho,\theta)$ of the Riemannian metric (2×2 matrix G) must satisfy the following conditions

$$g_{11} = 1, \quad g_{12} = g_{21} = 0, \quad \lim_{\rho\to 0} g_{22} = 0, \quad \lim_{\rho\to 0} (\sqrt{g_{22}})_\rho = 1\,. \tag{9}$$

A geodesic $\gamma : I \subset R \to M$ in polar coordinates, $\gamma(t) = (\rho(t), \theta(t))$ must satisfy

$$\begin{cases} \rho'' - \frac{1}{2}(g_{22})_\rho(\theta')^2 = 0\,, \\ \theta'' + \frac{(g_{22})_\rho}{g_{22}}\rho'\theta' + \frac{1}{2}\frac{(g_{22})_\theta}{g_{22}}(\theta')^2 = 0\,. \end{cases}$$

The distance between signal points in M is given by the length of the geodesic $\gamma(t)$, from t_1 to t_2, that is,

$$s = \int_{t_1}^{t_2} \sqrt{(\rho')^2 + g_{22}(\theta')^2}\, dt\,.$$

For the polar coordinate system g_{22} is the solution to

$$(\sqrt{g_{22}})_{\rho\rho} + K\sqrt{g_{22}} = 0 \,, \tag{10}$$

where K denotes the sectional curvature of M.

When M is 2-D, K is known as the **gaussian curvature**.

Assuming K constant, the solutions to (10) are

$$g_{22}(\rho,\theta) = \begin{cases} \rho^2 \,, & \text{if } K = 0 \text{ (Euclidean)}, \\ \frac{1}{K}\sin^2(\sqrt{K}\rho) \,, & \text{if } K > 0 \text{ (Elliptic)}, \\ \frac{1}{-K}\sinh^2(\sqrt{-K}\rho) \,, & \text{if } K < 0 \text{ (Hyperbolic)} \,. \end{cases} \tag{11}$$

Note that g_{22} must also satisfy (9), and $det(G) = g_{22}(\rho,\theta)$.

1- When $K = 0$, the **geodesic distance** between any two given points $z_1, z_2 \in \mathbb{R}^2$ is

$$d_{\mathbb{R}} = |z_1 - z_2|, \tag{12}$$

where $|\,|$ denotes the absolute value of $z_i = r_i e^{j\theta_i}$, $i = 1, 2$.

2- When $K > 0$, the **geodesic distance** between any two given points $z_1, z_2 \in \mathbb{S}^2$ is

$$d_{\mathbb{S}} = \frac{2\pi l}{\sqrt{K}} \pm \frac{1}{j\sqrt{K}} \log \frac{|1 + z_1\bar{z}_2| + j|z_1 - z_2|}{|1 + z_1\bar{z}_2| - j|z_1 - z_2|}, \tag{13}$$

where l is the number of times that a geodesic passes by the point z_1 or its antipodal, until arriving at z_2, $z_i = r_i e^{j\theta_i}$ and $r_i = -j(e^{j\rho_i\sqrt{K}} - 1)/(e^{j\rho_i\sqrt{K}} + 1)$, $i = 1, 2$.

3- When $K < 0$, the **geodesic distance** between any two given points $z_1, z_2 \in \mathbb{H}^2$ is

$$d_{\mathbb{H}} = \frac{1}{\sqrt{-K}} \log \frac{|1 - z_1\bar{z}_2| + |z_1 - z_2|}{|1 - z_1\bar{z}_2| - |z_1 - z_2|}, \tag{14}$$

where $z_i = r_i e^{j\theta_i}$ and $r_i = (e^{\rho_i\sqrt{-K}} - 1)/(e^{\rho_i\sqrt{-K}} + 1)$, $i = 1, 2$.

Therefore, $p(y/x_m)$ and σ^2 can be found for each one of the homogeneous spaces $\mathbb{R}^2$, $\mathbb{S}^2$ and $\mathbb{H}^2$. Under this assumption, k_1, k_2 and σ^2 are independent of the transmitted signal x_m.

We assume $x_m = (0,\theta)$. Hence, for $y = (\rho,\theta)$, we have

1. For $K = 0$, the **pdf** is

$$p_{\mathbb{R}}(\rho,\theta) = k_1 e^{-k_2\rho^2}\rho, \quad \text{(Rayleigh)}$$

   where $k_1 = k_2/\pi$ and $\sigma^2 = 1/k_2$.

2. For $K < 0$, the **pdf** is

$$p_{\mathbb{H}}(\rho,\theta) = \frac{k_1}{\sqrt{-K}} e^{-k_2\rho^2} \sinh\left(\sqrt{-K}\rho\right),$$

   with

$$k_1 = \frac{\pi^{-3/2} e^{K/4k_2}\sqrt{-Kk_2}}{\operatorname{erf}(\sqrt{-K}/2\sqrt{k_2})}, \text{and} \quad \sigma^2 = \frac{2\sqrt{-Kk_2}e^{K/4k_2} + \sqrt{\pi}\operatorname{erf}(\sqrt{-K/4k_2})(2k_2 - K)}{4k_2^2\sqrt{\pi}\operatorname{erf}(\sqrt{-K/4k_2})}.$$

3. For $K > 0$, the **pdf** is

$$p_{\mathbb{S}}(\rho,\theta) = \frac{k_1}{\sqrt{K}} e^{-k_2\rho^2} |\sin(\sqrt{K}\rho)| \,,$$

where

$$k_1 = \frac{\sqrt{K}}{2\pi} \left(\sum_{i=0}^{\infty} \int_{i\pi}^{(i+1)\pi} (-1)^i e^{-k_2\rho^2} \sin(\sqrt{K}\rho)\, d\rho \right)^{-1} .$$

For $k_2 \gg K$, k_1 may be approximated by

$$k_1 \approx \frac{i\pi^{-3/2} e^{K/4k_2} \sqrt{Kk_2}}{\mathrm{erf}(i\sqrt{K}/2\sqrt{k_2})}, \text{ and } \sigma^2 \approx \frac{2i\sqrt{Kk_2}e^{K/4k_2} + \sqrt{\pi}\mathrm{erf}(i\sqrt{K/4k_2})(2k_2 - K)}{4k_2^2\sqrt{\pi}\mathrm{erf}(i\sqrt{K/4k_2})} .$$

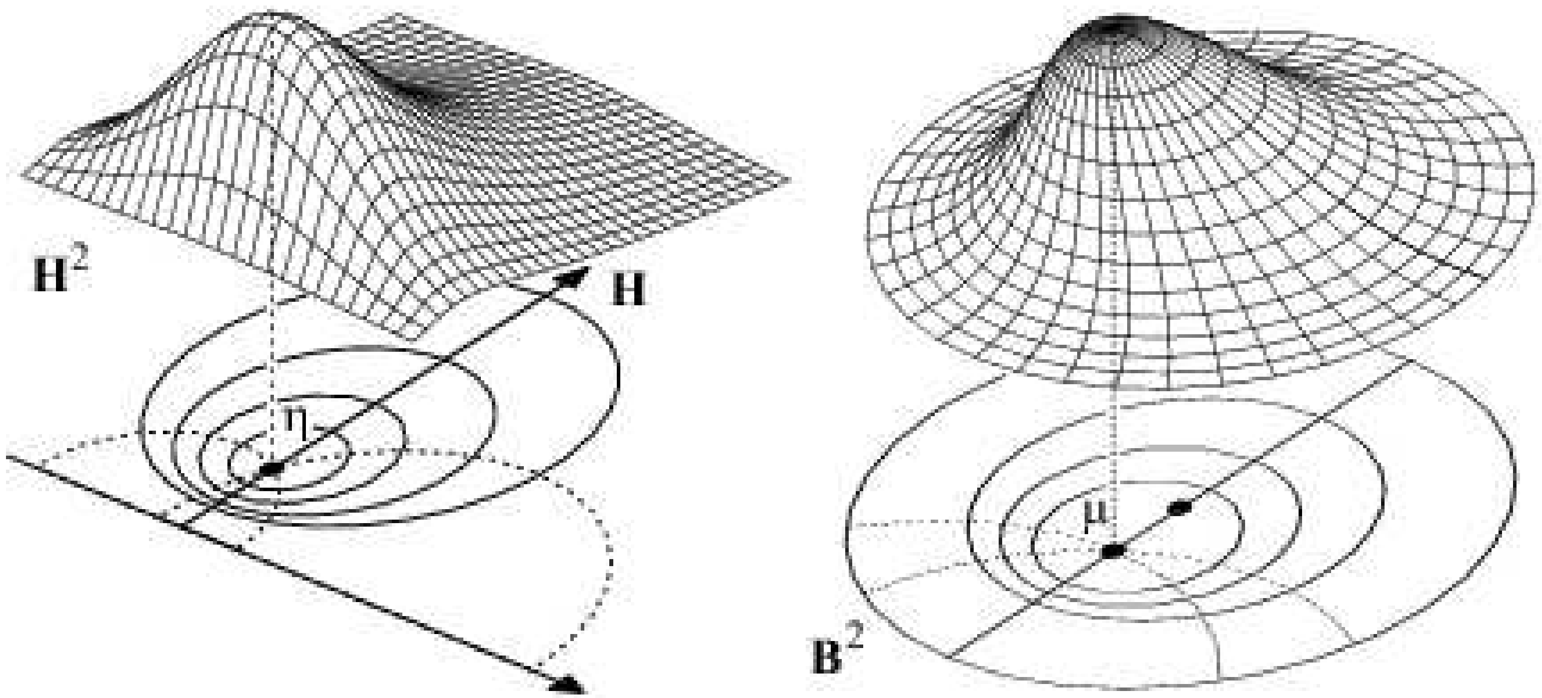


Figure 15: Hyperbolic gaussian pdf.

Performance Analysis of M-PSK in Spaces with K Constant

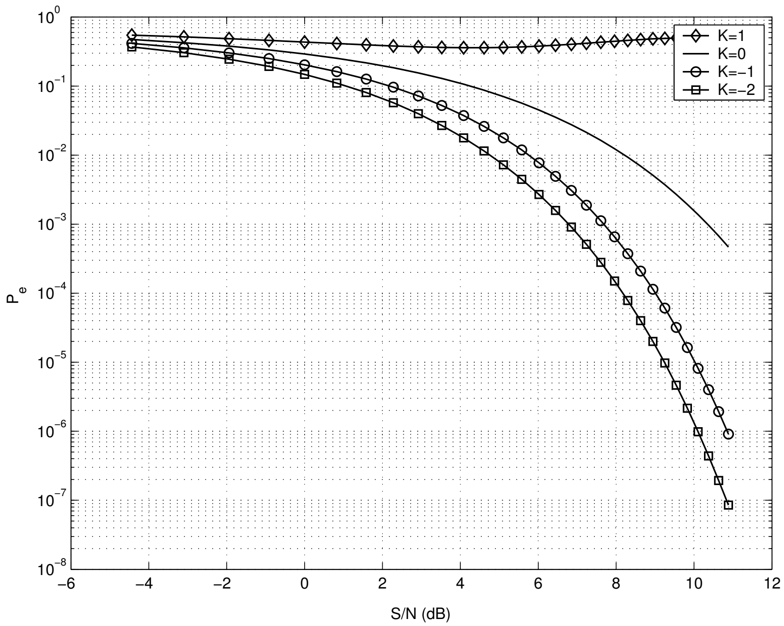


Figure 16: $P_e \times S/N$ for 4-PSK in spaces with curvatures 1, 0, -1 and -2.

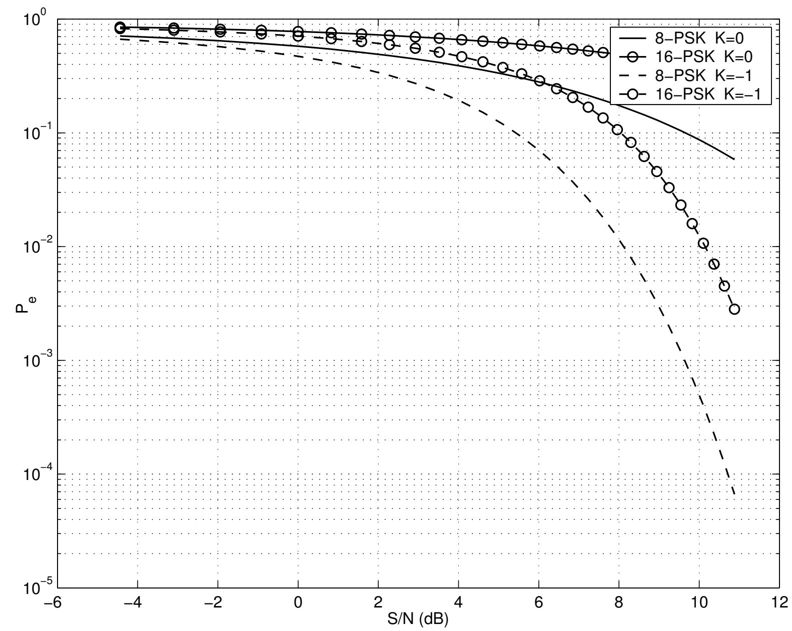


Figure 17: $P_e \times S/N$ for 8-PSK and 16-PSK in spaces with curvatures 0 and -1.

Fixing $E_t = (\pi/2)^2$ for an M-PSK, Fig. 18 shows d_K/d_0 versus M-PSK, where d_K is the minimum distance in a space with curvature K, and d_0 is the minimum distance in the Euclidean space, $K = 0$, d_0.

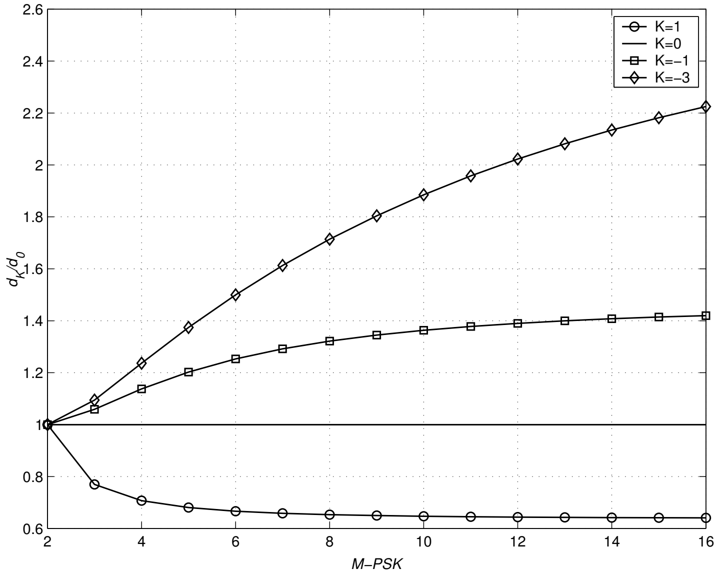


Figure 18: $d_K/d_0 \times M$ for $E_t = (\pi/2)^2$.

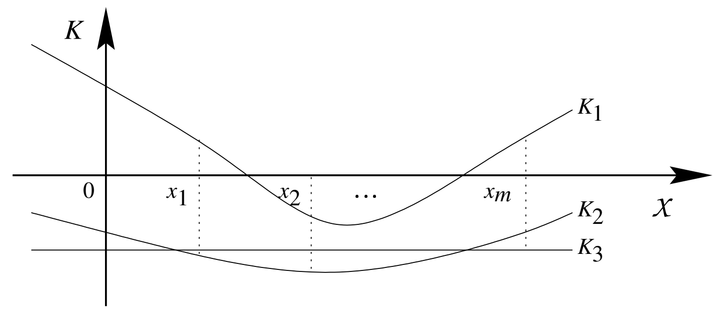


Figure 19: Sectional curvature versus signal set $\mathcal{X}$.

$$\text{Conclusion:} \quad K_1 < K_2 \quad \Rightarrow \quad P_e(K_1) < P_e(K_2)\ . \tag{15}$$

Fig. 20 shows $P_e \times K$ (4-PSK and 8-PSK) for $S/N = 4$dB. As expected, P_e diminishes with decreasing values of K.

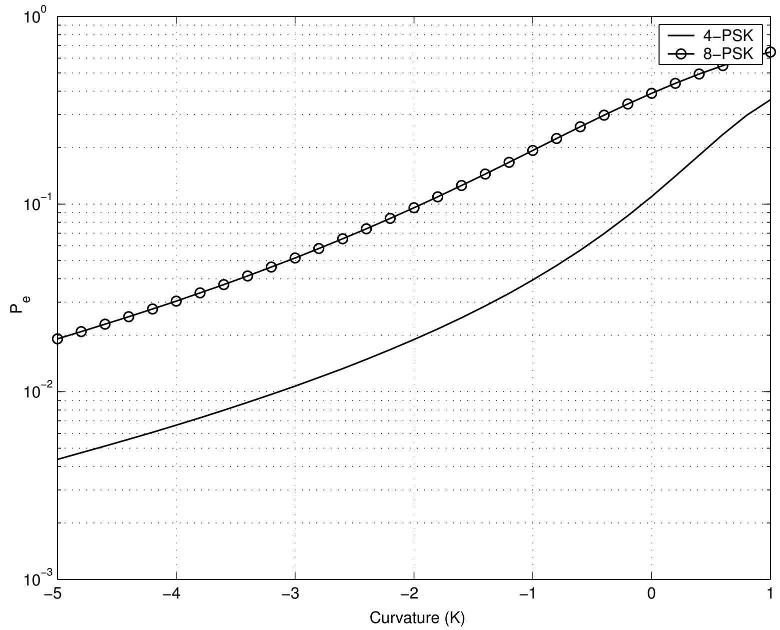


Figure 20: Error probability versus sectional curvature