Entropy Amplification by Aperiodic Noise and Applications to Side Information Problems

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Outline

- Aperiodic Noise
  - Definition
  - Difference Set Noise – Maximal Aperiodic Noise
  - Entropy Amplification Property
- Channel Coding with Side Info. (Writing on Dirty Paper)
  - Gaussian noise: No loss.
  - Aperiodic noise: Arbitrarily large loss.
- Source Coding with Side Information (Wyner-Ziv)
  - Gaussian source and squared-error distortion: No loss.
  - Aperiodic source and distortion: Arbitrarily large loss.

Aperiodic Noise

Definition: A set $\mathcal{Z} \subset \mathcal{G}$ has **unique differences** if

$$z_1 - z_2 = d$$

has at most one solution for $z_1, z_2 \in \mathcal{Z}$ and $d \neq 0$. Equivalently, if

$$|\mathcal{Z} \cap (\mathcal{Z} + d)| \leq 1, \quad \forall d \neq 0.$$

Definition: A random variable Z which is uniformly distributed on a set $\mathcal{Z}$ with unique differences is **aperiodic noise**.

- Autocorrelation of P_Z has minimal upper bound of $1/|\mathcal{Z}|^2$. (Except at zero.)

Difference Sets

Definition: A **planar difference set** $\mathcal{D} \subset \mathcal{G}_L$ satisfies

$$|\mathcal{D} \cap (\mathcal{D} + g)| = 1, \quad \forall g \neq 0.$$

- Maximal set with unique differences.
- Example: For $\mathcal{D} = \{1, 2, 4\} \subset \mathcal{G}_7$

$$\begin{aligned}
\mathcal{D} + 1 &= \{\mathbf{2}, 3, 5\} \\
\mathcal{D} + 2 &= \{3, \mathbf{4}, 6\} \\
\mathcal{D} + 3 &= \{\mathbf{4}, 5, 7\} \\
\mathcal{D} + 4 &= \{5, 6, \mathbf{1}\} \\
\mathcal{D} + 5 &= \{6, 7, \mathbf{2}\} \\
\mathcal{D} + 6 &= \{7, \mathbf{1}, 3\}
\end{aligned}$$

Result from Design Theory

Lemma: A planar difference set $\mathcal{D}$ exists with $|\mathcal{D}| = \alpha$ for

$$\alpha = p^m + 1, \ (p \text{ prime}, m \text{ arbitrary integer}).$$

The addition is over mod L, where

$$L = \alpha(\alpha - 1) + 1.$$

- Can generate from primitive cubic in $\mathbb{F}(p^m)$.
- Example in $\mathcal{G}_{307}$:

$\{1, 2, 4, 45, 57, 62, 68, 76, 83, 92, 96, 125, 161, 179, 201, 211, 238, 263\}$

Entropy Amplification Property

Theorem (EAP): If Z is aperiodic noise on $\mathcal{Z} \subset \mathcal{G}$ with $|\mathcal{Z}| = \alpha$, and X is any RV on $\mathcal{X} \subset \mathcal{G}$ with $|\mathcal{X}| = \beta \leq \alpha + 1$, then

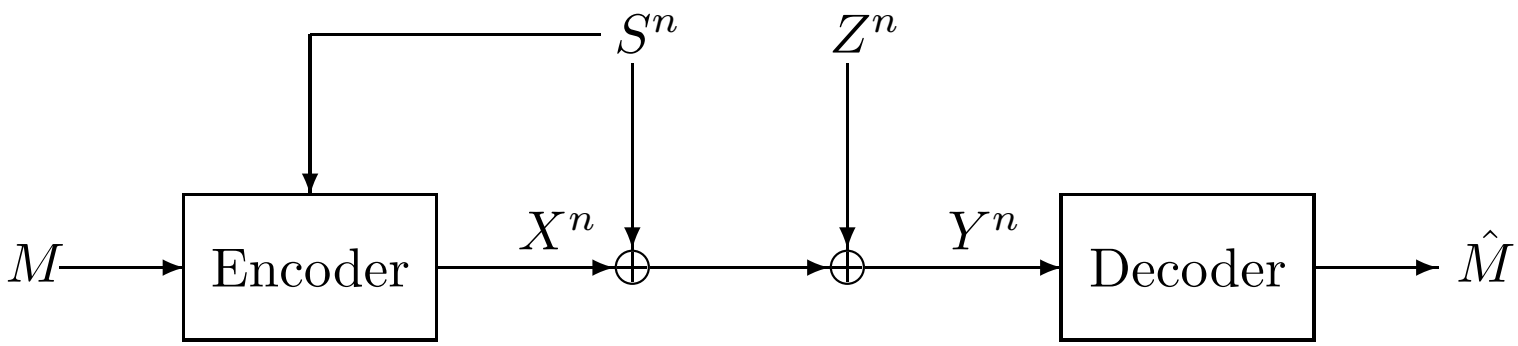
$$H(X) + H(Z) - \underbrace{\frac{\beta - 1}{\alpha}}_{\leq 1 \text{ bit}} \leq H(X + Z) \leq H(X) + H(Z).$$

- Lower bound tight for $\mathcal{Z}$ difference set, X uniform on $\mathcal{X} \subseteq \mathcal{Z}$.
- Upper bound asymptotically tight for X uniform on $\mathcal{X} \subseteq -\mathcal{Z}$.

Corollary: For any $\epsilon > 0$, there exists a planar difference set $\mathcal{D}$ such that if X and Z are i.i.d. uniform on $\mathcal{D}$, then

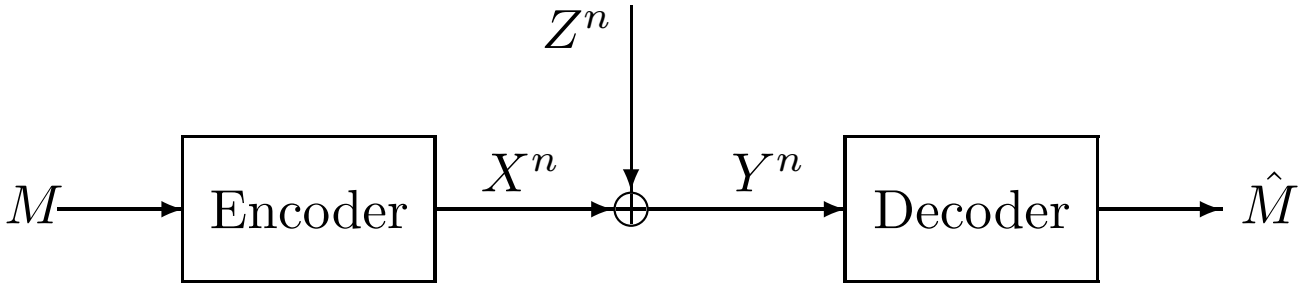
$$H(X - Z) - H(X + Z) \geq 1 - \epsilon \text{ bits}.$$

Writing on Dirty Paper



- Two independent additive effects:
  1. "Interference" (S): Known non-causally at encoder.
  2. "Noise" (Z): Not known directly.
- Constrained input (X).
- Coined by Costa in 1983.

Comparison System : Zero-Interference



- Same noise, same input constraint as WDP
- Equivalent to WDP with interference also at decoder
- Could be called "Writing on Clean Paper"

$$\text{Loss} = C_{\text{ZI}} - C_{\text{WDP}}$$

Some Prior Results

For real alphabets and power constraint ($\sum x_i^2 \leq nP$)

- S, Z Gaussian $\Rightarrow$ Loss $= 0$ [Costa '83]
- Z Gaussian $\Rightarrow$ Loss $= 0$ [Erez-Shamai-Z. '02, C.-Lapidoth '02]
- Loss $= 0$ (for many strategies) $\Rightarrow$ Z Gaussian [C.-Lapidoth '02]
- $E[Z^2] \leq P \Rightarrow$ Loss $\leq 1/2$ bit [Z. '02]

Discrete WDP

- All addition mod L with results in $\mathcal{G}_L = \{1, \ldots, L\}$.
- Hard input constraint: Each $x_i \in \mathcal{C} \subseteq \mathcal{G}_L$.
- Strong interference: $S_i \sim \text{Unif}(\mathcal{G}_L)$,

$$C_{\text{WDP}} = \sup_{P_V, Q(\cdot)} H(V) - H(Q(V) + Z),$$

  where $Q(v) - v \in \mathcal{C}$ for all v.

- Zero interference:

$$C_{\text{ZI}} = \sup_{P_X} H(X + Z) - H(Z),$$

  where P_X has support only on $\mathcal{C}$.

What causes large loss?

Noise must be "non-Gaussian"

- Prior results: No loss $\Leftrightarrow$ Gaussian noise.

Noise must be aperiodic. Consider:

- Noise Z only on $\{\beta, 2\beta, \ldots, L\}$.
- Constraint set $\mathcal{C} = \{1, \ldots, \beta\}$.
- Given $m \in \mathcal{C}$ and current interference s,

$$\underbrace{\Big(\overbrace{(m-s) \bmod \beta}^{\text{Input: } x \in \mathcal{C}} + s + Z\Big)}_{\text{Output: } y} \bmod \beta = m.$$

Need "aperiodic" noise...

Capacity with Aperiodic Noise

Theorem: For Z aperiodic noise on $\mathcal{Z}$, and any $\mathcal{C}$ with $|\mathcal{C}| \leq |\mathcal{Z}|$.

- With no interference,

$$C_{\text{ZI}} \geq \log |\mathcal{C}| - 1 \text{ bits/channel use.}$$

- With strong interference,

$$C_{\text{WDP}} \leq 2 \text{ bits/channel use.}$$

Implications:

- Loss at least $\log |\mathcal{C}| - 3$
- Loss can be arbitrarily large
- Loss can be arbitrarily close to 100%

Proof of Capacity Results

Applications of Entropy Amplification Property:

1. Let X be uniform over $\mathcal{C}$,

$$H(X+Z) - H(Z) \geq H(X) - 1 = \log|\mathcal{C}| - 1.$$

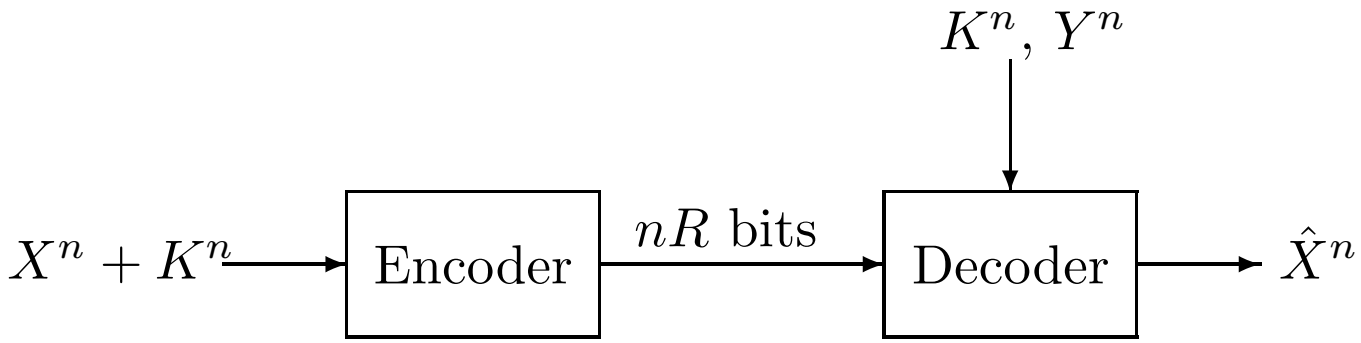
2. Let $K(\cdot)$ **re**-quantize $Q(V)$ with $|K^{-1}(k)| \leq |\mathcal{Z}|$,

$$\begin{aligned} H(V) - H(Q+Z) &\leq H(V) - H(Q+Z|K) \\ &\leq H(V) - H(Q|K) - H(Z) + 1. \end{aligned}$$

Remainder of proof:

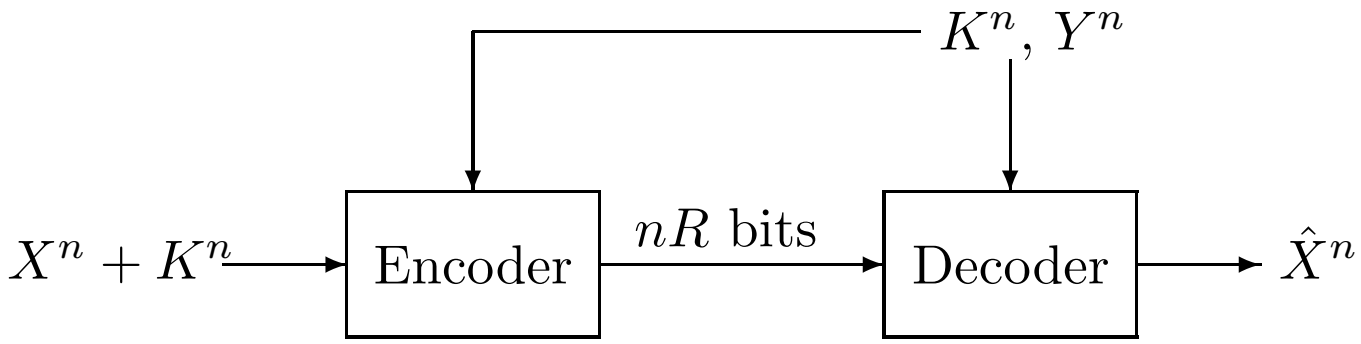
- Show $H(V) - H(Q|K) - H(Z) \leq 1$.

Variation on Wyner-Ziv Source Coding: Compressing an Encrypted Source



- Source X^n and correlated random variable Y^n.
- Key K^n, independent of (X^n, Y^n).
- Reconstruction $\hat{X}^n$.
  - Must satisfy distortion constraint.
  - Equivalent to reconstructing $X^n + K^n$.

Comparison System : Key at Encoder



- Same distributions and distortion constraint as before.
- Key does not play a role here.

$$\text{Loss} = R^{\text{WZ}}_{X+K|K,Y}(D) - R_{X+K|K,Y}(D)$$

Some Prior Results

For real alphabets, squared error distortion ($\sum(\hat{x}_i - x_i)^2 \leq nD$):

- X, K Gaussian (X and Y ind.) $\Rightarrow$ Loss $= 0$ [Wyner-Ziv '76].
- General distributions $\Rightarrow$ Loss ≤ 0.5 bits [Zamir '96].

For binary alphabets, Hamming distortion:

- Loss ≤ 0.22 bits [Zamir '96].

Compressing an Encrypted Aperiodic Source

Consider finite group $\mathcal{G}$, subset $\mathcal{S} \subset \mathcal{G}$ with unique differences.

- Let K be uniformly distributed over $\mathcal{G}$.
- Let

$$X \sim \begin{cases} \text{Unif}(\mathcal{S}), & \text{w.p. } 1-\epsilon \quad (Y=1), \\ \text{Unif}(\mathcal{G}), & \text{w.p. } \epsilon \quad (Y=2). \end{cases}$$

- Let

$$d(x,\hat{x}) = \begin{cases} 0, & x-\hat{x} \in \mathcal{S}, \\ \infty, & x-\hat{x} \notin \mathcal{S}. \end{cases}$$

Loss for Aperiodic Source

Let $U_{\mathcal{S}}$ and $U'_{\mathcal{S}}$ are i.i.d. uniformly distributed on $\mathcal{S}$.

$$\begin{aligned} \text{Loss} &= (1-\epsilon)\left[H(U_{\mathcal{S}} - U'_{\mathcal{S}}) - \log|\mathcal{S}|\right] \\ &\approx \log|\mathcal{S}|. \end{aligned}$$

Tight for ϵ small and $\mathcal{S}$ large difference set (by EAP).

- Loss can be arbitrarily large.
- Loss can be arbitrarily close to 100%.

Conclusions and Future Directions

Main tool: Aperiodic Noise and EAP.

- Generalize EAP to large support.
- Find other uses of entropy amplification.

Main result: 100% loss for WDP and for Wyner-Ziv.

- Extend to real alphabets and average distortion constraints.