A Classi£cation of Posets admitting MacWilliams Identity

Hyun Kwang Kim and Dong Yeol Oh

***Abstract*— In this paper all poset structures are classi£ed which admit the MacWilliams identity, and the MacWilliams identities for poset weight enumerators corresponding to such posets are derived. We prove that being a hierarchical poset is a necessary and suf£cient condition for a poset to admit MacWilliams identity. An explicit relation is also derived between $\mathbf{P}$-weight distribution of a hierarchical poset code and $\overline{\mathbf{P}}$-weight distribution of the dual code.**

***Index Terms*— MacWilliams identity, poset codes, $\mathbf{P}$-weight enumerator, leveled $\mathbf{P}$-weight enumerator, hierarchical poset.**

I. Introduction

LET $\mathbb{F}_q$ be the £nite £eld with q elements and $\mathbb{F}_q^n$ be the vector space of n-tuples over $\mathbb{F}_q$. Coding theory may be considered as the study of $\mathbb{F}_q^n$ when $\mathbb{F}_q^n$ is endowed with Hamming metric. Since the late 1980's several attempts have been made to generalize the classical problems of the coding theory by introducing a new non-Hamming metric on $\mathbb{F}_q^n$ (cf [8 - 10]). These attempts led Brualdi et al. [1] to introduce the concept of poset codes. First, we begin by brie¤y introducing the basic notions of poset code such as poset-weight and poset-distance. See [1] for details.

Let $\mathbb{F}_q^n$ be the vector space of n-tuples over a £nite £eld $\mathbb{F}_q$ with q elements. Let $\mathbf{P}$ be a partial ordered set, which will be abbreviated as a poset in the sequel, on the underlying set $[n] = \{1, 2, \ldots, n\}$ of coordinate positions of vectors in $\mathbb{F}_q^n$ with the partial order relation denoted by $\leq$ as usual. For $u = (u_1, u_2, \cdots, u_n) \in \mathbb{F}_q^n$, the support $supp(u)$ and $\mathbf{P}$-weight $w_{\mathbf{P}}(u)$ of u are de£ned to be

$$supp(u) = \{i \mid u_i \neq 0\} \text{ and } w_{\mathbf{P}}(u) = |< supp(u) >|,$$

where $< supp(u) >$ is the smallest ideal (recall that a subset I of $\mathbf{P}$ is an ideal if $a \in I$ and $b \leq a$, then $b \in I$) containing the support of u. It is well-known that for any $u, v \in \mathbb{F}_q^n$, $d_{\mathbf{P}}(u, v) := w_{\mathbf{P}}(u - v)$ is a metric on $\mathbb{F}_q^n$. The metric $d_{\mathbf{P}}$ is called $\mathbf{P}$-metric on $\mathbb{F}_q^n$. Let $\mathbb{F}_q^n$ be endowed with $\mathbf{P}$-metric. Then a (linear) code $\mathcal{C} \subseteq \mathbb{F}_q^n$ is called a (linear) $\mathbf{P}$-code of length n. The $\mathbf{P}$-weight enumerator of a linear $\mathbf{P}$-code $\mathcal{C}$ is de£ned by

$$W_{\mathcal{C},\mathbf{P}}(x) = \sum_{u \in \mathcal{C}} x^{w_{\mathbf{P}}(u)} = \sum_{i=0}^{n} A_{i,\mathbf{P}} x^i,$$

where $A_{i,\mathbf{P}} = |\{u \in \mathcal{C} \mid w_{\mathbf{P}}(u) = i\}|$.

This research was supported by the Com^{2}MaC-KOSEF and POSTECH BSRI research fund.

H.K.Kim and D.Y.Oh are with the Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Korea.

Remark : If $\mathbf{P}$ is an antichain, then $\mathbf{P}$-metric is equal to Hamming metric. So $\mathbf{P}$-weight enumerator of a linear code $\mathcal{C}$ becomes Hamming weight enumerator of $\mathcal{C}$.

The MacWilliams identity for linear codes over $\mathbb{F}_q$ is one of the most important identities in the coding theory, and it expresses Hamming weight enumerator of the dual code $\mathcal{C}^\perp$ of a linear code $\mathcal{C}$ over $\mathbb{F}_q$ in terms of Hamming weight enumerator of $\mathcal{C}$. Since Hamming metric is a special case of poset metrics, it is natural to attempt to obtain MacWilliams-type identity for certain $\mathbf{P}$-weight enumerators. See [3 - 5] for this direction of researches. Essentially, what enables us to obtain MacWilliams identity for Hamming metric is that Hamming weight enumerator of the dual code $\mathcal{C}^\perp$ is uniquely determined by that of $\mathcal{C}$. The following example suggests that we need some modi£cation to generalize MacWilliams identity for certain type of poset weight enumerators.

Example 1.1: Let $\mathbf{P} = \{1, 2, 3\}$ be a poset with order relation $1 < 2 < 3$ and $\overline{\mathbf{P}} = \{1, 2, 3\}$ be a poset with order relation $1 > 2 > 3$. Consider the following binary linear $\mathbf{P}$-codes:

$$\mathcal{C}_1 = \{(0,0,0),(0,0,1)\}\ ,\ \mathcal{C}_2 = \{(0,0,0),(1,1,1)\}.$$

It is easy to check that $\mathbf{P}$-weight enumerators of $\mathcal{C}_1$ and $\mathcal{C}_2$ are given by

$$W_{\mathcal{C}_1,\mathbf{P}}(x) = 1 + x^3 = W_{\mathcal{C}_2,\mathbf{P}}(x).$$

The dual codes of $\mathcal{C}_1$ and $\mathcal{C}_2$ are respectively given by

$$\mathcal{C}_1^\perp = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}$$

and

$$\mathcal{C}_2^\perp = \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}.$$

The $\mathbf{P}$-weight enumerators of $\mathcal{C}_1^\perp$ and $\mathcal{C}_2^\perp$ are given by

$$W_{\mathcal{C}_1^\perp,\mathbf{P}}(x) = 1 + x + 2x^2,\ W_{\mathcal{C}_2^\perp,\mathbf{P}}(x) = 1 + x^2 + 2x^3,$$

while $\overline{\mathbf{P}}$-weight enumerators of $\mathcal{C}_1^\perp$ and $\mathcal{C}_2^\perp$ are given by

$$W_{\mathcal{C}_1^\perp,\overline{\mathbf{P}}}(x) = 1 + x^2 + 2x^3 = W_{\mathcal{C}_2^\perp,\overline{\mathbf{P}}}(x).$$

As it is seen above, although $\mathbf{P}$-weight enumerators of the codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are the same, $\mathbf{P}$-weight of the dual codes may be different. Fortunately,however, $\overline{\mathbf{P}}$-weight enumerators of the dual codes are the same.

Feeding back this information we de£ne, for a given poset $\mathbf{P}$, the poset $\overline{\mathbf{P}}$ as follows:

$$\mathbf{P} \text{ and } \overline{\mathbf{P}} \text{ have the same underlying set and}$$

$$x \leq y \text{ in } \overline{\mathbf{P}} \Leftrightarrow y \leq x \text{ in } \mathbf{P}.$$

The poset $\overline{\mathbf{P}}$ is called the dual poset of $\mathbf{P}$.

De£nition 1.2: Let $\mathbf{P}$ be a poset on $[n]$. It is said that $\mathbf{P}$ admits MacWilliams identity if $\overline{\mathbf{P}}$-weight enumerator of the dual code $\mathcal{C}^\perp$ of a linear code $\mathcal{C}$ over $\mathbb{F}_q$ is uniquely determined by $\mathbf{P}$-weight enumerator of $\mathcal{C}$.

For an illustration of our de£nition, we give two classes of posets which admit MacWilliams identity.

In [11], Rosenbloom and Tsfasman introduced a new non-Hamming metric which is called the ρ-metric or the Rosenbloom-Tsfasman metric on linear spaces over £-nite £elds. The ρ-metric is de£ned on the linear space $Mat_{m,n}(\mathbb{F}_q)$, where $Mat_{m,n}(\mathbb{F}_q)$ is the set of all matrices with m-rows and n-columns over $\mathbb{F}_q$. For the sake of simplicity, we introduce it only in the case $m = 1$ and refer to [2], [12] for a general treatment. We remark that ρ-metric can be realized as a poset metric over the disjoint union of chains.

Now let $m = 1$. For $u = (u_1, u_2, \cdots, u_n) \in \mathbb{F}_q^n$, we set $\rho(0) = 0$ and $\rho(u) = max\{i \mid u_i \neq 0\}$ for $u \neq 0$. For a given linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$, we de£ne the ρ-weight enumerator for $\mathcal{C}$ by

$$W(\mathcal{C}|z) = \sum_{i=0}^{n} w_i(\mathcal{C}) z^i = \sum_{u \in \mathcal{C}} z^{\rho(u)},$$

where $w_i(\mathcal{C}) = |\{u \in \mathcal{C} \mid \rho(u) = i\}|, 0 \leq i \leq n$.
The following identity was obtained in [12, Theorem 4.4]:

$$\begin{aligned}&(qz-1)W(\mathcal{C}^{*\perp}|z) + 1 - z \\ &= |\mathcal{C}^{\perp}| z^{n+1}[q(1-z)W(\mathcal{C}|\frac{1}{qz}) + qz - 1], \qquad (1)\end{aligned}$$

where $\mathcal{C}^{*\perp} = \{v \in \mathbb{F}_q^n \mid < u, v >= 0 \text{ for all } u \in \mathcal{C}\}$, and $< u, v >= \sum_{i=1}^{n} u_i v_{n+1-i}$.

If we put $\mathbf{P} = \{1, 2, \ldots, n\}$ with order relation $1 < 2 < \ldots < n$, then ρ-metric becomes $\mathbf{P}$-metric and $W(\mathcal{C}^{*\perp}|z) = W_{\mathcal{C}^{\perp}, \overline{\mathbf{P}}}(z)$.

The MacWilliams identity for Hamming weight enumerators and the work of Skriganov [12, Theorem 4.4] state that antichain and chain on $[n], n \geq 1$, admit MacWilliams identity.

In this paper, we classify all poset structures which admit MacWilliams identity. We also derive MacWilliams identities for poset weight enumerators corresponding to such poset codes.

Section 2 gives a necessary condition for a poset $\mathbf{P}$ to admit MacWilliams identity. It will be proved that being a hierarchical poset is a necessary condition for a poset $\mathbf{P}$ to admit MacWilliams identity.

In section 3, MacWilliams identity for a hierarchical poset code is derived, and it will be proved that our necessary condition in Section 2 is also a suf£cient condition for admitting MacWilliams identity.

Section 4 examines the relationship between $\{A_{i,\mathbf{P}}\}_{i=0,\ldots,n}$ and $\{A'_{i,\overline{\mathbf{P}}}\}_{i=0,\ldots,n}$. More precisely, we will express explicitly $A'_{i,\overline{\mathbf{P}}}$ in terms of $A_{j,\mathbf{P}}, 0 \leq j \leq n$, using Krawtchouk polynomials.

II. Necessary condition for admitting MacWilliams identity

In this section, we will give a necessary condition for a poset $\mathbf{P}$ to admit MacWilliams identity. First, a hierarchical poset as the ordinal sum of antichains is introduced, and it will be proved that being a hierarchical poset is a necessary condition for a poset $\mathbf{P}$ to admit MacWilliams identity.

Let $n_1, n_2, \ldots, n_t$ be positive integers with $n_1 + n_2 + \cdots + n_t = n$. We de£ne the poset $\mathbb{H}(n; n_1, n_2, \ldots, n_t)$ on the set $\{(i, j) \mid 1 \leq i \leq t, 1 \leq j \leq n_i\}$ whose order relation is given by

$$(i, j) < (l, m) \Leftrightarrow i < l.$$

The poset $\mathbb{H}(n; n_1, n_2, \ldots, n_t)$ is called a hierarchical poset with t-levels and n-elements. For each $1 \leq i \leq t$, the subset $\{(i, j) \mid 1 \leq j \leq n_i\}$ of $\mathbb{H}(n; n_1, n_2, \ldots, n_t)$ is called i^{th}-level set of $\mathbb{H}(n; n_1, n_2, \ldots, n_t)$, and it is denoted by $\Gamma^i(\mathbb{H})$. Note that $\Gamma^i(\mathbb{H})$ is an antichain with cardinality n_i.

Let $\mathbb{H}(n; n_1, n_2, \ldots, n_t)$ be a hierarchical poset with t-levels and n-elements. From now on, we will identify the underlying set of $\mathbb{H}(n; n_1, n_2, \ldots, n_t)$ with the coordinate positions of vectors in $\mathbb{F}_q^n$ by identifying the subset $\{n_1 + n_2 + \cdots + n_{i-1} + 1, \ldots, n_1 + n_2 + \cdots + n_{i-1} + n_i\}$ of $[n]$ with the i^{th} level set $\Gamma^i(\mathbb{H})$ in an obvious way. By convention we set $n_0 = 0$.

For a poset $\mathbf{P}$, we de£ne $min(\mathbf{P}) = \{i \in \mathbf{P} \mid i$ is minimal in $\mathbf{P}\}$. The following lemma is an immediate consequence of the concepts developed so far and will be useful in the sequel.

Lemma 2.1: Let $\mathbf{P}$ be a poset on $[n]$ and $\overline{\mathbf{P}}$ be the dual poset of $\mathbf{P}$. For $u \in \mathbb{F}_q^n$, we have

$$w_{\overline{\mathbf{P}}}(u) = n \Leftrightarrow supp(u) \supseteq \min(\mathbf{P}).$$

For a given poset $\mathbf{P}$, we put $\mathbf{P}' = \mathbf{P} \setminus min(\mathbf{P})$. Then $\mathbf{P}'$ is also a poset under the partial order relation induced from that of $\mathbf{P}$.

Lemma 2.2: Let $\mathbf{P}$ be a poset of cardinality n. Suppose that $min(\mathbf{P})$ has n_1 elements. Then, for each vector $u \in \mathbb{F}_q^n$ satisfying $supp(u) \subseteq min(\mathbf{P})$,

$$q^{n-n_1} \mid |\{v \in \mathbb{F}_q^n \mid u \cdot v = 0 \text{ and } w_{\overline{\mathbf{P}}}(v) = n\}|,$$

where $a|b$ denotes that a divides b.

Proof : Without loss of generality, we may assume that $min(\mathbf{P}) = \{1, 2, \ldots, n_1\}$. Since $supp(u) \subseteq min(\mathbf{P})$, u can be written in the form $u = (a_1, \ldots, a_i, 0, \ldots, 0)$, where $0 \neq a_j \in \mathbb{F}_q$ for all $1 \leq j \leq i$ and $i \leq n_1$. Let A be the set of vectors over $\mathbb{F}_q$ of length i de£ned by

$$A := \{(b_1, \ldots, b_i) \in \mathbb{F}_q^i \mid a_1 b_1 + \cdots + a_i b_i = 0 \text{ and } b_j \neq 0 \text{ for } 1 \leq j \leq i\}.$$

Then we have

$$|\{v \in \mathbb{F}_q^n \mid u \cdot v = 0, w_{\overline{\mathbf{P}}}(v) = n\}| = |A| q^{n-n_1} (q-1)^{n_1 - i}.$$

Lemma 2.3: Suppose that $\mathbf{P}$ admits MacWilliams identity. Then, for each minimal element i in $\mathbf{P}' = \mathbf{P} \setminus min(\mathbf{P})$ and j in $min(\mathbf{P})$, we have $i \geq j$.

Proof : Let $|\mathbf{P}| = n$ and $|min(\mathbf{P})| = n_1$. If $n = n_1$, then the lemma is true. Hence we may assume that $n > n_1$.
We claim that $| < i > | = 1 + |min(\mathbf{P})|$ for each $i \in min(\mathbf{P}')$. Suppose not. Then we can choose $i \in min(\mathbf{P}')$ such that $| < i > | < 1 + |min(\mathbf{P})|$. Since $| < i > | < 1 + |min(\mathbf{P})|$, we can choose two vectors $u_1, u_2 \in \mathbb{F}_q^n$ such that $supp(u_1) = \{i\}, supp(u_2) \subseteq min(\mathbf{P})$, and $| < supp(u_1) > | = | < supp(u_2) > |$. Now we consider two linear codes $\mathcal{C}_1$ and $\mathcal{C}_2$ generated by u_1 and u_2, respectively. Since $| < supp(u_1) > | = | < supp(u_2) > |$, $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same $\mathbf{P}$-weight enumerator. It follows from our assumption that $\mathcal{C}_1^{\perp}$ and $\mathcal{C}_2^{\perp}$ have the same $\overline{\mathbf{P}}$-weight enumerator. Therefore we should have the following equation:

$$|\{v \in \mathcal{C}_1^{\perp} \mid w_{\overline{\mathbf{P}}}(v) = n\}| = |\{v \in \mathcal{C}_2^{\perp} \mid w_{\overline{\mathbf{P}}}(v) = n\}|.$$

It is immediate that

$$|\{v \in \mathcal{C}_1^\perp \mid w_{\overline{\mathbf{P}}}(v) = n\}| = q^{n-(n_1+1)}(q-1)^{n_1},$$

and it follows from Lemma 2.2 that

$$q^{n-n_1} \mid |\{v \in \mathcal{C}_2^\perp \mid w_{\overline{\mathbf{P}}}(v) = n\}|.$$

These yield that $q^{n-n_1} \mid q^{n-(n_1+1)}(q-1)^{n_1}$. However it is impossible, since q is power of a prime. This prove that $| < i > | = 1 + |min(\mathbf{P})|$ for each $i \in min(\mathbf{P}')$. The statement of Lemma 2.3 follows immediately from this fact.

Remark : If $i \in \mathbf{P}'$, then $i \geq k$ for some $k \in \min(\mathbf{P}')$. Therefore we have obtained : if $\mathbf{P}$ admits MacWilliams identity, then for $i \in \mathbf{P}'$ and $j \in min(\mathbf{P})$, we have $i \geq j$.

Lemma 2.4: If a poset $\mathbf{P}$ admits MacWilliams identity, then $\mathbf{P}'$ also admits MacWilliams identity.

Proof : Let $|\mathbf{P}| = n$ and $|min(\mathbf{P})| = n_1$. If $n = n_1$, then the lemma is true. Hence we may assume that $n > n_1$.

Let $\mathcal{C}_1', \mathcal{C}_2'$ be two linear codes of length $n - n_1$ with the same $\mathbf{P}'$-weight enumerators. We consider two linear codes of length n de£ned by

$$\mathcal{C}_i = \mathbb{F}_q^{n_1} \bigoplus \mathcal{C}_i' := \{(u, v) \mid u \in \mathbb{F}_q^{n_1}, v \in \mathcal{C}_i'\}, i = 1, 2.$$

It follows from the previous remark that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same $\mathbf{P}$-weight enumerators. Therefore $\mathcal{C}_1^\perp, \mathcal{C}_2^\perp$ have the same $\overline{\mathbf{P}}$-weight enumerators. Since $\mathcal{C}_i^\perp = \{(u, v) \mid u = 0 \in \mathbb{F}_q^{n_1}, v \in \mathcal{C}_i'^\perp\}$, for $i = 1, 2$, $\mathcal{C}_1'^\perp$ and $\mathcal{C}_2'^\perp$ have the same $\overline{\mathbf{P}'}$-weight enumerators. This proves that $\mathbf{P}'$ also admits MacWilliams identity.

From the above lemmas and inductive argument, we have the following theorem.

Theorem 2.5: If $\mathbf{P}$ admits MacWilliams identity, then $\mathbf{P}$ is a hierarchical poset.

III. MacWilliams identity for a hierarchical poset code

In this section, we will derive the MacWilliams identity for a hierarchical poset code. Let $\mathcal{C}$ be a linear $\mathbf{P}$-code of length n over $\mathbb{F}_q$. We £rst introduce the 'leveled' $\mathbf{P}$-weight enumerator $W_{\mathcal{C},\mathbf{P}}(x : y_0, y_1, \ldots, y_t)$ and obtain an equation which relates $W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x : z_{t+1}, z_t, \ldots, z_1)$ with variations of leveled $\mathbf{P}$-weight enumerator of $\mathcal{C}$. By specializing this equation, we will obtain the MacWilliams identity for a hierarchical poset code, and prove that our necessary condition in Section 2 is also a suf£cient condition for admitting the MacWilliams identity. In this section, $\mathbf{P}$ will denote a hierarchical poset with t-levels and n-elements unless otherwise speci£ed.

Let $\mathbf{P} = \mathbb{H}(n; n_1, n_2, \ldots, n_t)$ be a hierarchical poset with t-levels and n - elements on the set $[n] = \{1, 2, \ldots, n\}$. As mentioned earlier, we identify the underlying set of $\mathbf{P}$ with the coordinate positions of vectors in $\mathbb{F}_q^n$. Since $n = n_1 + \cdots + n_t$ and $\mathbb{F}_q^n = \mathbb{F}_q^{n_1} \bigoplus \mathbb{F}_q^{n_2} \bigoplus \cdots \bigoplus \mathbb{F}_q^{n_t}$, for $u \in \mathbb{F}_q^n$, we may write

$$u = (u_1, u_2, \ldots, u_t), \text{ and } \quad u_i \in \mathbb{F}_q^{n_i}.$$

For an integer $0 \leq i \leq t$, we also use the following notation:

$$\widehat{n_i} = n - (n_1 + \cdots + n_i) = n_{i+1} + \cdots + n_t,$$
$$\widetilde{u_{i+1}} = (u_{i+1}, \ldots, u_t) \in \mathbb{F}_q^{\widehat{n_i}}.$$

For a linear $\mathbf{P}$-code $\mathcal{C}$, we de£ne $\mathcal{C}_i$ and $\mathcal{C}_i^1$ as follows:

$$\mathcal{C}_i = \{u \in \mathcal{C} \mid u_{i+1} = \cdots = u_t = 0\}, \text{ and}$$
$$\mathcal{C}_i^1 = \{u \in \mathcal{C}_i \mid u_i \neq 0\}.$$

Let $\mathcal{C}$ be a linear $\mathbf{P}$-code of length n over $\mathbb{F}_q$. We introduce the 'leveled' $\mathbf{P}$-weight enumerator $W_{\mathcal{C},\mathbf{P}}(x : y_0, y_1, \ldots, y_t)$ of $\mathcal{C}$ as follows:

$$\begin{aligned}
W_{\mathcal{C},\mathbf{P}}(x : y_0, y_1, \ldots, y_t) &= \sum_{u \in \mathcal{C}} x^{w_P(u)} y_{s_P(u)} \\
&= A_{0,\mathbf{P}} y_0 + (A_{1,\mathbf{P}} x + \cdots + A_{n_1,\mathbf{P}} x^{n_1}) y_1 \\
&+ (A_{n_1+1,\mathbf{P}} x^{n_1+1} + \cdots + A_{n_1+n_2,\mathbf{P}} x^{n_1+n_2}) y_2 \\
&+ \cdots \\
&+ (A_{n_1+\cdots+n_{t-1}+1,\mathbf{P}} x^{n_1+\cdots+n_{t-1}+1} + \cdots \\
&+ A_{n_1+\cdots+n_t,\mathbf{P}} x^{n_1+\cdots+n_t}) y_t,
\end{aligned}$$

where $s_P(u) = max\{i | u_i \neq 0\}$ in the expression $u = (u_1, \ldots, u_t)$ and $A_{i,\mathbf{P}} = |\{u \in \mathcal{C} \mid w_\mathbf{P}(u) = i\}|$.

For the sake of simplicity in our calculation, we also introduce the i^{th}-level $\mathbf{P}$-weight enumerator $LW_{\mathcal{C},\mathbf{P}}^{(i)}(x), 1 \leq i \leq t$, as follows:

$$\begin{aligned}
LW_{\mathcal{C},\mathbf{P}}^{(i)}(x) &:= \sum_{j=1}^{n_i} A_{n_1+\cdots+n_{i-1}+j,\mathbf{P}} x^{n_1+\cdots+n_{i-1}+j} \\
&= (A_{n_1+\cdots+n_{i-1}+1,\mathbf{P}} x^1 + \cdots + A_{n_1+\cdots+n_i,\mathbf{P}} x^{n_i}) x^{n-\widehat{n_{i-1}}}.
\end{aligned}$$

By convention, we put $LW_{\mathcal{C},\mathbf{P}}^{(0)}(x) := A_{0,\mathbf{P}}$.

Remark : (a) If we put $y_0 = y_1 = \cdots = y_t = 1$, then the 'leveled' $\mathbf{P}$-weight enumerator of $\mathcal{C}$ becomes the 'usual' $\mathbf{P}$-weight enumerator of $\mathcal{C}$:

$$W_{\mathcal{C},\mathbf{P}}(x : 1, \ldots, 1) = W_{\mathcal{C},\mathbf{P}}(x) = \sum_{i=0}^{t} LW_{\mathcal{C},\mathbf{P}}^{(i)}(x). \quad (2)$$

(b) If we put $y_j = 1$ for $1 \leq j \leq i$ and $y_k = 0$ for $k > i$, then the 'leveled' $\mathbf{P}$-weight enumerator of $\mathcal{C}$ becomes the $\mathbf{P}$-weight enumerator of the subspace $\mathcal{C}_i$

(c) It is easy to see that

$$W_{\mathcal{C}_i,\mathbf{P}}(x) - W_{\mathcal{C}_{i-1},\mathbf{P}}(x) = LW_{\mathcal{C},\mathbf{P}}^{(i)}(x) = \sum_{u \in \mathcal{C}_i^1} x^{w_\mathbf{P}(u)}. \quad (3)$$

Recall that an additive character χ on $\mathbb{F}_q$ is just a homomorphism from the additive group of $\mathbb{F}_q$ into the multiplicative group of complex numbers of magnitude 1. We give the following lemmas about additive characters on $\mathbb{F}_q$ which play an important role in the proof of the main theorem without proof. See [6], [7] for detailed discussion on additive characters.

Lemma 3.1: Let χ be a nontrivial additive character of $\mathbb{F}_q$ and α be a £xed element of $\mathbb{F}_q$. Then

$$\sum_{\beta \in \mathbb{F}_q} \chi(\alpha\beta) = \begin{cases} q & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha \neq 0 \end{cases}.$$

Lemma 3.2: Let χ be a nontrivial additive character of $\mathbb{F}_q$. Then, for any linear code $\mathcal{C}$ over $\mathbb{F}_q$,

$$\sum_{v \in \mathcal{C}} \chi(u \cdot v) = \begin{cases} 0 & \text{if } u \notin \mathcal{C}^\perp \\ |\mathcal{C}| & \text{if } u \in \mathcal{C}^\perp \end{cases}.$$

Let f be a complex-valued function de£ned on $\mathbb{F}_q^n$. The Hadamard transform $\widehat{f}$ of f is de£ned by

$$\widehat{f}(u) = \sum_{v \in \mathbb{F}_q^n} \chi(u \cdot v) f(v).$$

The following lemma, which is called the discrete Poisson summation formula, is an easy consequence of Lemma 3.2.

Lemma 3.3: Let $\mathcal{C}$ be a linear code of length n over $\mathbb{F}_q$ and f be a function on $\mathbb{F}_q^n$. Then

$$\sum_{u \in \mathcal{C}^\perp} f(u) \ = \ \frac{1}{|\mathcal{C}|} \sum_{u \in \mathcal{C}} \widehat{f}(u).$$

Lemma 3.4: If a function f is de£ned on $\mathbb{F}_q^n$ by $f(u) = x^{w_H(u)}$, then its Hadamard transform $\widehat{f}$ of f is given by

$$\begin{aligned} \widehat{f}(u) &= \sum_{v \in \mathbb{F}_q^n} \chi(u \cdot v) f(v) \\ &= (1+(q-1)x)^{n-w_H(u)}(1-x)^{w_H(u)}. \end{aligned}$$

The MacWilliams identity for Hamming weight enumerators can be obtained by applying discrete Poisson summation formula to the complex-valued function $f(u) = x^{w_H(u)}$. We now apply discrete Poisson summation formula to the complex-valued function $f(u) = x^{w_{\overline{\mathbf{P}}}(u)} z_{s_{\overline{\mathbf{P}}}(u)}$, where $s_{\overline{\mathbf{P}}}(u) = min\{i \mid u_i \neq 0\}$ in the expression $u = (u_1, \ldots, u_t), u_i \in \mathbb{F}_q^{n_i}$. By convention, we set $s_{\overline{\mathbf{P}}}(0) = t+1$. We now analyze the value $\widehat{f}(u)$ in detail. For an integer $0 \le i \le t$, we put

$$B_i = \{u = (u_1, \ldots, u_t) \in \mathbb{F}_q^n \mid u_1 = \ldots = u_i = 0, \ \text{and} \ u_{i+1} \neq 0\}.$$

Note that $\mathbb{F}_q^n = \bigcup_{i=0}^{t} B_i$ is a disjoint union.

It follows from the above observation that

$$\begin{aligned} \widehat{f}(u) &= \sum_{v \in \mathbb{F}_q^n} \chi(u \cdot v) f(v) \\ &= \sum_{i=0}^{t} \sum_{v \in B_i} \chi(u \cdot v) x^{w_{\overline{\mathbf{P}}}(v)} z_{s_{\overline{\mathbf{P}}}(v)}. \qquad (4) \end{aligned}$$

Denote the inner sum in (4) by $S_i(u), 0 \le i \le t$. For $v \in B_i$ with $i < t$, we have $w_{\overline{\mathbf{P}}}(v) = n_{i+2} + \cdots + n_t + w_H(v_{i+1}) = \widehat{n_{i+1}} + w_H(v_{i+1})$ and $s_{\overline{\mathbf{P}}}(v) = i+1$, where $\widehat{n}_i = n - (n_1 + n_2 + \cdots + n_i)$. For $v \in \mathbb{F}_q^n$, we write $v = (v_1, v_2, \cdots, v_i, \widetilde{v_{i+1}})$, where $\widetilde{v_{i+1}} = (v_{i+1}, v_{i+2}, \ldots, v_t) \in \mathbb{F}_q^{\widehat{n_i}}$. Hence the inner sum $S_i(u)$ in (4) for $i < t$ is

$$\begin{aligned} S_i(u) &= \sum_{v \in B_i} \chi(u \cdot v) x^{w_{\overline{\mathbf{P}}}(v)} z_{s_{\overline{\mathbf{P}}}(v)} \\ &= x^{\widehat{n_{i+1}}} z_{i+1} \sum_{v \in B_i} \chi(u \cdot v) x^{w_H(v_{i+1})} \\ &= x^{\widehat{n_{i+1}}} z_{i+1} \sum_{\widetilde{v_{i+2}} \in \mathbb{F}_q^{\widehat{n_{i+1}}}} \chi(\widetilde{u_{i+2}} \cdot \widetilde{v_{i+2}}) \\ &\quad \times \sum_{v_{i+1} \neq 0 \in \mathbb{F}_q^{n_{i+1}}} \chi(u_{i+1} \cdot v_{i+1}) x^{w_H(v_{i+1})}. \end{aligned}$$

It follows from Lemma 3.4 that

$$\begin{aligned} &\sum_{v_{i+1} \neq 0 \in \mathbb{F}_q^{n_{i+1}}} \chi(u_{i+1} \cdot v_{i+1}) x^{w_H(v_{i+1})} \\ &= (\frac{1-x}{Q(x)})^{n_{i+1}} Q(x)^{w_H(u_{i+1})} - 1, \end{aligned}$$

where $Q(x) = \frac{1-x}{1+(q-1)x}$. Hence we have

$$\begin{aligned} S_i(u) &= x^{\widehat{n_{i+1}}} z_{i+1} \sum_{\widetilde{v_{i+2}} \in \mathbb{F}_q^{\widehat{n_{i+1}}}} \chi(\widetilde{u_{i+2}} \cdot \widetilde{v_{i+2}}) \\ &\quad \times \Big((\frac{1-x}{Q(x)})^{n_{i+1}} Q(x)^{w_H(u_{i+1})} - 1 \Big) \\ &= x^{\widehat{n_{i+1}}} z_{i+1} \Big((\frac{1-x}{Q(x)})^{n_{i+1}} Q(x)^{w_H(u_{i+1})} - 1 \Big) \\ &\quad \times \sum_{\widetilde{v_{i+2}} \in \mathbb{F}_q^{\widehat{n_{i+1}}}} \chi(\widetilde{u_{i+2}} \cdot \widetilde{v_{i+2}}). \end{aligned}$$

For $i < t$, it follows from the Lemma 3.2 that

$$S_i(u) = \begin{cases} 0 & \text{if } \widetilde{u_{i+2}} \neq 0 \in \mathbb{F}_q^{\widehat{n_{i+1}}} \\ (qx)^{\widehat{n_{i+1}}} z_{i+1} & \\ \times \Big((\frac{1-x}{Q(x)})^{n_{i+1}} Q(x)^{w_H(u_{i+1})} - 1 \Big) & \text{if } \widetilde{u_{i+2}} = 0 \ . \end{cases} \qquad (5)$$

For $i = t$, it is clear that $S_t(u) = z_{t+1}$.

Hence we have $\widehat{f}(u) = z_{t+1} + \sum_{i=0}^{t-1} S_i(u)$, where $S_i(u)$ is given by (5).

Let $\mathcal{C}$ be a linear $\mathbf{P}$-code of length n over $\mathbb{F}_q$, where $\mathbf{P} = \mathbb{H}(n : n_1, \ldots, n_t)$ is a hierarchical poset with t-levels and n-elements. For $0 \le i \le t$, we consider the subspace $\mathcal{C}_i$ of $\mathcal{C}$ de£ned by

$$\mathcal{C}_i = \{u = (u_1, \ldots, u_t) \in \mathcal{C} | u_{i+1} = \cdots = u_t = 0\}.$$

Note that $\mathcal{C}_i$ is the subset of the codewords u of $\mathcal{C}$ satisfying $\widetilde{u_{i+1}} = 0$. Therefore it follows from (5) that

$$\begin{aligned} &\sum_{u \in \mathcal{C}} S_i(u) = \sum_{u \in \mathcal{C}_{i+1}} S_i(u) \\ &= (qx)^{\widehat{n_{i+1}}} z_{i+1} \sum_{u \in \mathcal{C}_{i+1}} \Big((\frac{1-x}{Q(x)})^{n_{i+1}} Q(x)^{w_H(u_{i+1})} - 1 \Big) \qquad (6) \end{aligned}$$

Denote the right hand side of the sum in (6) by $S(\mathcal{C}_{i+1})$. Then,

$$\begin{aligned} &S(\mathcal{C}_{i+1}) \\ &= \sum_{u \in \mathcal{C}_{i+1}} \Big((\frac{1-x}{Q(x)})^{n_{i+1}} Q(x)^{w_H(u_{i+1})} - 1 \Big) \\ &= (1+(q-1)x)^{n_{i+1}} \sum_{u \in \mathcal{C}_{i+1}} Q(x)^{w_H(u_{i+1})} - |\mathcal{C}_{i+1}| \qquad (7) \end{aligned}$$

Put $\mathcal{C}_{i+1}^0 = \{u \in \mathcal{C}_{i+1} \mid u_{i+1} = 0\}$ and $\mathcal{C}_{i+1}^1 = \{u \in \mathcal{C}_{i+1} \mid u_{i+1} \neq 0\}$. For each $u \in \mathcal{C}_{i+1}^1$, we have

$$w_{\mathbf{P}}(u) = w_H(u_{i+1}) + n_1 + n_2 + \cdots n_i = w_H(u_{i+1}) + (n - \widehat{n}_i).$$

It follows from this observation that the inner sum in (7) is

$$(\tfrac{1+(q-1)x}{1-x})^{n-\widehat{n}_i} \sum_{u \in \mathcal{C}_{i+1}^1} (\tfrac{1-x}{1+(q-1)x})^{w_{\mathbf{P}}(u)} + |\mathcal{C}_i|.$$

It follows form (2) and (3) that

$$\sum_{u\in\mathcal{C}} S_i(u) = \sum_{u\in\mathcal{C}_{i+1}} S_i(u)$$
$$= (qx)^{\widehat{n_{i+1}}}(1+(q-1)x)^{n_{i+1}}(\frac{1}{Q(x)})^{n-\widehat{n_i}}$$
$$\times \quad z_{i+1} \sum_{u\in\mathcal{C}^1_{i+1}} Q(x)^{w_{\mathbf{P}}(u)}$$
$$+(qx)^{\widehat{n_{i+1}}} z_{i+1}\ \big(\ |\mathcal{C}_i|(1+(q-1)x)^{n_{i+1}} - |\mathcal{C}_{i+1}|\ \big)$$
$$= (\frac{qx}{1-x})^n(\frac{1+(q-1)x}{qx})^{n-\widehat{n_{i+1}}}(1-x)^{\widehat{n}_i}$$
$$\times\ LW^{(i+1)}_{\mathcal{C},\mathbf{P}}(Q(x))z_{i+1}$$
$$+(qx)^{\widehat{n_{i+1}}} z_{i+1}\ \big(\ |\mathcal{C}_i|(1+(q-1)x)^{n_{i+1}} - |\mathcal{C}_{i+1}|\ \big) \quad (8)$$

Since

$$\widehat{f}(u) = \sum_{i=0}^{t}\sum_{v\in B_i}\chi(u\cdot v)x^{w_{\overline{\mathbf{P}}}(v)}z_{s_{\overline{\mathbf{P}}}(v)} = z_{t+1} + \sum_{i=0}^{t-1} S_i(u),$$

we have

$$\sum_{u\in\mathcal{C}}\widehat{f}(u)$$
$$= |\mathcal{C}|z_{t+1} + \sum_{u\in\mathcal{C}}\sum_{i=0}^{t-1} S_i(u)$$
$$= |\mathcal{C}|z_{t+1} + (\frac{qx}{1-x})^n\sum_{i=0}^{t-1} a_i(x)LW^{(i+1)}_{\mathcal{C},\mathbf{P}}(Q(x))z_{i+1}$$
$$+\sum_{i=0}^{t-1} b_i(x)|\mathcal{C}_i|z_{i+1} - \sum_{i=0}^{t-1}(qx)^{\widehat{n_{i+1}}}|\mathcal{C}_{i+1}|z_{i+1}, \quad (9)$$

where $a_i(x) = (\frac{1+(q-1)x}{qx})^{n-\widehat{n_{i+1}}}(1-x)^{\widehat{n_i}}$ and $b_i(x) = (1+(q-1)x)^{n_{i+1}}(qx)^{\widehat{n_{i+1}}}$.

Since $W_{\mathcal{C},\mathbf{P}}(x : y_0,\ldots,y_t) = \sum_{i=0}^{t} LW^{(i)}_{\mathcal{C},\mathbf{P}}(x)y_i$, $Q(x) = \frac{1-x}{1+(q-1)x}$, and $a_i(x) = (\frac{1+(q-1)x}{qx})^{n-\widehat{n_{i+1}}}(1-x)^{\widehat{n_i}}$, the first summation in (9) becomes

$$(\frac{qx}{1-x})^n W_{\mathcal{C},\mathbf{P}}(\frac{1-x}{1+(q-1)x} : f_0, f_1,\ldots,f_t), \quad (10)$$

where

$$f_i = \begin{cases} 0 & \text{if } i=0 \\ (\frac{1+(q-1)x}{qx})^{n-\widehat{n_i}}(1-x)^{\widehat{n_{i-1}}}z_i & \text{if } i\geq 1\ . \end{cases} \quad (11)$$

Since $|\mathcal{C}_i| = A_{0,\mathbf{P}} + (A_{1,\mathbf{P}}+\cdots+A_{n_1,\mathbf{P}}) + \cdots + (A_{n_1+\cdots+n_{i-1}+1,\mathbf{P}}+\cdots+A_{n_1+\cdots+n_i,\mathbf{P}})$, we have the following equation:

$$\sum_{i=0}^{t-1} b_i(x)|\mathcal{C}_i|z_{i+1}$$
$$= b_0(x)|\mathcal{C}_0|z_1 + b_1(x)|\mathcal{C}_1|z_2 + \cdots + b_{t-1}(x)|\mathcal{C}_{t-1}|z_t$$
$$= A_{0,\mathbf{P}}(b_0(x)z_1 + b_1(x)z_2 + \cdots + b_{t-1}(x)z_t)$$
$$+(A_{1,\mathbf{P}}+\cdots+A_{n_1,\mathbf{P}})(b_1(x)z_2+\cdots+b_{t-1}(x)z_t)$$
$$+\cdots+$$
$$+(A_{n_1+\cdots+n_{t-2}+1,\mathbf{P}}+\cdots+A_{n_1+\cdots+n_{t-1},\mathbf{P}})b_{t-1}(x)z_t.$$

Let $g_j = \sum_{i=j}^{t-1} b_i(x)z_{i+1}$, for $0\leq j\leq t-1$ and $g_t = 0$. (Recall that $b_i(x) = (qx)^{\widehat{n_{i+1}}}(1+(q-1)x)^{n_{i+1}}$.)

Since $LW^{(i)}_{\mathcal{C},\mathbf{P}}(1) = |\mathcal{C}_i| - |\mathcal{C}_{i-1}| = A_{n_1+\cdots+n_{i-1}+1,\mathbf{P}}+\cdots+A_{n_1+\cdots+n_i,\mathbf{P}}$ and $W_{\mathcal{C},\mathbf{P}}(1 : y_0,\ldots,y_t) = \sum_{i=0}^{t} LW^{(i)}_{\mathcal{C},\mathbf{P}}(1)y_i$, the second summation in (9) becomes

$$\begin{aligned}\sum_{i=0}^{t-1} b_i(x)|\mathcal{C}_i|z_{i+1} &= \sum_{i=0}^{t} LW^{(i)}_{\mathcal{C},\mathbf{P}}(1)g_i \\ &= W_{\mathcal{C},\mathbf{P}}(1 : g_0, g_1,\ldots,g_t), \end{aligned} \quad (12)$$

where

$$g_j = \begin{cases} \sum_{i=j}^{t-1}(qx)^{\widehat{n_{i+1}}}(1+(q-1)x)^{n_{i+1}}z_{i+1} & \text{if } 0\leq j\leq t-1 \\ 0 & \text{if } j=t\ . \end{cases} \quad (13)$$

In the same manner, the last summation in (9) becomes

$$\sum_{i=0}^{t-1}(qx)^{\widehat{n_{i+1}}}z_{i+1}|\mathcal{C}_{i+1}| = W_{\mathcal{C},\mathbf{P}}(1 : h_0, h_1,\ldots,h_t) \quad (14)$$

where

$$h_j = \begin{cases} \sum_{i=j}^{t}(qx)^{\widehat{n_i}}z_i & \text{if } 1\leq j\leq t \\ \sum_{i=1}^{t}(qx)^{\widehat{n_i}}z_i & \text{if } j=0\ . \end{cases} \quad (15)$$

By applying discrete Poisson summation formula

$$\sum_{u\in\mathcal{C}^\perp} f(u) = \frac{1}{|\mathcal{C}|}\sum_{u\in\mathcal{C}}\widehat{f}(u),$$

we finally obtain the following theorem.

Theorem 3.5: Let $\mathbf{P} = \mathbb{H}(n : n_1,\ldots,n_t)$ be the hierarchical poset of n-elements with t-levels and $\mathcal{C}$ be a linear $\mathbf{P}$-code of length n over $\mathbb{F}_q$. Then

$$W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x : z_{t+1},\ldots,z_1) = \frac{1}{|\mathcal{C}|}\sum_{u\in\mathcal{C}}\widehat{f}(u)$$
$$= z_{t+1} + \frac{1}{|\mathcal{C}|}\ \big(\ (\frac{qx}{1-x})^n W_{\mathcal{C},\mathbf{P}}(Q(x) : f_0,\ldots,f_t)$$
$$+W_{\mathcal{C},\mathbf{P}}(1 : g_0,\ldots,g_t) - W_{\mathcal{C},\mathbf{P}}(1 : h_0,\ldots,h_t)\ \big),$$

where $Q(x) = \frac{1-x}{1+(q-1)x}$, and f_i, g_i, h_i are given by Equations $(11), (13)$ and (15).

If we put $z_1 = z_2 = \cdots = z_{t+1} = 1$ in Theorem 3.5, then $W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x : 1, 1,\ldots,1)$ becomes the 'usual' the $\overline{\mathbf{P}}$-weight enumerator $W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x)$ of the dual code $\mathcal{C}^\perp$ on the poset $\overline{\mathbf{P}}$. Hence the $\overline{\mathbf{P}}$-weight enumerator of the dual code $\mathcal{C}^\perp$ is uniquely determined by the $\mathbf{P}$-weight enumerator of $\mathcal{C}$ itself.

Combining this with Theorem 2.5, we obtain the following main theorem.

Theorem 3.6: A poset $\mathbf{P}$ admits MacWilliams identity if and only if $\mathbf{P}$ is a hierarchical poset.

As an illustration, we apply Theorem 3.5 to special cases, and compare our results with the previous result.

Let $\mathbf{P}$ be an antichain of n-elements, that is, $\mathbf{P}$ is a hierarchical poset with 1-level. Put $z_1 = z_2 = 1$. The equations (11), (13) and (15) can be written as follows:

$$f_0 = 0, \quad f_1 = \Big(\frac{1+(q-1)x}{qx} \Big)^n (1-x)^n, \tag{16}$$
$$g_0 = (1+(q-1)x)^n, \quad g_1 = 0, \tag{17}$$
$$h_0 = h_1 = 1. \tag{18}$$

After a simple calculation, we obtain the following corollary.

Corollary 3.7: Let $\mathbf{P}$ be an anti-chain of n-elements and $\mathcal{C}$ be a linear $\mathbf{P}$-code over $\mathbb{F}_q$. Then,

$$W_{\mathcal{C}^\perp}(x) = W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x:1,1) = \frac{1}{|\mathcal{C}|}(1+(q-1)x)^n W_{\mathcal{C}} \Big(\frac{1-x}{1+(q-1)x} \Big). \tag{19}$$

We remark that (19) is exactly the 'classical' MacWilliams identity for Hamming weight enumerators (cf [7, Ch5, Theorem 13]).

Let $\mathbf{P}$ be a chain of t-elements, that is, $\mathbf{P}$ is a hierarchical poset of t-levels and $n_1 = \cdots = n_t = 1$ so that $\widehat{n_i} = t-i$ for $0 \le i \le t$. Put $z_1 = z_2 = \ldots = z_t = 1$. Then we have the following equations:

$$f_i = \begin{cases} 0 & \text{if } i = 0 \\ (1-x)^{t+1} \Big(\frac{1+(q-1)x}{qx(1-x)} \Big)^i & \text{if } 1 \le i \le t \end{cases}, \tag{20}$$

$$g_i = \frac{1+(q-1)x}{qx-1}((qx)^{t-i}-1) \qquad \text{if } 0 \le i \le t, \tag{21}$$

$$h_i = \begin{cases} \frac{1}{qx-1}((qx)^{t-i+1}-1) & \text{if } 1 \le i \le t \\ \frac{1}{qx-1}((qx)^t-1) & \text{if } 1 \le i \le t. \end{cases} \tag{22}$$

From (20), (21), and (22), we have the followings:

$$\Big(\frac{qx}{1-x}\Big)^t W_{\mathcal{C},\mathbf{P}} \Big(\frac{1-x}{1+(q-1)x} : f_0, \ldots, f_t \Big) = (1-x)(qx)^t \Big(W_{\mathcal{C},\mathbf{P}}\big(\frac{1}{qx}\big) - 1 \Big), \tag{23}$$

$$W_{\mathcal{C},\mathbf{P}}(1 : g_0, \ldots, g_t) = \frac{1+(q-1)x}{qx-1} \Big((qx)^t W_{\mathcal{C},\mathbf{P}}\big(\frac{1}{qx}\big) - |\mathcal{C}| \Big), \tag{24}$$

$$W_{\mathcal{C},\mathbf{P}}(1 : h_0, \ldots, h_t) = \frac{(qx)^{t+1}}{qx-1} W_{\mathcal{C},\mathbf{P}}\big(\frac{1}{qx}\big) - \frac{1}{qx-1}|\mathcal{C}| - (qx)^t. \tag{25}$$

By applying (23) – (25) to Theorem 3.5, we have the followings:

$$W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x) = W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x:1,1,\ldots,1) = 1 - \frac{(q-1)x}{qx-1} + \frac{1}{|\mathcal{C}|} \Big(\frac{(qx)^{t+1}(1-x)}{qx-1} W_{\mathcal{C},\mathbf{P}}\big(\frac{1}{qx}\big) + x(qx)^t \Big). \tag{26}$$

Note that $|\mathcal{C}||\mathcal{C}^\perp| = q^t$ and some computations yield the following corollary.

Corollary 3.8: Let $\mathbf{P}$ be a chain of n-elements and $\mathcal{C}$ a linear $\mathbf{P}$-code over $\mathbb{F}_q$. Then,

$$(qx-1)W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x) + 1 - x = |\mathcal{C}^\perp| x^{t+1} \Big(q(1-x)W_{\mathcal{C},\mathbf{P}}\big(\frac{1}{qx}\big) + qx - 1 \Big). \tag{27}$$

This is the same as the result in [12, Theorem 4.4].

IV. Relationship between weight distributions

Let $\mathbf{P} = \mathbb{H}(n; n_1, \ldots, n_t)$ be a hierarchical poset of n-elements with t-levels and $\overline{\mathbf{P}}$ be its dual poset. Let $\mathcal{C}$ be a linear $\mathbf{P}$-code of length n over $\mathbb{F}_q$, and let $\{A_{i,\mathbf{P}}\}_{i=0,\ldots,n}$ (resp. $\{A'_{i,\overline{\mathbf{P}}}\}_{i=0,\ldots,n}$) be the weight distributions of the $\mathbf{P}$(resp. $\overline{\mathbf{P}}$) -code $\mathcal{C}$ (resp. $\mathcal{C}^\perp$), that is, $A_{i,\mathbf{P}} = |\{u \in \mathcal{C} \mid w_{\mathbf{P}}(u) = i\}|$ while $A'_{i,\overline{\mathbf{P}}} = |\{v \in \mathcal{C}^\perp \mid w_{\overline{\mathbf{P}}}(v) = i\}|$. In this section, we will study the relationship between $\{A_{i,\mathbf{P}}\}_{i=0,\ldots,n}$ and $\{A'_{i,\overline{\mathbf{P}}}\}_{i=0,\ldots,n}$. More precisely, we will express explicitly $A'_{i,\overline{\mathbf{P}}}$ in terms of $A_{j,\mathbf{P}}, 0 \le j \le n$, using Krawtchouk polynomials.

Before proceeding with hierarchical posets, we brie¤y review the relationship between $\{A'_i\}_{i=0,\ldots,n}$ and $\{A_i\}_{i=0,\ldots,n}$, where $A'_i = |\{u \in \mathcal{C}^\perp \mid w_H(u) = i\}|$ and $A_i = |\{u \in \mathcal{C} \mid w_H(u) = i\}|$. For convenience, we set $\gamma = q-1$ in this section.

De£nition 4.1: For any prime power q and positive integer n, the Krawtchouk polynomial is de£ned by

$$P_k(x:n) = \sum_{j=0}^{k} (-1)^j \gamma^{k-j} \binom{x}{j}\binom{n-x}{k-j}, k = 0, 1, \ldots, n.$$

These polynomials have the generating function

$$\big(1+\gamma x \big)^{n-i} (1-x)^i = \sum_{k=0}^{n} P_k(i:n)x^k, 0 \le i \le n. \tag{28}$$

Theorem 4.2: (Relationship between Hamming weight distributions) Let $\mathcal{C}$ be a linear code of length n over $\mathbb{F}_q$. Then

$$A'_k = \frac{1}{|\mathcal{C}|}\sum_{i=0}^{n} A_i P_k(i:n),$$

where $A'_k = |\{u \in \mathcal{C}^\perp \mid w_H(u) = k\}|$ and $A_i = |\{u \in \mathcal{C} \mid w_H(u) = i\}|$.

Let $\mathbf{P} = \mathbb{H}(n; n_1, \ldots, n_t)$ be a hierarchical poset of n-elements with t-levels and $\mathcal{C}$ be a linear $\mathbf{P}$-code of length n over $\mathbb{F}_q$. We de£ne $LW^{(i)}_{\mathcal{C},\mathbf{P}}(x,y)$ as follows:

$$LW^{(i)}_{\mathcal{C},\mathbf{P}}(x,y) := \sum_{j=1}^{n_i} A_{n_1+\cdots+n_{i-1}+j} x^{n_i-j} y^{n_1+\cdots+n_{i-1}+j}. \tag{29}$$

Then it is easy to see that

$$LW^{(i)}_{\mathcal{C},\mathbf{P}}(x,y) = W_{\mathcal{C}_i,\mathbf{P}}(x,y) - x^{n_i} W_{\mathcal{C}_{i-1},\mathbf{P}}(x,y). \tag{30}$$

The $LW^{(i)}_{\mathcal{C},\mathbf{P}}(x,y)$ is also called the i^{th} level $\mathbf{P}$-weight enumerator of $\mathcal{C}$.

By setting $z_1 = z_2 = \cdots = z_{t+1} = 1$ in Theorem 3.5, we obtain the following theorem.

Theorem 4.3: Let $\mathbf{P} = \mathbb{H}(n; n_1, \ldots, n_t)$ and $\mathcal{C}$ be a linear $\mathbf{P}$-code over $\mathbb{F}_q$. Then

$$\begin{aligned} & W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x) \\ &= 1 + \frac{1}{|\mathcal{C}|}\sum_{i=0}^{t-1}\frac{(qx)^{\widehat{n_{i+1}}}}{(1-x)^{n-\widehat{n}_i}} LW^{(i+1)}_{\mathcal{C},\mathbf{P}}(1+\gamma x, 1-x) \\ &+ \frac{1}{|\mathcal{C}|}\sum_{i=0}^{t-1}(qx)^{\widehat{n_{i+1}}}\ \left(\ (1+\gamma x)^{n_{i+1}}|\mathcal{C}_i| - |\mathcal{C}_{i+1}|\ \right). \quad (31) \end{aligned}$$

Since $n - \widehat{n}_i = n_1 + \cdots + n_i$, the following equation can be easily derived from $(28), (29)$, and (30):

$$\begin{aligned} & \frac{(qx)^{\widehat{n_{i+1}}}}{(1-x)^{n-\widehat{n}_i}} LW^{(i+1)}_{\mathcal{C},\mathbf{P}}(1+\gamma x, 1-x) \\ = \quad & (qx)^{\widehat{n_{i+1}}}\sum_{k=0}^{n_{i+1}}\left(\ \sum_{j=1}^{n_{i+1}} A_{n_1+\cdots+n_i+j} P_k(j : n_{i+1})\ \right) x^k. \end{aligned}$$

For convenience, we set

$$a_k(j : n_{i+1}) := \sum_{j=1}^{n_{i+1}} A_{n_1+\cdots+n_i+j} P_k(j : n_{i+1}).$$

Since $P_0(j : n_{i+1}) = 1$, we have

$$a_0(j : n_{i+1}) = \sum_{j=1}^{n_{i+1}} A_{n_1+\cdots+n_i+j} = |\mathcal{C}_{i+1}| - |\mathcal{C}_i|. \quad (32)$$

Therefore, the £rst summation in (31) becomes

$$\frac{1}{|\mathcal{C}|}\sum_{i=0}^{t-1}(qx)^{\widehat{n_{i+1}}}\ \left(\ \sum_{k=0}^{n_{i+1}} a_k(j : n_{i+1}) x^k\ \right). \quad (33)$$

It follows from the binomial series that the last summation in (31) becomes

$$\frac{1}{|\mathcal{C}|}\sum_{i=0}^{t-1}(qx)^{\widehat{n_{i+1}}}\ \left(\ |\mathcal{C}_i| - |\mathcal{C}_{i+1}| + \sum_{k=1}^{n_{i+1}}\binom{n_{i+1}}{k}\gamma^k |\mathcal{C}_i| x^k\ \right). \quad (34)$$

By (32), (33), and (34), the RHS of (31) in the Theorem 4.3 becomes

$$\begin{aligned} & 1 + \frac{1}{|\mathcal{C}|}\sum_{i=0}^{t-1}(qx)^{\widehat{n_{i+1}}} \\ & \times \sum_{k=1}^{n_{i+1}}\ \left(\ a_k(j : n_{i+1}) + \binom{n_{i+1}}{k}\gamma^k|\mathcal{C}_i|\ \right) x^k. \quad (35) \end{aligned}$$

On the other hand, the LHS of (31) in the Theorem 4.3 can be written as

$$\begin{aligned} & W_{\mathcal{C}^\perp,\overline{\mathbf{P}}}(x) \\ &= A'_{0,\overline{\mathbf{P}}} + A'_{1,\overline{\mathbf{P}}}x + \cdots + A'_{n_t,\overline{\mathbf{P}}}x^{n_t} \\ &+ \left(\ A'_{n_t+1,\overline{\mathbf{P}}}x + \cdots + A'_{n_t+n_{t-1},\overline{\mathbf{P}}}x^{n_{t-1}}\ \right) x^{n_t} \\ &+ \cdots \\ &+ \left(\ A'_{n_t+\cdots+n_2+1,\overline{\mathbf{P}}}x + \cdots + A'_{n_t+\cdots+n_1,\overline{\mathbf{P}}}x^{n_1}\ \right) x^{n_t+\cdots+n_2} \\ &= 1 + \sum_{i=0}^{t-1} x^{\widehat{n}_{i+1}} \sum_{k=1}^{n_{i+1}} A'_{n_t+\cdots+n_{i+2}+k,\overline{\mathbf{P}}} x^k. \quad (36) \end{aligned}$$

Since $a_k(j : n_{i+1}) = \sum_{j=1}^{n_{i+1}} A_{n_1+\cdots+n_i+j,\mathbf{P}} P_k(j : n_{i+1})$ and $|\mathcal{C}_i| = \sum_{k=0}^{n_1+\cdots+n_i} A_k$, we have the following theorem from (35) and (36). (Note $A'_{0,\overline{\mathbf{P}}} = A_{0,\mathbf{P}} = 1$.)

Theorem 4.4: Let $\mathbf{P} = \mathbb{H}(n; n_1, \ldots, n_t)$ be a hierarchical poset of n-elements with t-levels and $\mathcal{C}$ be a linear $\mathbf{P}$-code of length n over $\mathbb{F}_q$. Then, for each $0 \le i \le t-1$, $1 \le k \le n_{i+1}$,

$$\begin{aligned} & A'_{n_t+\cdots n_{i+2}+k,\overline{\mathbf{P}}} \\ &= \frac{q^{\widehat{n_{i+1}}}}{|\mathcal{C}|}\sum_{j=1}^{n_{i+1}} P_k(j : n_{i+1}) A_{n_1+\cdots+n_i+j,\mathbf{P}} \\ &+ \frac{q^{\widehat{n_{i+1}}}}{|\mathcal{C}|}\binom{n_{i+1}}{k}\gamma^k \sum_{j=0}^{n_1+\cdots+n_i} A_{j,\mathbf{P}}. \end{aligned}$$

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