# On Approximate Horn Minimization

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- 1. Preliminaries
- 2. Different versions of the problem
- 3. Approximability
- 4. Inapproximability

# Preliminaries

- Horn clause: at most one unnegated variable, e.g.
  - $a, b \rightarrow c$ or $\overline{a} \vee \overline{b} \vee c$
  - definite Horn clause: one unnegated variable
- (definite) Horn formula: conjunction of (definite) Horn clauses
  - $(a, b \rightarrow c) \wedge (a, c \rightarrow d) \wedge (d, e \rightarrow f)$
- size: number of clauses or variables
- implication (can be decided efficiently):
  - $(a, b \rightarrow c) \wedge (a, c \rightarrow d) \models (a, b \rightarrow d)$
- equivalence of $\varphi$ and $\psi$:
  - $\forall a \; \varphi(a) \Leftrightarrow \psi(a)$
  - $\forall C \; \varphi \models C \Leftrightarrow \psi \models C$

# Horn minimization

- given a Horn formula $\varphi$ and a number $k$, is there a Horn formula with at most $k$ clauses equivalent to $\varphi$?
- $x_i \rightarrow y_j, \quad y_1, \ldots, y_n \rightarrow x_i \qquad n^2 + n$ clauses
- $x_i \rightarrow x_{i+1}, \quad x_n \rightarrow y_i, \quad y_1, \ldots, y_n \rightarrow x_1 \qquad 2n$ clauses
- restricted version: given a Horn formula $\varphi$ and a number $k$, does $\varphi$ have a subformula with at most $k$ clauses equivalent to $\varphi$?

# Previous work

- Boros, Čepek (1994): NP-complete
- Hammer, Kogan (1993):
  - NP-complete if the number of literals is to be minimized; in P for quasi-acyclic formulas
  - (restricted) Horn minimization (number of clauses) has an $(n-1)$-approximation algorithm ($n$: number of different variables) $o(n)$?
- Maier (1983), Ausiello, D'Atri, Saccà (1986), Guigues, Duquenne (1986), Angluin, Frazier, Pitt (1992): minimization of the number of bodies can be done efficiently
- Boros, Čepek, Kogan (1997): iterative decomposition algorithm

# Transitive reduction

- given a directed graph $G$ and a number $k$, is there a directed graph with at most $k$ edges having the same transitive closure? - in P
- given a directed graph $G$ and a number $k$, does $G$ have a subgraph with at most $k$ edges having the same transitive closure? - NP-complete
- Berman, DasGupta, Karpinski (2007): 1.5-approximation algorithm
- definite Horn clauses of size 2

# $\mathcal{R}$-equivalence

- Flögel, Kleine Büning, Lettmann (1993): introduce new variables
- $x_1/y_1, \ldots, x_n/y_n \to u$ $\quad$ $2^n$ clauses
- $x_i \to z_i, \;\; y_i \to z_i, \;\; z_1, \ldots, z_n \to u$ $\quad$ $2n + 1$ clauses
- same set of consequences over the original variables
- $\varphi \sim_{\mathcal{R}} \psi$, where $\mathcal{R}$ is a set of variables: same set of consequences over $\mathcal{R}$
- $co - NP$-complete even in rather restricted cases

# Narrow extension

- introduce new variables in a restricted way
- $x_i \to y_j, \;\; i = 1, \ldots, n, \; j = 1, \ldots, m$ $\quad$ $nm$ clauses
- $x_i \to z, \;\; z \to y_j, \;\; i = 1, \ldots, n, \; j = 1, \ldots, m$ $\quad$ $n + m$ clauses
- new variables can be heads, or singleton bodies with old heads
- $\varphi \sim_{\mathcal{R}} \psi$, where variables new are introduced by narrow extension: can be decided efficiently

# A $o(n)$ approximation algorithm

**Theorem**

*There is an efficient $O(\frac{n}{\log n})$-approximation algorithm for narrow extended Horn minimization.*

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- find an equivalent Horn formula with the minimal number of bodies
- find a decomposition of the bipartite graph between heads and bodies

# Hardness of approximation

**Theorem**

*If NP* $\not\subseteq$ *DTIME*$(n^{polylog(n)})$ *then for every* $0 < \delta < 1$ *restricted Horn minimization is not*

$$2^{\log^{\delta} size(\varphi)}$$

*approximable.*

- approximation preserving reduction from the MINREP problem

# Decomposition of graphs

- Chung, Erdős, Spencer (1983), Bublitz (1986): every $n$-vertex graph can be partitioned into complete bipartite graphs with $O(n^2/\log n)$ vertices altogether

- Tuza (1984): every $(a, b)$-bipartite graph can be partitioned into complete bipartite graphs with $O(ab/\log(a+b))$ vertices altogether

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- Kővári, Sós, Turán (1954): every $(n, n)$-bipartite graph with $c_q n^{2-1/q}$ edges contains a complete $(q, q)$-bipartite graph (Zarankiewicz problem)
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- Kirchner (2008): efficient algorithm to find a complete $(\sqrt{\log n}, \sqrt{\log n})$ bipartite graph in a dense $n$-vertex graph

# Efficient decomposition

## Theorem

*There are efficient algorithms for finding*

- *a complete bipartite subgraph with parts about*

$$\frac{\log n}{\log(n^2/m)}$$

*in $n$-vertex graphs with $m$ edges ($m > n^{3/2}$)*

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- *a decomposition of $n$-vertex graphs into complete bipartite graphs with $O(n^2/\log n)$ vertices altogether*

- *a decomposition of $(a, b)$-bipartite graphs into complete bipartite graphs with $O(ab/\log(a + b))$ vertices altogether*