Non coherent space-time coding

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Center for Discrete Mathematics & Theoretical Computer Science
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Aim of this talk

- ✓ Consider a wireless system where very short packets as well as longer ones are allowed
  - ✗ For example, wireless IP
  - ✗ Or any other system where transmission is packet-oriented with packet of any size
- ✓ Consider a full rate MIMO system with 4 transmit antennas and using 16 QAM symbols.
  - ✗ The spectral efficiency of such a system would be 16 bits p.c.u.
  - ✗ A packet of length 128 bits would correspond to a space-time codeword of length 8 channel uses (**very short!!**)

We should be able to transmit very short codewords at any time, without knowing channel coefficients.

Non Coherent reception and space-time coding

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- ✗ Pseudo-coherent reception with training sequences, pilot symbols, ... [HH00, GDE, TB03]
- ✗ Differential reception with differential space-time codes [HH02, HS00, Hug00, TJ00]
- ✗ Purely non coherent reception with unitary codewords [ARU01, HMR+00] (do not try to estimate the channel; construct a code which does not care of the channel)

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- ✓ We are interested in the pure non coherent case
  - ✗ Zheng and Tse [ZT02] used the Grassmann manifold to adress the non coherent case problem (Information Theory)
  - ✗ We are able to construct full rate fully diverse non coherent codes as “packings” in the Grassmann manifold

Outline

- ✓ Introduction
- ✓ Non coherent reception
  - ✗ Differential detection and degrees of freedom
  - ✗ GLRT detector

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- ✓ Grassmann packings on $G_{T,M}(\mathbb{C})$
  - ✗ The Grassmann manifold
  - ✗ Principal angles and Product “distance”
  - ✗ Parameterization of $G_{T,M}(\mathbb{C})$

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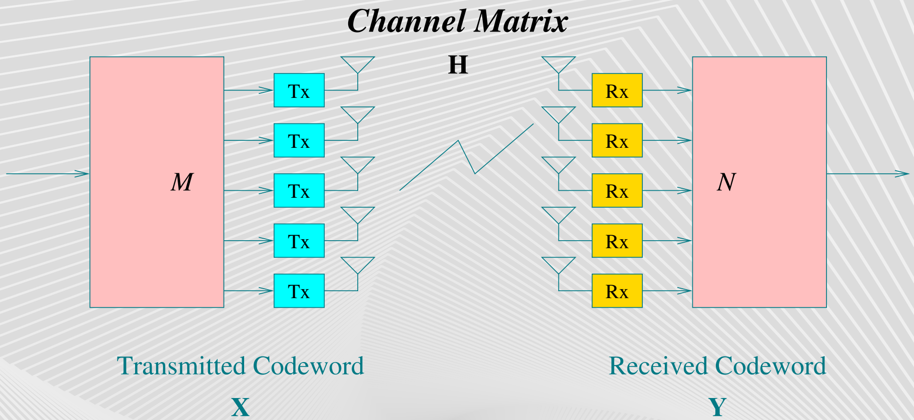
  - ✗ Principal angles and Product “distance”
  - ✗ Parameterization of $G_{T,M}(\mathbb{C})$
- ✓ The case $G_{T,1}(\mathbb{C})$ (one single antenna): spherical codes
  - ✗ Construction
  - ✗ An example
- ✓ The general case

System model



- ✓ Received signal (quasi-static channel)

$$\mathbf{Y}_{T\times N} = \mathbf{X}_{T\times M}.\mathbf{H}_{M\times N} + \mathbf{W}_{T\times N} \tag{1}$$

  with $\mathbf{H}$
  - ✗ perfectly known at the receiver (coherent codes)
  - ✗ completely unknown at the receiver end (differential or non coherent codes)
- ✓ We are interested in non coherent space-time codes with $M = N$, $T \geq 2M$ and high spectral efficiency.

Design methodology

✓ Choose the number of degrees of freedom ς (symbols per channel use) as a function of M, N, T and the type of code (coherent, differential or non coherent). Table 1 gives ς_{opt} for each case.

✓ We construct a code which satisfies to the asymptotic design criterion

- ✗ Diversity
- ✗ Coding advantage based on a product "pseudo-distance"

✓ Aim: Find codes with large minimum product "pseudo-distance"

Coherent STC	Differential STC	Non Coherent STC
$\min(M, N)$	$\frac{1}{2}\min(M, N)$	$M^{\star} \cdot \left(1 - \frac{M^{\star}}{T}\right)$

Table 1: Optimal number of degrees of freedom ς_{opt} per channel use $M^{\star} = \min\left(M, N, \lfloor \frac{T}{2} \rfloor\right)$

Differential detection

- ✓ Differential codes are associated to a maximal number of degrees of freedom

$$\varsigma_{\text{opt}} = \frac{1}{2} \min\left(M, N\right) = \frac{M}{2}$$

if $M = N$.

- ✓ Short blocks decrease ς_{opt} whereas the allocated number of degrees of freedom, when $\mathbf{H}$ unknown, is

$$M \cdot \left(1 - \frac{M}{T}\right)$$

when $M = N$ and $T \geq 2M$

- ✓ To increase the total number of degrees of freedom

$$\downarrow$$

Non coherent detection

Non Coherent Detection

- ✓ ML detection is equivalent to GLRT detection when
  - ✗ Word $\mathbf{X}_{T\times M}$ is unitary
  - ✗ Coefficients of matrix $\mathbf{H}_{M\times N}$ are uncorrelated
- ✓ GLRT decision is [WM02],

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}\in\mathcal{C}} \inf_{\mathbf{H}} \|\mathbf{Y} - \mathbf{X}\cdot\mathbf{H}\|_F^2 \tag{2}$$

which can be rewritten as

$$\hat{\mathbf{X}} = \arg\max_{\mathbf{X}\in\mathcal{C}} \mathrm{Trace}\left(\mathbf{Y}\mathbf{Y}^{\dagger}\cdot\mathbf{X}\mathbf{X}^{\dagger}\right) \tag{3}$$

where † is for “transpose + conjugate”

The Grassmann Manifold (I)

✓ **Principle:** Use a constructive method to find codes on the Grassmann manifold. "Constructive counterpart" to the geometric interpretation of [ZT02].

✓ **Change of coordinates:** Codeword $\mathbf{X}_{T\times M}$ is a basis of the M dimensional subspace $\mathbf{\Omega_X}$.

✗ **Transformation**

$$\mathbf{X} \mapsto (\mathbf{F_X}, \mathbf{\Omega_X}) \tag{4}$$

where $\mathbf{F_X} \in \mathbb{C}^{M\times M}$ is a change of basis of $\mathbf{\Omega_X}$

$$\mathbb{C}^{T\times M} \rightarrow \mathbb{C}^{M\times M} \times G_{T,M}(\mathbb{C})$$

where $G_{T,M}(\mathbb{C})$ is a Grassmann manifold, *i.e.* the set of all M dimensional subspaces in $\mathbb{C}^T$

✗ $\mathbf{H}$ in eq. (1) only affects matrix $\mathbf{F_X}$.

The Grassmann Manifold (I cont'd)

✓ $G_{T,M}(\mathbb{C})$ is the set of all M dimensional subspaces in $\mathbb{C}^T$

✗ It is a differentiable manifold with dimension $M \cdot (T - M)$

✗ Some authors have already worked on packings for the Grassmann manifold [CHS96, BN02] for some metrics (chordal distance, geodesic distance, ...)

✗ But as it is often the case in Rayleigh fading channels, our metric is related to a so-called "product distance" and a packing in the Grassmann manifold remains an open question (till ***yesterday*** [Slo03])

The Grassmann Manifold (II)

✓ Packings on the Grassmann manifold with a distance criterion derived from the pairwise error probability of the GLRT detector [BV01]

✗ If $\mathbf{X}_i$ and $\mathbf{X}_j$ are two distinct codewords $(\in \mathbb{C}^{T\times M})$ associated to subspaces $\mathbf{\Omega}_{\mathbf{X}_i}$ and $\mathbf{\Omega}_{\mathbf{X}_j}$, then construct the matrix

$$\begin{bmatrix} \mathbf{X}_i^\dagger \\ \mathbf{X}_j^\dagger \end{bmatrix} \cdot \begin{bmatrix} \mathbf{X}_i & \mathbf{X}_j \end{bmatrix} = \begin{bmatrix} I & \mathbf{R}_{ij}^\dagger \\ \mathbf{R}_{ij} & I \end{bmatrix}$$

✓ The expression of the *asymptotic pairwise error probability* is

$$P(\mathbf{X}_i \to \mathbf{X}_j) \simeq \frac{\Gamma^{-MN} \begin{pmatrix} 2MN-1 \\ MN \end{pmatrix}}{\det\left(I - \mathbf{R}_{ij}^\dagger \mathbf{R}_{ij}\right)^N}$$

where Γ is the average signal to noise ratio.

Principle angles

- ✓ Matrix $\mathbf{R}_{ij}^{\dagger}\mathbf{R}_{ij}$ has eigenvalues $\cos^2\left[\theta_k\left(\mathbf{\Omega}_{\mathbf{X}_i}, \mathbf{\Omega}_{\mathbf{X}_j}\right)\right]$, $k = 1, \ldots, M$ where $\theta_k\left(\mathbf{\Omega}_{\mathbf{X}_i}, \mathbf{\Omega}_{\mathbf{X}_j}\right)$ is the k^{th} principal angle between subspaces $\mathbf{\Omega}_{\mathbf{X}_i}$ and $\mathbf{\Omega}_{\mathbf{X}_j}$[CHS96].
  - ✗ Minimization of P.E.P. is equivalent to the maximization of

$$\det\left(I - \mathbf{R}_{ij}^{\dagger}\mathbf{R}_{ij}\right) = \prod_{k=1}^{M} \sin^2\theta_k \tag{5}$$

    which can be viewed as a kind of product distance [BVRB96]

- ✓ For high rate codes, construction of the code must take into account maximization of

$$\min_{\substack{\mathbf{X}_i, \mathbf{X}_j \in \mathcal{C} \\ \mathbf{X}_i \neq \mathbf{X}_j}} \prod_{k=1}^{M} \theta_k\left(\mathbf{\Omega}_{\mathbf{X}_i}, \mathbf{\Omega}_{\mathbf{X}_j}\right) \tag{6}$$

Parameterization of the Grassmann Manifold (I)

- It is shown in [EAS98] that

$$G_{T,M}(\mathbb{C}) \cong U_T(\mathbb{C}) / (U_M(\mathbb{C}) \times U_{T-M}(\mathbb{C})) \tag{7}$$

where $U_n(\mathbb{C})$ is the group of n dimensional complex unitary matrices.

  - That means that each subspace in $G_{T,M}(\mathbb{C})$ can be represented by a unitary transform in $U_T(\mathbb{C}) / (U_M(\mathbb{C}) \times U_{T-M}(\mathbb{C}))$ applied to a reference M-dimensional subspace
  - Hence (see [EAS98]) $G_{T,M}(\mathbb{C})$ can be represented by the $T \times M$ matrix

$$\mathbf{G} = \left[\exp\begin{pmatrix} 0 & \mathbf{B} \\ -\mathbf{B}^\dagger & 0 \end{pmatrix}\right] \cdot \mathbf{I}_{T,M} \tag{8}$$

where $\mathbf{B}$ is any $M \times (T-M)$ complex matrix.

- Dimension of $G_{T,M}(\mathbb{C})$ is $M \cdot (T-M) \Rightarrow M \cdot \left(1 - \frac{M}{T}\right)$ degrees of freedom p.c.u. (see table 1)

Parameterization of the Grassmann Manifold (II)

- ✓ Singular values decomposition of

$$\mathbf{B} = \mathbf{U}_{M\times M} \cdot \Lambda_{M\times(T-M)} \cdot \mathbf{V}^{\dagger}_{(T-M)\times(T-M)} \tag{9}$$

with

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & \lambda_M & 0 & \cdots & 0 \end{pmatrix}$$

- ✓ In that case, by applying eq. (8), codeword $\mathbf{X}$ is

$$\mathbf{X} = \begin{pmatrix} \mathbf{U}\cdot\mathbf{C}\cdot\mathbf{U}^{\dagger} \\ \mathbf{V}\cdot\mathbf{S}\cdot\mathbf{U}^{\dagger} \end{pmatrix}_{T\times M} \tag{10}$$

with

$$\mathbf{C} = \begin{pmatrix} \cos\lambda_1 & & \\ & \ddots & \\ & & \cos\lambda_M \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} \sin\lambda_1 & & & \\ & \ddots & & \mathbf{0} \\ & & \sin\lambda_M & \end{pmatrix}^{\mathrm{T}}$$

Link with the Cayley codes of [JH03]

✓ Parameterization is

$$\mathbf{X} = \left[\left(\mathbf{I} + \begin{pmatrix} 0 & \mathbf{B} \\ -\mathbf{B}^{\dagger} & 0 \end{pmatrix} \right)^{-1} \cdot \left(\mathbf{I} - \begin{pmatrix} 0 & \mathbf{B} \\ -\mathbf{B}^{\dagger} & 0 \end{pmatrix} \right) \right] \cdot \mathbf{I}_{T,M} \qquad (11)$$

✓ After some calculations,

$$\mathbf{X} = \begin{pmatrix} \mathbf{U} \cdot \frac{1-\mathbf{\Lambda}^2}{1+\mathbf{\Lambda}^2} \cdot \mathbf{U}^{\dagger} \\ \mathbf{V} \cdot \frac{2\mathbf{\Lambda}}{1+\mathbf{\Lambda}^2} \cdot \mathbf{U}^{\dagger} \end{pmatrix}_{T \times M} \qquad (12)$$

which is the parameterization of $\mathbf{C}$ and $\mathbf{S}$ with

$$\mathbf{\Lambda} = \tan \frac{\mathbf{\Theta}}{2}$$

One single antenna: Spherical codes

- ✓ $G_{T,1}(\mathbb{C})$ is isomorphic to $\mathcal{S}_T / \{\exp i\varphi\}, \varphi \in [0, 2\pi[$
- ✓ Cosine-Sine decomposition of eq. (10) gives the codeword

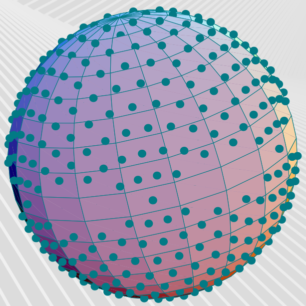
$$\mathbf{X}^{\mathrm{T}} = \begin{pmatrix} \cos\rho & b_1 \frac{\sin\rho}{\rho} & \cdots & b_{T-1}\frac{\sin\rho}{\rho} \end{pmatrix} \quad (13)$$

  with $\mathbf{B} = \begin{pmatrix} b_1 & b_2 & \cdots & b_{T-1} \end{pmatrix}$ and $\rho = \sqrt{\sum_{i=1}^{T-1} |b_i|^2}$.
- ✓ There is only one principal angle θ between two straight complex lines. Principal angle between $\mathbf{X}$ and the reference line is $\rho \leq \frac{\pi}{2}$.
  - ✗ So, some spherical shaping must be done on constellation of vectors of type $\mathbf{B}$.
- ✓ This construction is very similar to that giving rise to "wrapped spherical codes" [HZ97]. The difference is that in [HZ97], the sphere is a pure sphere and not $G_{T,1}(\mathbb{C})$.

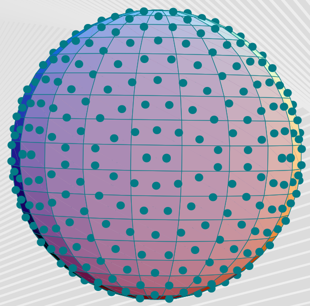
Spherical codes: An illustration

✓ An example: $T = 3$; on $G_{3,1}(\mathbb{R})$, use an hexagonal constellation for vector $\mathbf{B}$ (finite part of the A_2 lattice)

✗ With hexagonal (Voronoï constellation [For89]) and spherical shaping [LFT94]



(a) Hexagonal Shaping



(b) Spherical Shaping

Figure 1: Wrapped A_2 lattice with hexagonal and spherical shaping

Spherical Codes: An example

✓ Consider a code on $G_{5,1}(\mathbb{C})$ constructed by wrapping the E_8 Gosset lattice

- ✗ Shaping is the cubical one. Spectral efficiency: 2.4 bits p.c.u.

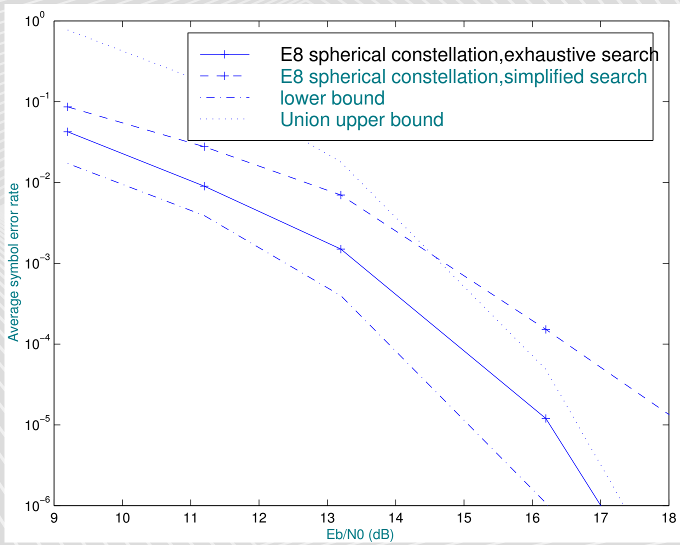


Figure 2: Simulation results for the wrapped E_8 lattice

- ✗ Exhaustive GLRT and suboptimal decoding (on the tangent subspace to $G_{T,1}(\mathbb{C})$) are compared

The General Case: Choice of matrix B

✓ Subspace $\mathbf{\Omega_X}$ is generated by the orthonormal basis

$$\mathbf{X} = \left[\exp\begin{pmatrix} 0 & \mathbf{B} \\ -\mathbf{B}^{\dagger} & 0 \end{pmatrix}\right] \cdot \mathbf{I}_{T,M}$$

Proposition. *Let* β_k *be the* $k^{\mathbf{th}}$ *principal angle between* $\mathbf{\Omega_X}$ *and the reference subspace represented by* $\mathbf{I}_{T,M}$*. Then* β_k *is the* $k^{\mathbf{th}}$*singular value of* $\mathbf{B}$ *and*

$$\prod_{k=1}^{M} \sin^2 \beta_k \neq 0, \forall \mathbf{X} \in \mathcal{C}$$

iff the coherent code defined by the $\mathbf{B}$ *matrices is fully diverse (see [KB03a])*

✓ With a coherent code, it is quite easy to construct a non coherent code. But the diversity property of this code needs another result.

Fully diverse non coherent code

✓ In order to complete the diversity proof,

$$\prod_{k=1}^{M} \sin^2 \theta_k \left(\mathbf{X}_i, \mathbf{X}_j\right) \neq 0, \forall \mathbf{X}_i, \mathbf{X}_j \in \mathcal{C}, \mathbf{X}_i \neq \mathbf{X}_j$$

we need another property, namely,

$$\max_{k=1}^{M} \max_{\mathbf{X} \in \mathcal{C}} \lambda_k \left(\mathbf{B}_{\mathbf{X}}\right) \leq \frac{\pi}{2} - \epsilon \qquad (14)$$

with $\lambda_k \left(\mathbf{B}_{\mathbf{X}}\right)$ being singular values of matrix $\mathbf{B}_{\mathbf{X}}$ and ϵ is a properly chosen constant related to the structure of the coherent code used to construct the Grassmann code.

Proposition. *A* Grassmann code *defined by the exponential mapping on a* fully diverse coherent $M \times (T - M)$ *code such that inequality (14) is satisfied is a fully diverse non coherent code.*

Some results

- ✓ Simulation results are presented for Grassmann manifolds $G_{4,2}(\mathbb{C})$ (with a coherent 2×2 code [DTB02]), $G_{6,2}(\mathbb{C})$ (with a coherent 2×4 code [KB03a]) and $G_{6,3}(\mathbb{C})$ (with a coherent 3×3 code [ED03]). All these coherent codes can be seen as finite subsets of cyclic algebras [BR03, SRS03]. QPSK symbols are used.
- ✓ $\mathbf{B}$ matrix must be a word of a full rate, fully diverse coherent code
  - ✗ For $G_{4,2}(\mathbb{C})$, take for instance [DTB02],

$$\mathbf{B} = \begin{bmatrix} s_1 + \theta s_2 & \phi\left(s_3 + \theta s_4\right) \\ \phi\left(s_3 - \theta s_4\right) & s_1 - \theta s_2 \end{bmatrix}$$

    with $\phi^2 = \theta = e^{i\frac{\pi}{4}}$ and $s_i, i = 1, \cdots, 4$ are the 4 information QPSK symbols.
  - ✗ For $G_{6,2}(\mathbb{C})$, take for instance [KB03b]

$$\mathbf{B} = \begin{bmatrix} s_1 + \theta s_2 & \phi\left(s_3 + \theta s_4\right) & \phi^2\left(s_5 + \theta s_6\right) & \phi^3\left(s_7 + \theta s_8\right) \\ \phi^3\left(s_7 - \theta s_8\right) & s_1 - \theta s_2 & \phi\left(s_3 - \theta s_4\right) & \phi^2\left(s_5 - \theta s_6\right) \end{bmatrix}$$

    with $\phi^2 = \theta = e^{i\frac{\pi}{4}}$ and and $s_i, i = 1, \cdots, 8$ are the 8 information QPSK symbols.

✗ For $G_{6,3}(\mathbb{C})$, take for instance,

$$\mathbf{B} = \begin{bmatrix} s_1+\theta s_2+\theta^2 s_3 & \phi\left(s_4+\theta s_5+\theta^2 s_6\right) & \phi^2\left(s_7+\theta s_8+\theta^2 s_9\right) \\ \phi^2\left(s_7+j\theta s_8+j^2\theta^2 s_9\right) & s_1+j\theta s_2+j^2\theta^2 s_3 & \phi\left(s_4+j\theta s_5+j^2\theta^2 s_6\right) \\ \phi\left(s_4+j^2\theta s_5+j\theta^2 s_6\right) & \phi^2\left(s_7+j^2\theta s_8+j\theta^2 s_9\right) & s_1+j^2\theta s_2+j\theta^2 s_3 \end{bmatrix}$$

with $\phi^3 = \theta = e^{i\frac{\pi}{9}}$ and and $s_i, i = 1, \cdots, 9$ are the 9 information QPSK symbols. [ED03]

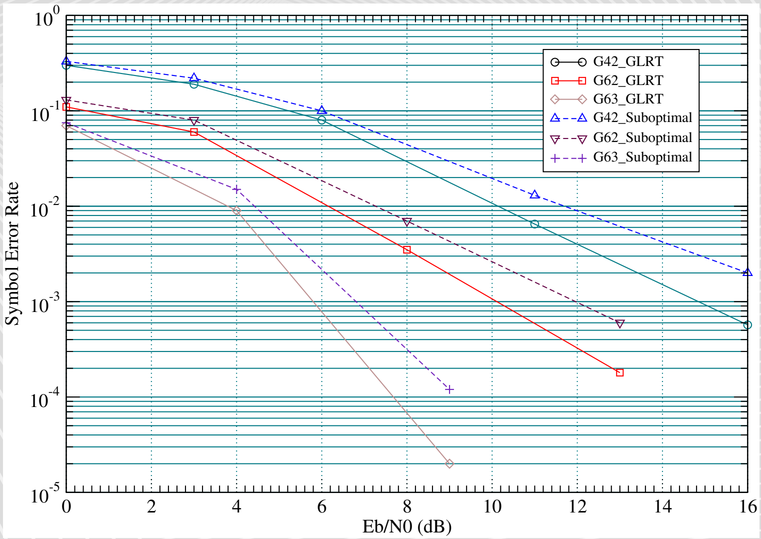


Figure 3: $G_{4,2}$: 2 bits p.c.u., $G_{6,2}$: 2.66 bits p.c.u., $G_{6,3}$: 3 bits p.c.u.

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