The Over-Concentrating Nature of Simultaneous Ascending Auctions*

Charles Zhóuchéng Zhèng†

First draft: June 30, 2004
This draft: October 3, 2004

Abstract

This paper analyzes simultaneous ascending auctions of two different items, viewed as complements by multi-item bidders. The finding is that such auctions overly concentrate the goods to a multi-item bidder and never overly diffuse them to single-item bidders. The main reason is that some bidders strictly want to jump-bid and jump-bidding allows the game to mimic a package auction, where single-item bidders cannot fully cooperate among themselves to bid against multi-item bidders. The second reason is that over-concentration causes resale and there is an equilibrium where a multi-item bidder becomes the reseller and chooses to under-sell the goods.

*This is developed from a note "Decentralized Auctions of Complements" presented in the Joint CSIO/IDEI Industrial Organization Workshop in 2004 at Northwestern University. I thank Gregory Pavlov, Gábor Virág, and Michael Whinston for comments, and the NSF for grant SES-0214471.

†Department of Economics, Northwestern University, 2003 Sheridan Road, Evanston, IL 60208. Email: czheng@northwestern.edu. Web: http://faculty-web.at.northwestern.edu/economics/zheng/.

1 Introduction

Simultaneous ascending auctions of heterogeneous items have caught much attention from researchers ever since the United States government, influenced by economists, started using these auctions to allocate radio frequencies in early 1990s. Even before that major application, economists had taken simultaneous ascending auctions as natural conceptual constructs to understand decentralized markets. Indeed, when there is no central coordination on the sales of multiple goods separately owned by different entities, the efficient Vickrey-Clarke-Groves mechanisms are unlikely to be used, and it is natural to assume that separate initial owners sell their goods separately. To capture the interactions among different sectors of an economy without artificially ranking one sector over another, it is natural to assume that these separate auctions start simultaneously. The open-outcry ascending-bid feature of these auctions provides a transparent setup to understand the process of price formation.

Researchers have found that simultaneous ascending auctions can achieve efficient outcomes if the items on sale are mutual substitutes (Gul and Stacchetti [8] and Milgrom [14]). However, when the items may be complements, these auctions are not found to achieve efficiency (Gul and Stacchetti [9] and Milgrom [14]), although efficiency can be achieved by a centralized bidding process (Ausubel [2] and Bikhchandani, de Vries, Schummer and Vohra [5]). To capture the decentralized nature of competitive markets, we need a theory of simultaneous ascending auctions of possibly complementary goods without central coordination. Although these auctions are already known to be probably inefficient, researchers have not found a pattern of the inefficiency. The hurdle is that inefficiency may take various forms, all parameter-dependent, so it is difficult to make predictions. These auctions are known to suffer an *exposure problem*: a bidder may have bought an item at a price above its standalone value and fail to acquire its complements (e.g., Bykowsky, Cull, and Ledyard [6], and Milgrom [14]). Worried by this problem, a bidder who considers multiple items as complements may underbid before he acquires any item and overbid for the rest after he has acquired some. Then the goods may be *over-concentrated* to a single bidder while efficiency requires that the goods go to different bidders, or the goods may be *over-diffused* to separate owners while efficiency requires that the goods go to a single bidder. Both kinds of inefficiency seem to be probable and we may not know which one is dominant without knowing specific parameters.

The thesis of this paper is that, once we take into account of the transparent feature of simultaneous ascending auctions, the prediction of these auctions becomes qualitatively unambiguous: their inefficiency takes the form of probable over-concentration and never over-diffusion. The reason is that bidders may signal through jump-bidding, so a bidder who values multiple items can infer whether he will profitably acquire the entire package before committing to buying any item. Thus, the exposure problem is eliminated, and the only remaining source of inefficiency is that bidders who value only single items cannot fully cooperate with each other to compete against bidders who value multiple items. This kind of inefficiency is the well-known *threshold problem* for package auctions, where bids are contingent on packages of items (e.g., [6] and [14]). That leads to probable over-concentration and never over-diffusion. Over-concentration creates a strict incentive for resale, but it is found in this paper that the same kind of inefficiency persists when resale is allowed.

In our model, there are only two items, A and B, and three bidders, a *local* bidder who values only A, another who values only B, and a *global* bidder who values both as complements. Bidders know who is global and who is local but do not know others' valuations. The primitives are listed in §2. The basic mechanism that bans jump-bidding and cross-bidding (bidding for an unvalued item) is analyzed in §3. This part is related to the asymmetric-information analysis of simultaneous auctions in the literature such as Krishna and Rosenthal [12] (sealed-bid second-price), Rosenthal and Wang [16] (sealed-bid first-price), and Albano, Germano and Lovo [1] (ascending-bid, two items, and uniformly distributed values); none of them consider cross- or jump-bidding.

The paper then turns to jump bidding in §4. When a local bidder is the first to drop out from an item say A, the other two bidders each strictly want to jump-bid for B in order to determine the winner of B before the global bidder commits to buying A (Lemmas 6 and 8). Consequently, the global bidder learns whether he can profitably acquire the whole package before buying any item (Proposition 1). If he finds it unprofitable to continue, the global bidder withdraws his bids, so the local bidder who is the first to stop making higher bids may win due to the other local bidder's high jump-bids. That leads to a local bidder's free-riding incentive and consequently a positive probability of over-concentration (Proposition 2). This section is slightly related to the jump-bidding literature such as Avery [4] and Gunderson and Wang [10], which have shown that jump-bidding may reduce the demand from one's rival. None of them consider multiple heterogeneous items.

The analysis is then extended to the case where cross-bidding is also allowed (§5). Cross-bidding needs to be considered because, conditional on the equilibrium in the no-jump- and no-cross-bidding case (Lemma 1), a local bidder wishes to bid for his unvalued item in order to prevent the global bidder from becoming more aggressive after winning that item. Yet in equilibrium cross-bidding mitigates the global bidder's exposure problem because his winning an item implies that both local bidders are outbid on that item and so he will face less intense competition for its complement. Hence there is an equilibrium where local bidders do not cross-bid (Proposition 3) and the incentive and consequence of jump bidding remain the same as before. Here we obtain a somewhat surprising result that the simultaneous ascending auctions can replicate the allocation of any undominated-strategy equilibrium of an ascending package auction (Proposition 4).

As probable over-concentration leads to a strict incentive for resale, the analysis is also extended to a model where resale is allowed and any bidder, if winning both items in earlier auctions, gets to commit to a selling mechanism for possible resale. In §6, an equilibrium is constructed where the global bidder acts as the middleman and, becoming a monopolist, over-concentrates the goods in his own hands (Proposition 5).

2 The primitives

There are two items, A and B. There are three bidders: a local bidder α who values only item A, a local bidder β who values only item B, and a global bidder γ who views both items as complements. The following table lists their valuations:

	$\emptyset$	A	B	A & B
local α	0	t_α	0	t_α
local β	0	0	t_β	t_β
global γ	0	0	0	t_γ

For each $i \in \{\alpha, \beta, \gamma\}$, t_i is a random variable whose realized value is bidder i's the private information and is independently drawn from a distribution F_i, with continuous positive density f_i and support $[0, \bar{t}_i]$. A bidder's payoff is equal to his valuation of the package he acquires minus his total payment.

The solution concept is *undominated strategy equilibrium*, perfect Bayesian equilibrium that never uses any action or strategy that is weakly dominated from the standpoint of any continuation game. Call it *equilibrium* briefly.

If $g(x, y)$ and $h(z)$ are real functions of variables x, y, and z, let $\mathrm{E}[g(\mathbf{x}, y) \mid h(\mathbf{z}) \geq 0]$ denote the expected value of $g(x, y)$, with the random variables boldfaced in the bracket, conditional on $h(z) \geq 0$. Let $1_S(x)$ denote the indicator function of random variable x that satisfies condition S. Let $z^+ := \max\{z, 0\}$.

3 A basic analysis of the exposure problem

3.1 The basic mechanism

The two items are auctioned off via separate clock auctions that start simultaneously. Prices start at zero. For each item k, the price p_k for item k rises continuously at an exogenous positive speed $\dot{p}_k$ until k is sold. Bidder α can bid only for item A, bidder β only for B, and γ can bid for both items. Ties are broken by coin toss.

To be eligible for an item, a bidder needs to participate in its auction from the start. Once he *quits* (drops out) from an item, a bidder cannot raise his bid for that item any more. If a bidder does not quit from an item, we say he *continues* or *stays* or *remains* for it. The auction of an item ends when all but one bidder has quit the item; immediately the remaining bidder buys the item at its current price.[1] The good cannot be returned for refund. Bidders' actions are commonly observed.

[1] This decentralized closing rule is aligned with this paper's focus on the decentralized nature of markets. The simultaneous auctions used by FCC have a centrally coordinated closing rule (Milgrom [14]).

3.2 The equilibrium

Restricted to bidding only for his valued item, a local bidder finds it dominant to be straightforward, i.e., to bid for his desired item up to its true value. This is not so for the global bidder γ, because he takes into account the exposure problem that he may end with buying an item at a price above its standalone value and failing to acquire its complement. The next lemma finds the global bidder's best reply to local bidders' dominant strategy.

Lemma 1 *For any $(p_A, p_B) \in [0, \bar{t}_\alpha] \times [0, \bar{t}_\beta]$ and type $t_\gamma \in [0, \bar{t}_\gamma]$, define*

$$v_A(t_\gamma, p_B) := \mathrm{E}\left[(t_\gamma - \mathbf{t}_\beta)^+ \,|\mathbf{t}_\beta \geq p_B\right]; \tag{1}$$

$$v_B(t_\gamma, p_A) := \mathrm{E}\left[(t_\gamma - \mathbf{t}_\alpha)^+ \,|\mathbf{t}_\alpha \geq p_A\right]. \tag{2}$$

If cross-bidding and jump-bidding are banned, straightforward bidding is weakly dominant for each local bidder. Given any current (p_A, p_B), the best reply from the global bidder γ is:

1. *If neither A nor B has had a winner, continue bidding for both items if $v_A(t_\gamma, p_B) > p_A$ and $v_B(t_\gamma, p_A) > p_B$, and quit from both auctions if one of the inequalities fails.*

2. *If item A or B has been won by someone else, quit from both auctions immediately.*

3. *If item A (or B) has been won by bidder γ, continue bidding for item B (or A) until its current price p_B (or p_A) reaches t_γ.*

Proof Plan 2 in the above strategy is obviously optimal for bidder γ: the price for an item say A is for sure higher than its standalone value 0, since local bidder α's value is for sure positive. Plan 3 in the strategy is also obviously optimal, since the payment for the already acquired item is sunk. Thus, we need only to examine plan 1.

Consider the event for plan 1, with current prices (p_A, p_B) and both local bidders remaining. From the fact that bidder α has not quit, bidder γ learns that α's value t_α exceeds item A's current price p_A. If bidder γ has bought A, we are in the event for plan 3 and hence he wins item B if $t_\gamma - t_\beta > 0$ and loses B if the inequality is reversed. If he does win B and hence gets both items, bidder γ's total profit is equal to $t_\gamma - t_\beta - p_A$ since bidder β

is straightforward. If bidder γ loses B, his total profit is equal to $-p_A$. Thus, when both local bidders are still active, bidder γ's expected profit from buying item A at the current instant is equal to

$$v_A(t_\gamma, p_B) - p_A, \tag{3}$$

and analogously his expected profit from buying item B at the current instant is equal to

$$v_B(t_\gamma, p_A) - p_B. \tag{4}$$

Note: as type distributions have no atom and no gap, (3) and (4) are continuous and strictly decreasing functions of (p_A, p_B) and hence shrink continuously with the time counted by the clocks in the auction game.

Let us prove the optimality of plan 1. At any instant in the event for plan 1, either (a) both inequalities in plan 1 hold or (b) one of them does not hold. In case (a), by continuity of (3) and (4) with respect to time, these inequalities continue to hold for a sufficiently short while. Recall that (3) stands for bidder γ's expected profit from buying A conditional on not yet quitting B, and recall the analogous interpretation for (4). Thus, at the current instant it is dominated to quit from one item and continue with the other. It is also dominated to quit both items, because doing so gives zero payoff while not doing so ensures a positive expected payoff. Hence bidder γ continues on both items in this case.

In case (b), one of the inequalities in plan 1 fails. Say $v_A(t_\gamma, p_B) \leq p_A$. As (3) is strictly decreasing in time, bidder γ's expected profit from buying item A is negative from now on if he does not quit B. If he quits B, then he also quits A by plan 2. Thus, he quits at least from A. Then by plan 2 bidder γ quits B at the same time, as plan 1 prescribes. The case when the other inequality fails is analogous. Hence plan 1 is optimal. ■

3.3 Various kinds of inefficiency

Let us examine the allocation induced by the above perfect Bayesian equilibrium. By its definition (1) and the atomless and gapless type distributions, the function $v_A(t_\gamma, \cdot)$ is continuous and strictly decreasing; when p_B decreases from $\min\{t_\gamma, \bar{t}_\beta\}$ to zero, $v_A(t_\gamma, p_B)$ rises from $[t_\gamma - \bar{t}_\beta]^+$ to $\mathrm{E}[t_\gamma - \mathbf{t}_\beta]^+$ (Figure 1). Thus, in R^2, given any $t_\gamma \in [0, \bar{t}_\gamma]$, the ray

$$\left\{(p_A, p_B) \in [0, \infty)^2 : p_B = (\dot{p}_B / \dot{p}_A) p_A\right\} \tag{5}$$

and the continuous path

$$\left\{(v_A(t_\gamma, p_B), p_B) : p_B \in [0, \min\{t_\gamma, \bar{t}_\beta\}]\right\} \cup \left\{\left(p_A, \min\{t_\gamma, \bar{t}_\beta\}\right) : p_A \in [0, (t_\gamma - \bar{t}_\beta)^+]\right\} \tag{6}$$

have exactly one common point, denoted by $(p'_A(t_\gamma), p'_B(t_\gamma))$ (Figure 1). Analogously, (5) and the path

$$\left\{(p_A, v_B(t_\gamma, p_A)) : p_A \in [0, \min\{t_\gamma, \bar{t}_\alpha\}]\right\} \cup \left\{\left(\min\{t_\gamma, \bar{t}_\alpha\}, p_B\right) : p_B \in [0, (t_\gamma - \bar{t}_\alpha)^+]\right\} \tag{7}$$

have exactly one common point, denoted by $(p''_A(t_\gamma), p''_B(t_\gamma))$ (Figure 1). Note that (5) represents the ray along which (p_A, p_B) rises when both auctions are still going on. Hence at the point $(p'_A(t_\gamma), p'_B(t_\gamma))$, either bidder γ becomes indifferent between winning and losing A conditional on staying for B, or the price of B for sure stops rising ($p'_B(t_\gamma) = \bar{t}_\beta$). Likewise, at $(p''_A(t_\gamma), p''_B(t_\gamma))$, either bidder γ becomes indifferent about winning B conditional on staying for A, or p_A for sure stops rising ($p''_A(t_\gamma) = \bar{t}_\alpha$). Let

$$p^*_A(t_\gamma) := \min\left\{p'_A(t_\gamma), p''_A(t_\gamma)\right\} \quad \& \quad p^*_B(t_\gamma) := \min\left\{p'_B(t_\gamma), p''_B(t_\gamma)\right\}.$$

Since the slope of the price ray $p_B = (\dot{p}_B / \dot{p}_A) p_A$ is positive,

$$(p^*_A(t_\gamma), p^*_B(t_\gamma)) = (p'_A(t_\gamma), p'_B(t_\gamma)) \quad \text{or} \quad (p^*_A(t_\gamma), p^*_B(t_\gamma)) = (p''_A(t_\gamma), p''_B(t_\gamma)). \tag{8}$$

Note that $(p^*_A(t_\gamma), p^*_B(t_\gamma))$ is the instant at which global bidder γ quits both items, unless he has already won an item.

The equilibrium allocation is: If $t_\alpha > p^*_A(t_\gamma)$ and $t_\beta > p^*_B(t_\gamma)$, item A goes to local bidder α and item B goes to local β. If $t_\alpha < p^*_A(t_\gamma)$ and $t_\beta < t_\gamma$, both items go to global bidder γ (plans 1 and 3 of Lemma 1). If $t_\alpha < p^*_A(t_\gamma)$ and $t_\beta > t_\gamma$, item A goes to γ and B goes to β. If $t_\beta < p^*_B(t_\gamma)$, then γ wins both items if $t_\alpha < t_\gamma$ and wins only B and loses A to α if $t_\alpha > t_\gamma$. Ties occur with zero probability, as type distributions are atomless and functions $v_A(\cdot, p_B)$ and $v_B(\cdot, p_A)$ are continuous.

Lemma 2 *If $t_\gamma > 0$, then $p^*_A(t_\gamma) > 0$ and $p^*_B(t_\gamma) > 0$; if also $t_\gamma \neq \bar{t}_\alpha + \bar{t}_\beta$, then $t_\gamma > p^*_A(t_\gamma) + p^*_B(t_\gamma)$.*

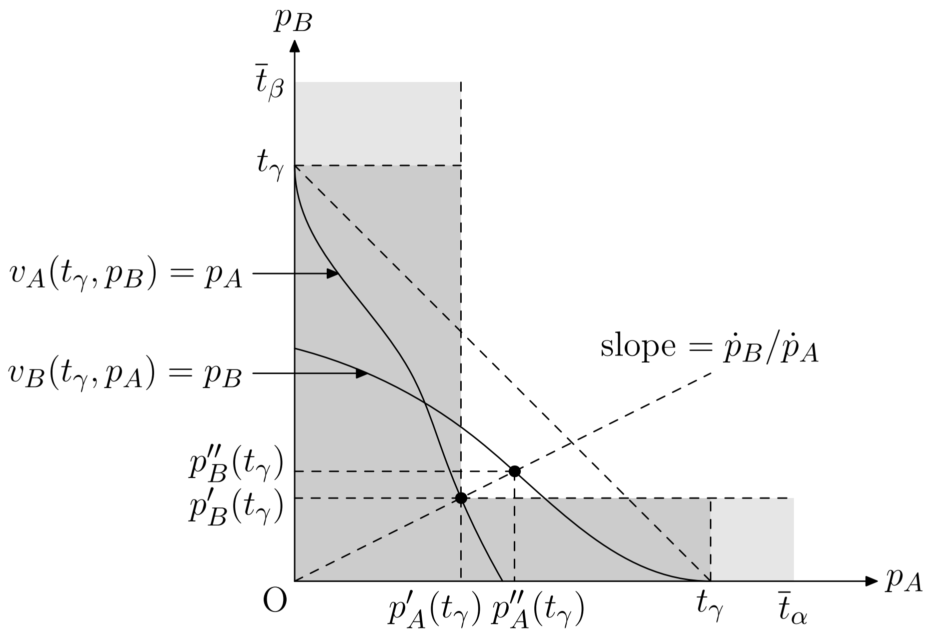

Figure 1: Dark: $\{A, B\} \to \gamma$; grey: $\{A, B\} \to \alpha$ or β; white: $A \to \alpha$ & $B \to \beta$.

Proof Since $0 < \dot{p}_B/\dot{p}_A < \infty$, it is obvious that $p^*_A(t_\gamma) > 0$ and $p^*_B(t_\gamma) > 0$ for all $t_\gamma > 0$. To prove the rest of the lemma, recall definition (1) and the assumption that the distribution of t_γ has no gap. Then $v_A(t_\gamma, p_B) < t_\gamma - p_B$ unless $p_B = \bar{t}_\beta$, and $v_B(t_\gamma, p_A) < t_\gamma - p_A$ unless $p_A = \bar{t}_\alpha$. Thus, by (8), the desired inequality $t_\gamma > p^*_A(t_\gamma) + p^*_B(t_\gamma)$ follows unless

$$(p'_A(t_\gamma), p'_B(t_\gamma)) = (t_\gamma - \bar{t}_\beta, \bar{t}_\beta) = (\bar{t}_\alpha, t_\gamma - \bar{t}_\alpha) = (p''_A(t_\gamma), p''_B(t_\gamma)),$$

which implies $t_\gamma = \bar{t}_\alpha + \bar{t}_\beta$. ■

Inefficiency of the equilibrium takes three different forms, each probable. One is *over-diffusion*: item A goes to local bidder α and B goes to local β, while efficiency requires that both items go to the global bidder. This in our equilibrium is the event

$$t_\alpha > p^*_A(t_\gamma) \ \ \& \ \ t_\beta > p^*_B(t_\gamma) \ \ \& \ \ t_\alpha + t_\beta < t_\gamma,$$

which occurs with a positive probability because $t_\gamma > p^*_A(t_\gamma) + p^*_B(t_\gamma)$ (Lemma 2) and type distributions have no gap. The second kind of inefficiency is *over-concentration*: one bidder wins both items while efficiency requires that they go to different bidders. This in our equilibrium is the event

$$[t_\alpha < p^*_A(t_\gamma) \ \ \& \ \ t_\beta < t_\gamma \ \ \& \ \ t_\alpha + t_\beta > t_\gamma] \quad \text{or} \quad [t_\beta < p^*_B(t_\gamma) \ \ \& \ \ t_\alpha < t_\gamma \ \ \& \ \ t_\alpha + t_\beta > t_\gamma].$$

This event occurs with a positive probability because $p_A^*(t_\gamma) > 0$ and $p_B^*(t_\gamma) > 0$ (Lemma 2). The third kind of inefficiency is incomplete diffusion: the global bidder wins exactly one item while efficiency requires both items go to local bidders. This is the event

$$[t_\alpha < p_A^*(t_\gamma) \;\;\&\;\; t_\beta > t_\gamma] \quad \text{or} \quad [t_\beta < p_B^*(t_\gamma) \;\;\&\;\; t_\alpha > t_\gamma],$$

which occurs with a positive probability, again because $p_A^*(t_\gamma) > 0$ and $p_B^*(t_\gamma) > 0$.

Thus, the exposure problem leads to various kinds of inefficient outcomes. Such ambiguity, however, is only because our analysis so far has not fully exploited the transparent nature of simultaneous ascending auctions. With actions commonly observed, bidders might be able to avoid the exposure problem via signaling such as jump-bidding.

4 Jump bidding eliminates the exposure problem

Let us consider the moment when local bidder α is quitting at p_A. Now global bidder γ is on the verge of buying A without knowing how much he will have to pay for its complement B. Suppose the other local bidder β could credibly reveal his value t_β to bidder γ at this moment. Then bidder γ would know that the price for item B will be t_β. If his value is less than $p_A + t_\beta$, γ's profit will be negative if he is to buy both items, and he would not be able to avoid such loss if he buys A now, because he will bid for B up to t_γ once he has bought A. Thus, if $t_\gamma < p_A + t_\beta$, bidder γ wishes to quit both items immediately and yield the right for item A to bidder α. Then the global bidder could avoid the exposure problem, and local bidder β's winning event could be expanded from $\{t_\gamma : t_\beta > t_\gamma\}$ to $\{t_\gamma : t_\beta > t_\gamma - p_A\}$. Such arrangement would need local bidder β to reveal his type credibly. That can be done, as we will see soon, by a jump bid for item B, i.e., an amount of payment that he promises to deliver if he wins B right now.

Thus, conditional on the previously assumed simple rule that prices have to rise continuously at exogenous speeds, bidders have a strict incentive to deviate from it. The deviation would allow them to signal via jump-bidding and to reduce losses by withdrawing one's bids. Even if auctioneers do not allow such deviation, bidders' strict incentive makes it costly to maintain the prohibition. Thus, we amend the mechanism as follows—

4.1 A model that allows jump-bidding

At any instant during the auction of any item, an active bidder chooses whether to *continue* or *jump-bid* or *stop* or *withdraw*. To continue, the bidder keeps pressing his button for the item. To jump-bid, the bidder cries out a bid (for this item) higher than its current price indicated by the clock. A bidder remains *active* if and only if he continues or jump-bids. To stop, the bidder releases the button and forever forfeits the right to raise his bid for the item. In withdrawing, a bidder will never get the good and he may need to compensate the seller for the difference between his highest bid and the final selling price if this difference is positive: if some other bidder continues after this bidder withdraws, this difference is zero and hence the withdrawing bidder pays zero; if all other bidders withdraw immediately after this bidder withdraws, these bidders each pay an equal share of the difference.

If a bidder say i stops or withdraws or jump-bids in the auction for an item, the price clock for this item pauses for at most δ seconds for any active bidder to react. The pause ends if all such bidders have reacted or if δ seconds has passed.

If a bidder say i stops or withdraws at the auction for an item when its current price is p, during the pause of the price clock for this item, any active bidder can withdraw. If all remaining bidders withdraw during the pause, the good is sold to bidder i at the price p if i did not withdraw, and the good is not sold, with withdrawal penalty divided among all withdrawing bidders, if i did withdraw. If exactly one active bidder does not withdraw during the pause, the good is sold to this active bidder at price p. If more than one active bidder does not withdraw in the pause, the price clock resumes from the level p.

If a bidder say i submits a jump bid b for an item, during the pause of the price clock for this item, every other active bidder decides whether to stop or *match* b (with a bid equal to b) or *top* b (with a higher jump bid). If someone tops b with a higher bid b', the process repeats with the new jump bid b'. If someone matches a jump bid and no one tops it, the pause ends and the price clock resumes from the current highest jump bid. If all but the jump-bidder stops, the jump bidder buys the item at his most current jump bid.

Note that the above amendments do not require any coordination between auctioneers of different goods. Hence the model continues to capture the decentralized nature of markets.

Should central coordination be available, the exposure problem can be eliminated trivially: when local bidder α drops out, pause the auction of item A until the auction of item B ends and then let the global bidder decide whether to buy A or not. A main point of the next subsection is that central coordination is completely unnecessary.

4.2 Jump bidding in the decisive moment

A bidder is called *the first dropout* if he stops or withdraws from the item(s) for which he has been bidding while none of other bidders have stopped or withdrawn. If a local bidder say α, who has been bidding for item A, is the first dropout, the *decisive moment* refers to the minute interval after α's dropout action and before global bidder γ has decided whether to withdraw from item A or not. If γ does not withdraw from A during this moment, he buys item A when the pause caused by α's dropout ends. Since γ's maximum willingness-to-pay for item B jumps when he buys item A , local bidder β wants to influence γ's decision in the decisive moment through jump bidding for B. Such jump bidding eliminates the exposure problem for the global bidder:

Proposition 1 *Assume that it takes less than half of the maximum time (δ seconds) of a decisive moment to cry out a bid and register it. At any equilibrium of the simultaneous-auctions game, if the global bidder wins an item at a positive price, then he wins its complement and, before buying any of them, he knows the total price for both items.*

This proposition follows from Lemmas 6 and 8 that will be proved in this subsection. To prove these lemmas, we shall analyze the continuation game after a local bidder say α becomes the first dropout from item A. We shall see that this continuation game turns into a very fast English auction for item B that completes within the decisive moment. During this English auction, the active local bidder β's maximum willingness-to-pay ($MWTP$) for item B is simply his value t_β, but the global bidder γ's MWTP for item B is less than γ's value t_γ, since he can withdraw from A during the decisive moment.

Lemma 3 *If a local bidder say α is the first dropout when the current price for item A is p_A,* *then, during the decisive moment, local bidder β's MWTP for item B is*

$$w_\beta := \tilde{w}_\beta(t_\beta) := t_\beta, \tag{9}$$

and global bidder γ's MWTP for item B is

$$w_\gamma := \tilde{w}_\gamma(t_\gamma, p_A, \lambda) := t_\gamma - \lambda p_A, \tag{10}$$

where

$$\lambda := \begin{cases} 1 & \text{if } \alpha\text{'s action is "stop"} \\ 1/2 & \text{if } \alpha\text{'s action is "withdraw".} \end{cases}$$

Proof Consider the case where bidder α's dropout action is "stop". Then α cannot raise his bid for A any more, so bidder γ can buy A at its current price p_A by the action "continue". Thus, if γ buys item B at some price p_B during the decisive moment, he will buy A at the end of the moment and get a total profit $t_\gamma - p_A - p_B$. As α's action is not "withdraw", bidder γ can also ensure a zero payoff by withdrawing from A, for then item A will be sold to bidder α at its current price and so γ does not need to pay any withdrawal penalty. Hence bidder γ buys item B in the decisive moment if and only if p_B is less than $t_\gamma - p_A$, as claimed. The case where bidder α's dropout action is "withdraw" is similar: if he buys B in the decisive moment, his payoff is $t_\gamma - p_A - p_B$; else (via withdrawing from A) his payoff is $-p_A/2$, since he needs to pay half of the bid p_A that bidder α and he both withdraw. Then bidder γ buys B in the decisive moment if and only if its price is less than $t_\gamma - p_A/2$. ■

Lemma 4 *If a local bidder say α is the first dropout when the current price for item A is p_A, then, at any continuation equilibrium on whose path the winner of item B is determined during the decisive moment, item B goes to the bidder whose MWTP during the decisive moment for B is higher.*

Proof This is similar to the dominance solvability argument of second-price auctions, except that the continuation game after α's dropout may involve signaling through open outcries. Given any continuation equilibrium e, let $W_i(e,h)$ denote the posterior support of bidder i's ($i = \beta, \gamma$) MWTP conditional on current history h. Let p_B denote the current price of B. Then obviously it is weakly dominated for bidder i to stop or withdraw from item B when

$$w_i > \max\{p_B, \inf W_{-i}(e,h)\}.$$

It is also weakly dominated for bidder i to submit a bid above w_i, because a winner has to pay his own (jump) bid during the decisive moment. Therefore, coupled with the rational expectations $w_{-i} \geq \inf W_{-i}(e, h)$ at any equilibrium, the lemma follows. ■

Lemma 5 *If a local bidder α is the first dropout when the current price for item A is p_A, then, at any continuation equilibrium on whose path the winner of item B is determined in the decisive moment, each remaining bidder's expected payment (viewed at the start of the decisive moment) conditional on winning B is uniquely determined: if bidder i's ($i \in \{\beta, \gamma\}$ and $-i$ is this set minus i) MWTP during the decisive moment is w_i (Lemma 3) and if h denotes the history up to the start of this moment, bidder i's expected payment is equal to*

$$\mathcal{P}_i(w_i) := \mathrm{E}\left[\mathbf{w}_{-i} \mid \mathbf{w}_{-i} \leq w_i; h\right]. \tag{11}$$

Proof At any such continuation equilibrium, the allocation is uniquely determined by Lemma 4. As bidders' payoff functions are in the standard quasilinear form (recalling (10)), the payoff-equivalence theorem in auction theory implies this lemma. ■

Lemma 6 *Suppose, conditional on any event that a local bidder for an item k is the first dropout when its price is positive, there exists a continuation equilibrium on whose path the winner of the other item is determined during the decisive moment. Then, at any equilibrium of the simultaneous-auctions game, whenever a local bidder is the first dropout when prices are positive, the winner(s) of both items are determined in the same decisive moment.*

Proof Without loss of generality, let bidder α be the first dropout when $p_A > 0$. It suffices to show that each of bidders β and γ strictly prefers having the winner of B determined during the decisive moment to after the moment. This suffices because: (i) an active bidder, through jump-bidding, can unilaterally initiate the process of determining the winner of B in the decisive moment; (ii) once initiated, this process keeps going unless an active bidder chooses not to top his rival's bid or unless the decisive moment ends; and (iii) the process can be completed during the decisive moment due to the lemma's assumed existence of the desired continuation equilibrium.

Let us demonstrate such preference for bidder β. If the winner of B is determined during the decisive moment, β's winning event is $\{t_\gamma : \tilde{w}_\gamma(t_\gamma, p_A, \lambda) < t_\beta\}$ and his payment conditional on winning is (11) (Lemmas 4 and 5). If the winner of B is not determined in the decisive moment, bidder γ buys A after the moment (if γ withdraws from A during the moment then he would also have dropped out from B, thereby determining the winner of B) and then will bid for B up to his value t_γ; hence β's winning event is $\{t_\gamma : t_\gamma < t_\beta\}$ and his payment conditional on winning is $\mathrm{E}[\mathbf{t}_\gamma \mid \mathbf{t}_\gamma < t_\beta]$. Since $p_A > 0$, $\tilde{w}_\gamma(t_\gamma, p_A, \lambda) < t_\gamma$ (Eq. (10)), hence bidder β's expected payoff in the former case is higher.

Let us show such preference for bidder γ for the case where bidder α's dropout action is "withdraw" (the case where α's action is "stop" is simpler). If the winner of B is determined in the decisive moment, bidder γ's payoff is either $t_\gamma - p_A - t_\beta$ if γ wins B (if he wins B then he buys A) or $-p_A/2$ if γ loses B (if he loses B then he withdraws from A, paying half of the withdrawal penalty); i.e., γ's payoff is $(t_\gamma - t_\beta - p_A/2)^+ - p_A/2$. If the winner of B is determined after the decisive moment, bidder γ's payoff is $(t_\gamma - t_\beta)^+ - p_A/2 - p_A/2$. As $p_A > 0$, $(t_\gamma - t_\beta - p_A/2)^+ - p_A/2 \geq (t_\gamma - t_\beta)^+ - p_A/2 - p_A/2$ for all possible t_β and strictly so for some t_β. Hence bidder γ has our desired preference. ■

Lemma 7 *For each $i = \beta, \gamma$, the function $\mathcal{P}_i$ defined in (11) is weakly increasing; furthermore, given any history h up to the instant when α becomes the first dropout, for every x_i in the range of $\mathcal{P}_i$ and for almost every possible w_{-i} (relative to the posterior given h),*

$$\inf \mathcal{P}_i^{-1}(x_i) \geq w_{-i} \quad or \quad \sup \mathcal{P}_i^{-1}(x_i) \leq w_{-i}. \tag{12}$$

Proof By definition (11), the function $\mathcal{P}_i$ is weakly increasing. It is not necessarily strictly increasing only because the posterior distribution of $\mathbf{w}_{-i}$ conditional on history h may have gaps: By (11), $\mathcal{P}_i(w_i) = \mathcal{P}_i(w_i')$ if and only if this distribution has zero weight strictly between w_i and w_i'; i.e., for any x_i in the range of $\mathcal{P}_i$, the event "w_{-i} belongs to the interior of $\mathcal{P}_i^{-1}(x_i)$" has zero probability. Hence (12) is true almost surely conditional on h. ■

Before proving Proposition 1, we need to construct a continuation equilibrium that determines the winner of B within the decisive moment. Here is the idea of the construction. As already shown, once bidder α becomes the first dropout (from A), the other two bidders are both willing to speed up the auction for item B. Hence one of them immediately cries out a jump bid that fully reveals his MWTP. If this revealed value exceeds the other bidder's MWTP, the latter immediately drops out; else the latter tops the former with a bid equal to this revealed value, which makes the former immediately drop out. Thus, on equilibrium path, the winner of B is determined with at most two jump bids. This is physically feasible as long as crying out a jump bid takes sufficiently less time than the δ-second pause. To ensure incentive compatibility, we construct a bidder's jump bid as the expected value of his rival's MWTP conditional on the rival's defeat. In expectation, a bidder cannot do better than bidding this amount: in the English auction, he cannot do better than achieving the Vickrey outcome where he wins if and only if his MWTP is higher than his rival's and he pays the rival's MWTP if he wins.[2] By Lemma 7, the bid amount constructed in this fashion fully reveals a jump-bidder's private information almost surely.

Lemma 8 *If a local bidder say α is the first dropout when the current price for item A is $p_A > 0$, and if crying out a bid takes less than half of the maximum time (δ seconds) of the decisive moment, then there exists a continuation equilibrium on whose path the winner of item B is determined during the decisive moment.*

Proof Let h denote the history up to the start of the decisive moment, and let $W_i(h)$ be the support of bidder i's MWTP at the start of this moment conditional on h. We construct a continuation equilibrium:

a. Bidder β with MWTP $w_\beta := t_\beta$: if $w_\beta \leq p_B$, immediately withdraw; if $w_\beta > p_B$, immediately make a jump bid for B equal to $\mathcal{P}_\beta(w_\beta)$ defined by (11).

b. Bidder γ with MWTP w_γ given by (10): If bidder β has withdrawn, buy both items immediately. If β has made a jump bid x_β for B:

   i. If x_β belongs to the range $\mathcal{P}_\beta(W_\beta(h))$ of $\mathcal{P}_\beta$:

      - if $w_\gamma > \inf \mathcal{P}_\beta^{-1}(x_\beta)$, top x_β with a jump bid equal to

$$\tilde{\mathcal{P}}_\gamma(w_\gamma \mid x_\beta) := \mathrm{E}\left[\mathbf{w}_\beta \mid \mathbf{w}_\beta \leq w_\gamma; \mathbf{w}_\beta \in \mathcal{P}_\beta^{-1}(x_\beta); h\right]; \qquad (13)$$

[2]The bidders have no unilateral incentive to collude by dropping out simultaneously: Since dropout is irrevocable, a colluder cannot retaliate the other colluder for deviation.

- else withdraw from both items.

ii. If $x_\beta \notin \mathcal{P}_\beta(W_\beta(h))$, which is off-path, then update $\mathcal{P}_\beta^{-1}(x_\beta) := \{(x_\beta + w_\gamma)/2\}$ and then follow the previous plan (i) with h removed.

c. If bidder γ responds to β's jump bid with a bid x_γ, bidder β replies by following plans (i) and (ii) in the previous item, with the substitutions $W_\beta(h) \to W_\gamma(h)$, $\mathcal{P}_\beta^{-1}(x_\beta) \to \tilde{\mathcal{P}}_\gamma^{-1}(x_\gamma \mid x_\beta)$, $\tilde{\mathcal{P}}_\gamma(w_\gamma \mid x_\beta) \to \tilde{\mathcal{P}}_\beta(w_\beta \mid x_\beta, x_\gamma)$, $\mathbf{w}_\beta \to \mathbf{w}_\gamma$, $w_\gamma \to w_\beta$, and $x_\beta \to x_\gamma$.

d. If bidder β does not act immediately at the start of the decisive moment, bidder γ immediately acts by following plans (a)–(c) with the roles of β and γ switched.

On the path of this proposed equilibrium, almost surely the winner of item B is determined before the decisive moment ends: Once bidder β has cried out his initial jump bid x_β (in less than $\delta/2$ seconds), almost surely $\inf \mathcal{P}_\beta^{-1}(x_\beta) \geq w_\gamma$ or $\sup \mathcal{P}_\beta^{-1}(x_\beta) \leq w_\gamma$ ((12)). In the first case, γ withdraws immediately (plan b-i). In the second case, γ replies (in less than $\delta/2$ seconds) with a jump bid $\tilde{\mathcal{P}}_\gamma(w_\gamma \mid x_\beta)$ according to Eq. (13); seeing this bid, bidder β learns that $\sup \mathcal{P}_\beta^{-1}(x_\beta) \leq w_\gamma$; as $w_\beta \leq \sup \mathcal{P}_\beta^{-1}(x_\beta)$, bidder β drops out (plan c). Thus, as long as bidder γ postpones his decision on item A to the end of the δ-second pause, the winner of B is determined within the decisive moment.

Before checking the equilibrium conditions, let us prove the following claims for each bidder $i \in \{\beta, \gamma\}$ during the decisive moment, given rival $-i$'s strategy.

1. If the other bidder $-i$'s MWTP w_{-i} has been fully revealed to i, then bidder i's best reply is: bid w_{-i} if $w_i > w{-}i$ and withdraw if $w_i \leq w{-}i$.

2. Suppose w_{-i} has not been fully revealed to i. Then bidder i knows: if he bids now and if rival $-i$'s immediate response is a bid x_{-i} instead of dropout, then the bid x_{-i} fully reveals w_{-i}.

3. If w_{-i} has not been fully revealed to i, then bidder i knows: if he bids now, if rival $-i$'s immediate response is a bid instead of dropout, and if bidder i eventually wins B during the decisive moment, then $w_i \geq w{-}i$ and bidder i's payment for B is w_{-i}.

Proof of claim 1: Bidding above w_{-i} is obviously dominated. If his bid b_i is below w_{-i}, then b_i is outside the range of i's bids (Eqs. (11) and (13)), hence the other bidder $-i$ will follow plan b.ii and hence will cry out a bid strictly between b_i and w_{-i}, so that bidder i cannot win immediately. Hence bidding below w_{-i} does not make i better off and it makes i worse off if the decisive moment ends (Lemma 6).

Proof of claim 2: Let W_{-i} denote the nondegenerate support of rival $-i$'s MWTP at an instant during the decisive moment. Then the set W_{-i} is commonly known at this instant; otherwise, the set contains bidder i's private information and hence, by plans b.i and b.ii in the proposed equilibrium, the set is singleton (the midpoint between i's MWTP and $-i$'s most current bid) and hence degenerate. With equilibrium expectation about rival $-i$, bidder i knows: if i bids an amount x_i greater than or equal to the expected value of $\mathbf{w}_{-i}$ conditional on W_{-i}, then rival $-i$ will immediately quit (similar to the second case in the previous paragraph on the equilibrium path). Thus, bidder i knows that if $-i$ does not quit immediately then x_i has to be less than this expected value, which is commonly known as W_{-i} is so, then x_i has to be outside the commonly known range of i's bid function. By updating rule b.ii in the proposed equilibrium, rival $-i$'s posterior is "$w_i = (x_i + w_{-i})/2$." If $-i$ does not immediately quit after i bids x_i, then $(x_i + w_{-i})/2 < w_{-i}$ and $-i$ will respond by bidding $(x_i + w_{-i})/2$, which fully reveals w_{-i} to bidder i, as claimed.

Proof of claim 3: In the future event that rival $-i$ does not quit after bidder i's current bid and bidder i still can win during the decisive moment, claim 2 implies that w_{-i} will be fully revealed to bidder i before i makes the winning bid. Then claim 1 implies claim 3 since bidder i knows he himself will best reply at that future event.

With the claims proved above, let us verify the equilibrium condition for each bidder $i \in \{\beta, \gamma\}$. Consider any instant during the decisive moment. If the other bidder $-i$'s MWTP has been fully revealed, then the best reply described in claim 1 is exactly the action prescribed by plans b.i and b.ii in the proposed equilibrium. Hence suppose that $-i$'s MWTP has not been fully revealed. Let $\tilde{\mathcal{P}}_i(\cdot \mid h')$ denote the bid function for bidder i conditional on the history h' up to the current instant (e.g., Eq. (13)). For any $\hat{w}_i$ in the posterior support of i's MWTP given h', let

$$[\hat{w}_i] := \tilde{\mathcal{P}}_i^{-1}\left(\tilde{\mathcal{P}}_i(\hat{w}_i \mid h') \;\middle|\; h'\right).$$

Let $[\hat{w}_i] \geq w_{-i}$ denote the event that every element in $[\hat{w}_i]$ is greater than or equal to w_{-i}. We shall show that it is optimal for bidder i to bid according to function $\tilde{\mathcal{P}}_i(\cdot \mid h')$. Bidding outside the range of the bid function is dominated, by a reasoning similar to the proof of claim 1. Bidding within the range of $\tilde{\mathcal{P}}_i(\cdot \mid h')$ is equivalent to picking a $\hat{w}_i$ from the current support of $\mathbf{w}_i$ and announcing that his MWTP belongs to $[\hat{w}_i]$ and promising to pay $\tilde{\mathcal{P}}_i(\hat{w}_i \mid h')$ if he wins immediately. Let $u_i(\hat{w}_i, w_i)$ denote bidder i's expected payoff from this action, conditional on current history h' and his true MWTP w_i. We prove next that, given w_i, $u_i(\hat{w}_i, w_i)$ is maximized when $\hat{w}_i = w_i$.

If $\hat{w}_i < w_i$, then item B goes to i if and only if: (i) either the other bidder $-i$ drops out immediately, i.e., $[\hat{w}_i] \geq w_{-i}$ by (12), or (ii) bidder $-i$ does not drop out immediately but does so in a later round before the decisive moment ends; by (12) and claim 3, case (ii) is contained by the event $\{w_{-i} : [\hat{w}_i] < w_{-i} \leq w_i\}$. Thus,

$$\begin{aligned}
u_i(\hat{w}_i, w_i) &\leq \mathrm{E}\left[1_{[\hat{w}_i]\geq \mathbf{w}_{-i}}(\mathbf{w}_{-i})\left(w_i - \tilde{\mathcal{P}}_i(\hat{w}_i \mid h')\right) + 1_{[\hat{w}_i]<\mathbf{w}_{-i}\leq w_i}(\mathbf{w}_{-i})(w_i - \mathbf{w}_{-i})^+ \middle| h'\right] \\
&= \mathrm{E}\left[1_{\hat{w}_i\geq \mathbf{w}_{-i}}(\mathbf{w}_{-i})\left(w_i - \tilde{\mathcal{P}}_i(\hat{w}_i \mid h')\right) + 1_{\hat{w}_i<\mathbf{w}_{-i}\leq w_i}(\mathbf{w}_{-i})(w_i - \mathbf{w}_{-i})^+ \middle| h'\right] \\
&= w_i \mathrm{E}\left[1_{w_i\geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h'\right] - \mathrm{E}\left[\mathbf{w}_{-i} 1_{w_i\geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h'\right] \\
&= u_i(w_i, w_i),
\end{aligned}$$

where the first equality uses (12) and the second uses (13). Thus, bidder i cannot gain from under-reporting his type.

If $\hat{w}_i \geq w_i$, then item B goes to i if and only if the other bidder $-i$ immediately drops out after i has jump-bid $\tilde{\mathcal{P}}_i(\hat{w}_i \mid h')$. (By claim 3, if i cannot outbid $-i$ with $\hat{w}_i$ now, he cannot outbid $-i$ with his lower true value w_i afterwards.) This winning event is $\{w_{-i} : \hat{w}_i \geq w_{-i}\}$ by (12). Thus,

$$\begin{aligned}
u_i(\hat{w}_i, w_i) &= \left(w_i - \tilde{\mathcal{P}}_i(\hat{w}_i \mid h')\right) \mathrm{E}\left[1_{\hat{w}_i\geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h'\right] \\
&= w_i \mathrm{E}\left[1_{\hat{w}_i> \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h'\right] - \mathrm{E}\left[(\mathbf{w}_{-i}) 1_{\hat{w}_i\geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h'\right] \\
&= (w_i - \hat{w}_i)\mathrm{E}\left[1_{\hat{w}_i\geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h'\right] + \int_{p_B}^{\hat{w}_i} \mathrm{E}\left[1_{z_i\geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h'\right] dz_i,
\end{aligned}$$

where the second equality uses (12) and the third uses integration by parts. As the probability $\mathrm{E}\left[1_{\hat{w}_i\geq \mathbf{w}_{-i}}(\mathbf{w}_{-i}) \mid h'\right]$ is weakly increasing in $\hat{w}_i$, the above equation implies that picking $\hat{w}_i = w_i$ maximizes $u_i(\cdot, w_i)$ (Myerson [15, Lemma 2]). Thus, bidder i cannot gain from over-reporting. It follows that the bid $\tilde{\mathcal{P}}_i(\hat{w}_i \mid h')$ is bidder i's best reply, as desired. ■

Proof of Proposition 1 Suppose global bidder γ wins an item. Then he cannot be the first dropout; otherwise, he would have withdrawn from both items. Hence bidder α or β is the first dropout. Without loss of generality, let α be the first dropout when the current price for item A is p_A. By Lemmas 6 and 8, item B is won by either β or γ during the decisive moment. If β wins B, bidder γ withdraws from A in this moment, so the conclusion of this proposition is vacuously true; if γ wins B at some price p_B, he buys A at the price p_A, which has been frozen since α's dropout. Hence bidder γ knows the total price $p_A + p_B$ when he buys any of the items, and the conclusion of this proposition is again true. ■

Corollary 1 *If it takes less than half of the maximum time of the decisive moment to submit a bid, then "withdraw" is weakly dominated by "stop" for a local bidder when he becomes the first dropout and when the price is positive and less than or equal to the bidder's value.*

Proof Without loss of generality, let α be the first dropout when the current price for item A is p_A. By Lemmas 6 and 8, item B is won by either β or γ during the decisive moment. If global bidder γ wins B, then he continues on A, so "stop" and "withdraw" both yield zero payoff for bidder α. If γ loses B, then he withdraws from A, so bidder α gets a nonnegative payoff from "stop" and gets a negative payoff (penalty $-p_A/2$) from "withdraw". ■

4.3 Jump bidding leads to over-concentration

Proposition 1 implies that the global bidder knows whether he can profitably acquire both items before he commits to buying one of them. Hence he faces no exposure problem and is effectively bidding for the entire package $\{A, B\}$, so he would not underbid. The local bidders, in contrast, do not always bid up to their true values: A local bidder who drops out may win his desired item because the other local bidder submits a jump bid that may force the global bidder to quit during the decisive moment. Hence a local bidder wishes to free ride the other. This *threshold problem* is exactly the same as the classic public goods problem with private information. By the uniqueness of the equilibrium allocation in the continuation game after a local bidder becomes the first dropout (Proposition 1), this threshold problem cannot be eliminated no matter how bidders signal to each other.

Let $\prec$ denote an allocation, with $t_\gamma \prec (t_\alpha, t_\beta)$ denoting the event "bidder α wins A or bidder β wins B." Given an equilibrium-feasible allocation $\prec$ and for each local bidder i, let $P_i(t_i, \prec)$ denote the expected value of i's equilibrium payment, viewed at the start of the game, given his type t_i. The next lemma is a reinterpretation of the impossibility of efficient provision of public goods (Krishna and Perry [11, §8.2]), with the cost of public goods being the global bidder's value that local bidders' combined bid needs to top.

Lemma 9 *If the global bidder always truthfully reports his value t_γ, then it is impossible to have an equilibrium-feasible allocation $\prec$ that is almost surely ex post efficient and*

$$\mathrm{E}\left[P_\alpha(\mathbf{t}_\alpha, \prec) + P_\beta(\mathbf{t}_\beta, \prec)\right] \geq \mathrm{E}\left[\mathbf{t}_\gamma 1_{\mathbf{t}_\gamma \prec (\mathbf{t}_\alpha, \mathbf{t}_\beta)}(\mathbf{t}_\gamma, \mathbf{t}_\alpha, \mathbf{t}_\beta)\right]. \tag{14}$$

Proof From the quasilinear utility functions, the equilibrium condition, and the efficiency of allocation $\prec$, one can prove, with standard mechanism-design techniques, that

$$P_i(t_i, \prec) \leq \mathrm{E}\left[(\mathbf{t}_\gamma - \mathbf{t}_{-i}) 1_{0 < \mathbf{t}_\gamma - \mathbf{t}_{-i} < t_i}(\mathbf{t}_\gamma, \mathbf{t}_{-i})\right]$$

for each local bidder i ($-i$ denotes the other local bidder). Then, denoting $1_S := 1_S(\mathbf{t}_\gamma, \mathbf{t}_\alpha, \mathbf{t}_\beta)$,

$$\begin{aligned}
&\mathrm{E}\left[P_\alpha(\mathbf{t}_\alpha, \prec) + P_\beta(\mathbf{t}_\beta, \prec)\right] \\
&\quad \leq \mathrm{E}\left[\left[(\mathbf{t}_\gamma - \mathbf{t}_\alpha)^+ + (\mathbf{t}_\gamma - \mathbf{t}_\beta)^+\right] 1_{\mathbf{t}_\gamma < \mathbf{t}_\alpha + \mathbf{t}_\beta}\right] \\
&\quad = \mathrm{E}\left[\left(1_{\mathbf{t}_\gamma < \mathbf{t}_\alpha + \mathbf{t}_\beta}\right)\left[\mathbf{t}_\gamma - \left[\begin{array}{c} (\mathbf{t}_\alpha + \mathbf{t}_\beta - \mathbf{t}_\gamma) 1_{\mathbf{t}_\gamma > \max\{\mathbf{t}_\alpha, \mathbf{t}_\beta\}} + \mathbf{t}_\alpha 1_{\mathbf{t}_\beta > \mathbf{t}_\gamma > \mathbf{t}_\alpha} \\ + \mathbf{t}_\beta 1_{\mathbf{t}_\alpha > \mathbf{t}_\gamma > \mathbf{t}_\beta} + \mathbf{t}_\gamma 1_{\mathbf{t}_\gamma < \min\{\mathbf{t}_\alpha, \mathbf{t}_\beta\}} \end{array}\right]\right]\right] \\
&\quad < \mathrm{E}\left[\mathbf{t}_\gamma 1_{\mathbf{t}_\gamma < \mathbf{t}_\alpha + \mathbf{t}_\beta}\right].
\end{aligned}$$

This contradicts (14), since $t_\gamma \prec (t_\alpha, t_\beta) \Leftrightarrow t_\gamma < t_\alpha + t_\beta$ by the efficiency of allocation $\prec$. ∎

Proposition 2 *If jump-bidding is allowed, then, in any equilibrium of the simultaneous-auctions game, the allocation is over-concentrated with a positive probability and is never over-diffused.*

Proof Take any equilibrium specified by the hypothesis and let $\prec$ denote its allocation. By Proposition 1, the global bidder does not quit until the total price of both items has reached his value or until he knows the total price will reach his value. Hence it suffices to show that there is a positive probability with which some local bidder's equilibrium dropout price is less than his value. Suppose that this probability were zero, then if local bidder say α is the first dropout then $p_A = t_\alpha$ almost surely. It follows from Lemma 4 that the equilibrium allocation is ex post efficient almost surely. By Lemma 9, we will have a desired contradiction if (14) holds.

To prove (14), note that the event $t_\gamma \prec (t_\alpha, t_\beta)$ means: global bidder γ either (i) quits before winning any item or (ii) quits after winning one but not both. In case (i), he quits only if $p_A + p_B \geq t_\gamma$ currently or a local bidder say α quits at p_A and the other local bidder β outbids γ in the jump-bidding subgame. The inequality $p_A + p_B \geq t_\gamma$ automatically holds in the first subcase and it holds in the second subcase by Lemma 4 and Corollary 1 (the first dropout stops rather than withdraws, so $w_\gamma = t_\gamma - p_A$). In case (ii), having bought an item, γ bids for the other up to his value t_γ, hence the winning local bidder's payment is equal to t_γ. Again $p_A + p_B \geq t_\gamma$ holds. Hence (14) follows, as desired. ■

5 Partial extension to cross-bidding

Cross-bidding means bidding for an item which always has zero value for the bidder, e.g., bidder α bidding for B or β bidding for A. The previous sections assume that cross-bidding is not allowed. That assumption, at lease when jump-bidding is banned, is not innocuous, because a local bidder may wish to cross-bid: In the equilibrium in Lemma 1, before winning any item, the global bidder's highest total bid for both items is less than his valuation of the whole package ($t_\gamma > p^*_A(t_\gamma) + p^*_B(t_\gamma)$ in Lemma 2). But once he has won an item, his highest bid for its complement jumps to his valuation of both items (plan 3 in Lemma 1). Hence a local bidder say α wishes to bid for the zero-value item B in order to prevent the global bidder from becoming aggressive after winning B when local bidder β quits.

5.1 When jump bidding is banned

The model in this subsection is the same as the basic mechanism in §3.1 except that cross-bidding is allowed. It turns out that the equilibrium in Lemma 1 remains valid on path:

Despite local bidders' intention, cross-bidding mitigates the global bidder's exposure problem and hence makes him less willing to quit before the local bidders. Knowing this, each local bidder would rather not cross-bid. However, there is possibly another equilibrium where cross-bidding does occur: After a local bidder has quit, the other local bidder may stay cross-bidding and implicitly threaten to quit only one item at a time. Then the exposure problem may come back to suppress the global player's bid if he updates beliefs in a certain way. If this effect dominates the previous mitigating effect, local bidders prefer cross-bidding.

Lemma 10 *If cross-bidding is allowed and jump-bidding is not, and if local bidders have to cross-bid, then listed below are the only two undominated strategies for local bidder α, and the case for local bidder β is symmetric by switching the roles A$\leftrightarrow$B and $\alpha \leftrightarrow \beta$.*

a. *Keep bidding for both items until—*

   i. *if the global bidder quits before others, then quit B and continues A; or*

   ii. *if $p_A \geq t_\alpha$, then quit both items immediately; or*

   iii. *if the other local bidder β has quit B and $p_A + p_B \geq t_\alpha$, quit both items immediately;*

   iv. *if having quit A somehow, quit B immediately.*

b. *Follow the same plan as in (a) except that (a-iii) is changed to*

   iii* *if the other local bidder β has quit B and $p_A + p_B \geq t_\alpha$, then immediately quit B and stay for A until $p_A \geq t_\alpha$.*

Proof For local bidder α, plans (a-i), (a-ii), and (a-iv) are obviously dominant. We need only to consider the case for plans (a-iii) and (a-iii*), when α is alone against γ (as β has quit B, by iterated truncation of dominated actions he has quit A):

When $p_A + p_B < t_\alpha$, it is obviously dominated to quit both items or to quit A and continue B. Quitting B and staying for A is also weakly dominated: doing so yields a payoff $(t_\alpha - t_\gamma)^+$ for bidder α, since bidder γ, after winning B, will bid for A up to his value t_γ. In contrast, if α stays for both until the total price reaches t_α, he gets a payoff $(t_\alpha - \hat{t}_\gamma)^+$, where $\hat{t}_\gamma$ is the level of the total price at which γ quits. Obviously, $\hat{t}_\gamma \leq t_\gamma$; moreover, $\hat{t}_\gamma < t_\gamma$

if bidder γ is worried by the exposure problem when α is cross-bidding: γ may think it probable that α will quit B and stay for A up to t_α and hence γ may quit before the total price reaches t_γ. Thus, it is weakly dominated for α to quit any item at this point.

If $p_A + p_B \geq t_\alpha$, it is dominated to continue both items or to continue B and quit A. Hence α either quits both items or quits B and stays for A. They are indifferent to α: quitting both items yields zero payoff; quitting B and staying for A also yields zero payoff, because the fact that γ has not quit implies $t_\gamma \geq p_A + p_B$ and hence $t_\gamma \geq t_\alpha$; having won B, bidder γ will bid up to t_γ and hence α's payoff, whether he wins or not, will be zero. Hence (a-iii) and (a-iii*) are the only undominated plans when α is alone against γ. ■

Lemma 11 *If local bidders cross-bid and play strategy (a) in Lemma 10, global bidder γ has a unique best reply and it is—*

- *c-i. if one local bidder has quit both items and the other local bidder is cross-bidding, bid for both items until their total price reaches t_γ and then immediately quit both;*
- *c-ii. if both local bidders are staying for at least an item, follow the strategy in Lemma 1 with this revision: if α is cross-bidding, replace $v_B(t_\gamma, p_A)$ in plan 1 by $t_\gamma - p_A$; if β is cross-bidding, replace $v_A(t_\gamma, p_B)$ in plan 1 by $t_\gamma - p_B$.*

Proof Note: if a local bidder is cross-bidding and plays strategy (a), he quits both items simultaneously if he quits an item before global bidder γ quits. Hence plan (c-i) is opitmal.

To demonstrate (c-ii), let (p_A, p_B) be the current prices when both local bidders are bidding for something. First, suppose that both local bidders have been cross-bidding up to now. If γ wins item B now, bidder β must be quitting A and bidder α must be quitting both items right now (both following strategy (a)), hence γ immediately wins item A and gets a profit $t_\gamma - p_A - p_B$. Thus, conditional on staying for A, bidder γ stays for B if and only if $t_\gamma - p_A > p_B$. By the same argument, conditional on staying for B, γ stays for A if and only if $t_\gamma - p_B > p_A$. Hence his dropout strategy when both locals are cross-bidding is precisely the strategy in Lemma 1 with the substitutions $v_B(t_\gamma, p_A) \to t_\gamma - p_A$ and $v_A(t_\gamma, p_B) \to t_\gamma - p_B$.

Next suppose that, currently, local bidder α is cross-bidding and β is not. By the same reasoning in the previous paragraph, winning item B now implies a profit $t_\gamma - p_A - p_B$ for γ and hence $v_B(t_\gamma, p_A)$ in plan 1 is replaced by $t_\gamma - p_A$. In contrast, since β is bidding only for B, almost surely β stays for B after bidder γ has won A. Hence γ's expected profit from buying A at the current instant is equal to $v_A(t_\gamma, p_B) - p_A$, as in Lemma 1. Hence his dropout strategy in this case is precisely the strategy in Lemma 1 with the only substitution $v_B(t_\gamma, p_A) \to t_\gamma - p_A$. The case where only β is cross-bidding is symmetric. The only other case where neither local bidder is cross-bidding is identical to the case in Lemma 1. ■

To find the global bidder's best reply to strategy (b), define the following for any $(p_A, p_B) \in [0, \bar{t}_\alpha] \times [0, \bar{t}_\beta]$ and $t_\gamma \in (0, \bar{t}_\gamma]$:

$$\tilde{v}_A(t_\gamma, p_A, p_B) := \mathrm{E}\left[(t_\gamma - \mathbf{t}_\beta)^+ \mid p_B \le \mathbf{t}_\beta \le p_A + p_B\right]; \tag{15}$$

$$\tilde{v}_B(t_\gamma, p_A, p_B) := \mathrm{E}\left[(t_\gamma - \mathbf{t}_\alpha)^+ \mid p_A \le \mathbf{t}_\alpha \le p_A + p_B\right]; \tag{16}$$

$$x'(t_\gamma) := \sup\left\{x \in \left[0, \min\left\{\bar{t}_\alpha/\dot{p}_A, \bar{t}_\beta/\dot{p}_B, t_\gamma/(\dot{p}_A + \dot{p}_B)\right\}\right] : \tilde{v}_A(t_\gamma, x\dot{p}_A, x\dot{p}_B) > x\dot{p}_A\right\}; \tag{17}$$

$$x''(t_\gamma) := \sup\left\{x \in \left[0, \min\left\{\bar{t}_\alpha/\dot{p}_A, \bar{t}_\beta/\dot{p}_B, t_\gamma/(\dot{p}_A + \dot{p}_B)\right\}\right] : \tilde{v}_B(t_\gamma, x\dot{p}_A, x\dot{p}_B) > x\dot{p}_B\right\}; \tag{18}$$

$$x^*(t_\gamma) := \min\{x'(t_\gamma), x''(t_\gamma)\}. \tag{19}$$

Lemma 12 *For any fixed positive speeds $\dot{p}_A$ and $\dot{p}_B$ of the prices, and for any $t_\gamma \in (0, \bar{t}_\gamma]$, the cutoffs $x'(t_\gamma)$ and $x''(t_\gamma)$ defined above exist and are each unique and less than $t_\gamma/(\dot{p}_A + \dot{p}_B)$; moreover, for all $x \in \left[0, \min\left\{\bar{t}_\alpha/\dot{p}_A, \bar{t}_\beta/\dot{p}_B, t_\gamma/(\dot{p}_A + \dot{p}_B)\right\}\right]$, $\tilde{v}_A(t_\gamma, x\dot{p}_A, x\dot{p}_B) < x\dot{p}_A$ if $x > x'(t_\gamma)$, and $\tilde{v}_B(t_\gamma, x\dot{p}_A, x\dot{p}_B) < x\dot{p}_B$ if $x > x''(t_\gamma)$.*

Proof We shall prove the lemma for the case of $\tilde{v}_A(t_\gamma, x\dot{p}_A, x\dot{p}_B)$ and $x'(t_\gamma)$; the case of $\tilde{v}_B(t_\gamma, x\dot{p}_A, x\dot{p}_B)$ and $x''(t_\gamma)$ is analogous. By (15) and (16), $\tilde{v}_A(t_\gamma, 0, 0) = \tilde{v}_B(t_\gamma, 0, 0) > 0$ if $t_\gamma > 0$, hence $x'(t_\gamma)$ defined in (17) exists. As F_β has no gap, (15) implies

$$\left[p_A > 0 \;\&\; p_B < t_\gamma \;\&\; p_B < \bar{t}_\beta\right] \Longrightarrow \tilde{v}_A(t_\gamma, p_A, p_B) < t_\gamma - p_B.$$

Hence $\tilde{v}_A(t_\gamma, x\dot{p}_A, x\dot{p}_B) < x\dot{p}_A$ for all $x \in \left[0, \min\left\{\bar{t}_\alpha/\dot{p}_A, \bar{t}_\beta/\dot{p}_B, t_\gamma/(\dot{p}_A + \dot{p}_B)\right\}\right]$ that are sufficiently close to $t_\gamma/(\dot{p}_A + \dot{p}_B)$. Thus, $x'(t_\gamma) < t_\gamma/(\dot{p}_A + \dot{p}_B)$, as claimed. To complete the proof, we only need to show that $\tilde{v}_A(t_\gamma, x\dot{p}_A, x\dot{p}_B)$ is strictly decreasing in x when x ranges in $\left[0, \min\left\{\bar{t}_\alpha/\dot{p}_A, \bar{t}_\beta/\dot{p}_B, t_\gamma/(\dot{p}_A + \dot{p}_B)\right\}\right]$. With x in this interval,

$$\tilde{v}_A(t_\gamma, x\dot{p}_A, x\dot{p}_B) = t_\gamma - \mathrm{E}\left[\mathbf{t}_\beta \mid x\dot{p}_B \le \mathbf{t}_\beta \le x\dot{p}_A + x\dot{p}_B\right].$$

Hence we need only to prove that $\mathrm{E}[\mathbf{t}_\beta \mid x\dot{p}_B \le \mathbf{t}_\beta \le x\dot{p}_A + x\dot{p}_B]$ is strictly increasing in x when x ranges in this interval. When $x\dot{p}_A + x\dot{p}_B \ge \bar{t}_\beta$, $\mathrm{E}[\mathbf{t}_\beta \mid x\dot{p}_B \le \mathbf{t}_\beta \le x\dot{p}_A + x\dot{p}_B]$ is obviously strictly increasing in x. Hence the proof is complete if

$$\frac{d}{dx}\left(\mathrm{E}[\mathbf{t}_\beta \mid x\dot{p}_B \le \mathbf{t}_\beta \le x\dot{p}_A + x\dot{p}_B]\right) > 0$$

when $x \in (0, \min\{t_\gamma, \bar{t}_\beta\}/(\dot{p}_A + \dot{p}_B))$: This derivative is equal to a positive term times

$$\begin{array}{c}[(\dot{p}_A + \dot{p}_B)^2 x f_\beta((\dot{p}_A + \dot{p}_B)x) - (\dot{p}_B)^2 x f_\beta(\dot{p}_B x)]\,[F_\beta((\dot{p}_A + \dot{p}_B)x) - F_\beta(\dot{p}_B x)] \\ -\,[(\dot{p}_A + \dot{p}_B) f_\beta((\dot{p}_A + \dot{p}_B)x) - \dot{p}_B f_\beta(\dot{p}_B x)] \int_{\dot{p}_B x}^{(\dot{p}_A+\dot{p}_B)x} t_\beta dF_\beta(t_\beta).\end{array} \quad (20)$$

By the mean value theorem, the above integral is equal to

$$(\dot{p}_B + \xi)x\,[F_\beta((\dot{p}_A + \dot{p}_B)x) - F_\beta(\dot{p}_B x)]$$

for some $\xi \in [0, \dot{p}_A]$. Thus, (20) is equal to

$$x\,[F_\beta((\dot{p}_A + \dot{p}_B)x) - F_\beta(\dot{p}_B x)]\,[(\dot{p}_A - \xi)(\dot{p}_A + \dot{p}_B) f_\beta((\dot{p}_A + \dot{p}_B)x) + \xi \dot{p}_B f_\beta(\dot{p}_B x)]\,,$$

which is positive, as desired. ■

Lemma 13 *If local bidders cross-bid and play strategy (b) in Lemma 10, global bidder γ has a unique best reply and it is—*

d. If an item has had a winner, global bidder γ follows plans 2 and 3 in Lemma 1. If neither item has had a winner, γ's strategy is:

i. if α has not quit A and β has not quit B, stay for both items until instant $x^*(t_\gamma)$ (i.e., when $(p_A, p_B) = (x^*(t_\gamma)\dot{p}_A, x^*(t_\gamma)\dot{p}_B)$) and quit at that instant;

ii. if α has quit A (and hence B) and if β stays for both items, continue for both items if $t_\gamma > 2p_A + p_B$ and quit both items if the inequality does not hold;

iii. if β has quit B (and hence A) and if α stays for both items, continue for both items if $t_\gamma > p_A + 2p_B$ and quit both items if the inequality does not hold.

Proof The actions prescribed by strategy (b) are obviously dominant except when neither item has had a winner. Hence consider such an instant with current prices (p_A, p_B):

First, suppose α has not quit A and β has not quit B. If global bidder γ wins A right now, then right now local bidder β must be quitting A and hence $p_A + p_B \geq t_\beta$ by strategy (b-iii*); however, almost surely β stays for A at this moment, for otherwise plans (b-iii*) and (b-ii) would imply that $p_B = t_\beta$, which is a zero-probability event conditional on the fact that the other local bidder α quits at this moment ($p_A = t_\alpha$). In other words, if he wins A now, global bidder γ learns that $p_B \leq t_\beta \leq p_A + p_B$; hence his expected profit from buying A now is equal to $\tilde{v}_A(t_\gamma, p_A, p_B) - p_A$. By the same argument, his expected profit from buying B now is equal to $\tilde{v}_B(t_\gamma, p_A, p_B) - p_B$. By Lemma 12, as prices ascend, the expected profit from buying A at the current prices stays positive up to a unique instant $x'(t_\gamma)$ (unless a local bidder has quit already) and then stays negative forever. Thus, conditional on staying for B, bidder γ stays for A up to this instant and then immediately quits at least A. By the same reasoning, conditional on staying for A, bidder γ stays for B up to the instant $x''(t_\gamma)$ (defined in (18)) and then immediately quits at least B. Thus, by the last two paragraphs in the proof of Lemma 1, with the substitutions $v_A \to \tilde{v}_A$ and $v_B \to \tilde{v}_B$, it is uniquely optimal for γ to stay for both items up to the instant $x^*(t_\gamma)$ (defined in (19)) and then immediately quit both items. Hence (d-i) is uniquely optimal given strategy (b).

Suppose α has quit A (and hence B) and β stays for both items. If γ wins A right now, he learns from strategy (b-iii*) that $t_\beta = p_A + p_B$ and hence his profit is equal to $t_\gamma - p_A - t_\beta = t_\gamma - 2p_A - p_B$. With prices ascending, this instantaneous profit is decreasing. Thus, it is uniquely optimal for bidder γ to follow plan (d-ii), i.e., bid for both items up to the instant where $t_\gamma = 2p_A + p_B$ and then immediately quit both. As plan (d-iii) is analogous to (d-ii), we have completed the proof. ■

Proposition 3 *If cross-bidding is allowed and jump-bidding is not, then:*

i. there exists a perfect Bayesian equilibrium (PBE) where cross-bidding does not occur on path: each local bidder bids only for his valued item up to its value and does not cross-bid; if no one is cross-bidding, global bidder γ follows the strategy (plans 1–3) in Lemma 1; if someone is cross-bidding, γ expects the cross-bidder to play strategy (a) in Lemma 10 and γ plays strategy (c) in Lemma 11;

ii. in any PBE where both local bidders cross-bid on path, they play strategy (b) in Lemma 10 *and the global bidder plays strategy (d) in Lemma 13.*

Proof Claim (ii) follows directly from Lemmas 10 and 13. By Lemmas 10 and 11, claim (i) is true if we prove two claims: First, local bidders' strategy (a) best replies global bidder γ's strategy (c); second, given γ's strategy (c), cross-bidding is suboptimal fro local bidders.

First, we show that (a) best replies (c). By Lemmas 10, it suffices to show that a cross-bidder does not have a profitable deviation from strategy (a-iii). A deviation from (a-iii) means: when a local bidder say α is alone competing with γ for both items, instead of staying for both items until their total price reaches t_α and then quitting both simultaneously, bidder α quits at least one of them now. For the deviation to be undominated, α quits B and stays for A. The fact that bidder γ has been staying up to now, with current prices (p_A, p_B), tells bidder α that $t_\gamma > p_A + p_B$. Thus, there exists a sufficiently $\epsilon > 0$ such that γ's expected profit from buying B right now, conditional on the posterior belief "$t_\alpha \in [p_A, p_A + \epsilon]$," is positive. Expecting bidder γ to take this posterior belief conditional on α's deviation, α expects that γ will not quit B when α deviates by quitting B and staying for A. After winning B, γ will bid for A up to t_γ. Thus, α's deviation yields a profit $(t_\alpha - t_\gamma)^+$, which is equal to his profit from following (a-iii), as bidder γ bids for both items until their total price reaches t_γ (plan (c-i)).

Given (a) and (c), let us find γ's *dropout point*, the first instant where he quits both items when each local bidder is bidding for some item. If both local bidders cross-bid, then by (c-ii) γ continues for both items if $t_\gamma > p_A + p_B$ and else quits both, so γ's dropout point is the unique intersection between the line $p_A + p_B = t_\gamma$ and the price path $p_B = (\dot{p}_B/\dot{p}_A)p_A$; denote this point by $(p_A^o(t_\gamma), p_B^o(t_\gamma))$. If α cross-bids and β does not, then γ continues for both items if $v_A(t_\gamma, p_B) > p_A$ and $t_\gamma > p_A + p_B$ (note that the first inequality implies the second) and else quits both, hence his dropout point is $(p'_A(t_\gamma), p'_B(t_\gamma))$ (intersection of (5) and (6)). If α does not cross-bid and β does, γ continuous for both items if $v_B(t_\gamma, p_A) > p_B$ (and hence $t_\gamma > p_A + p_B$) and else quits both, hence his dropout point is $(p''_A(t_\gamma), p''_B(t_\gamma))$ (intersection of (5) and (7)). If neither local bidders cross-bid, then γ's dropout point is the same as in Lemma 1, $(p^*_A(t_\gamma), p^*_B(t_\gamma))$. Note: $(p''_A(t_\gamma), p''_B(t_\gamma)) < (p^o_A(t_\gamma), p^o_B(t_\gamma))$ unless the two points coincide, and $(p^*_A(t_\gamma), p^*_B(t_\gamma)) < (p'_A(t_\gamma), p'_B(t_\gamma))$ unless they coincide.

Finally, we show that, given (a) and (c), cross-bidding is weakly dominated by not cross-bidding for a local bidder say α. First consider the case where bidder β cross-bids. If $(p''_A(t_\gamma), p''_B(t_\gamma)) = (p^o_A(t_\gamma), p^o_B(t_\gamma))$, α is indifferent about cross-bidding, because his winning event and payment are unaffected. Suppose the two points do not coincide. When $t_\beta \leq p''_B(t_\gamma)$, bidder α is again indifferent: whether α cross-bids or not, β quits before γ and then γ will bid up to t_γ, so α's winning event is $t_\alpha > t_\gamma$ and he pays t_γ if he wins. When $p''_B(t_\gamma) < t_\beta \leq p^o_B(t_\gamma)$, bidder α is better-off not cross-bid than cross-bid: if α cross-bids, β quits before γ, and so α's winning event is $t_\alpha > t_\gamma$ and he pays t_γ if he wins; if α does not cross-bid, γ quits before β, so α's winning event is $t_\alpha > p''_A(t_\gamma)$ and he pays $p''_A(t_\gamma)$ if he wins. Since $t_\gamma > p''_A(t_\gamma)$, cross-bidding is worse off. When $t_\beta > p^o_B(t_\gamma)$, bidder α again prefers not cross-bidding: γ's maximum bid for A is $p^o_A(t_\gamma)$ if α cross-bids and is a less amount $p''_A(t_\gamma)$ if α does not cross-bid. Thus, bidder α prefers not cross-bidding to cross-bidding when bidder β cross-bids. The case when bidder β does not cross-bid is similar: by switching from cross-bidding to not cross-bidding, α moves γ's dropout point from $(p'_A(t_\gamma), p'_B(t_\gamma))$ down to $(p^*_A(t_\gamma), p^*_B(t_\gamma))$; as in the previous case, this increases α's winning probability and decreases his payment if he wins. Thus, cross-bidding is suboptimal for a local bidder. ■

The existence of a cross-bidding equilibrium, which is unique according to Proposition 3-ii, may depend on the parameters, since a local bidder does not necessarily prefer cross-bidding at the start of the auctions. That is because, in the subgame where they do cross-bid, the global bidder's dropout point when both locals are active is higher than his dropout point in the subgame where local bidders do not cross-bid. This follows from the fact that $\tilde{v}_A(t_\gamma, x\dot{p}_A, x\dot{p}_B) > v_A(t_\gamma, x\dot{p}_B)$ and $\tilde{v}_B(t_\gamma, x\dot{p}_A, x\dot{p}_B) > v_B(t_\gamma, x\dot{p}_A)$ for all instants x when both auctions are still ongoing. Intuitively, since local bidder β is cross-bidding A, the global bidder γ's winning A implies both locals are quitting A, so β's value is not that high and hence γ's future competition with β for item B will not be that severe. On the other hand, this mitigating effect from cross-bidding may be counterbalanced by the other effect when one local bidder has quit and the other is cross-bidding. Then the global bidder is worried by the exposure problem and would quit before the total price reaches his value. We may not know which effect is dominant without knowing the specific type-distributions.

In contrast, the equilibrium in Proposition 3-i is relatively parameter-independent. In its off-path subgame where local bidders cross-bid, cross-bidding completely eliminates the global bidder's exposure problem. Hence cross-bidding increases the global bidder's maximum bids. Knowing this, local bidders would rather not cross-bid. As this equilibrium is on-path identical to the equilibrium in Lemma 1 when cross-bidding is banned, the equilibrium allocation is the same. All three kinds of inefficiency are probable: over-concentration, over-diffusion, and incomplete diffusion.

5.2 When jump bidding is allowed

Let us amend the model in the previous subsection with the protocol in §4.1 to allow jump bidding.

5.2.1 The incentive of jump bidding

If a local bidder has not been cross-bidding, then he cannot switch to cross-bidding later, as to be eligible for an item he needs to participate in its auction from the start. Thus, as in Lemma 6, this bidder strictly wants to jump-bid when the other local bidder is the first dropout (from both items); once the local bidder has initiated a jump-bidding process, it will determine the winner within the decisive moment.

A cross-bidder's jump-bidding incentive is more complicated. Say local bidder α is the first dropout with current prices (p_A, p_B) and local bidder β has been cross-bidding up to now. If by now $p_A + p_B \geq t_\beta$, then β must quit A immediately and stay for B (the fact that β has not quit by now implies $t_\beta > p_B$); hence it is dominant for him to jump-bid, as in Lemma 6. However, if $p_A + p_B < t_\beta$ still holds, bidder β does not need to quit A now and does not necessarily want to jump-bid: If he is expected to play strategy (b) in Lemma 10, bidder β can make no jump bid and stay cross-bidding, then global bidder γ quits before the total price reaches t_γ (Lemma 13-d-ii). In doing so, bidder β gains from suppressing his rival's maximum bid but loses the chance to couple his bid with the quitting local bidder α's bid. Thus, when parameters permit, bidder β may choose not to jump-bid.

However, if a cross-bidder is expected to play strategy (a) in Lemma 10 when he is alone competing with the global bidder, the global bidder's maximum bid is not suppressed by cross-bidding (Lemma 11); hence a local bidder would play strategy (a) if he did cross-bid and he finds it suboptimal to cross-bid (Proposition 3-i). Thus, he jump-bids when the other local bidder quits. The next subsections will formalize this observation.

5.2.2 An ascending package auction

If there is an equilibrium where jump-bidding always occurs when a local bidder becomes the first dropout, then the analysis in §4.2 suggests that the equilibrium turns the simultaneous auctions into an auction where the global bidder bids for the entire package $\{\mathrm{A}, \mathrm{B}\}$. Hence we consider a simple model of such package auctions next:

The price speeds $\dot{p}_A$ and $\dot{p}_B$ are exogenously given. Local bidder α bids only for package $\{\mathrm{A}\}$, β only for $\{\mathrm{B}\}$, and global bidder γ bids only for $\{\mathrm{A}, \mathrm{B}\}$. The price for a package starts at zero and rises continuously, with jump-bidding banned, until all but one bidder for the package have quit.

1. If no one has quit, the price p_A (or p_B) for package $\{\mathrm{A}\}$ (or $\{\mathrm{B}\}$) rises at speed $\dot{p}_A$ (or $\dot{p}_B$), and the price p_{AB} for $\{\mathrm{A}, \mathrm{B}\}$ rises at speed $\dot{p}_A + \dot{p}_B$.

2. If global bidder γ quits before local bidders α and β, item A is sold to bidder α at the current price p_A, and B is sold to β at the current price p_B. (Note that $p_A + p_B$ is equal to bidder γ's dropout price p_{AB}.)

3. If bidder α quits at price p_A before bidders β and γ, stop raising p_A, and raise p_B and p_{AB} at the same speed.

   a. If subsequently bidder β quits at price p_B before the global bidder γ, sell both items to γ at a total price equal to $p_A + p_B$.

   b. If subsequently bidder γ quits at price p_{AB} before local bidder β, sell item B to bidder β at its current price p_B, and sell A to bidder α at his dropout price p_A. (Note that $p_{AB} = p_A + p_B$.)

4. If bidder β quits at price p_B before bidders α and γ, do the same thing as in provision 3 by switching between A and B and between α and β.

5. If bidders α and β quit simultaneously before global bidder γ, γ wins.

In this package auction, the global bidder's weakly dominant strategy is to bid for the entire package until $p_{AB} = t_\gamma$, since the game to him is an English auction of a single bundle $\{\mathrm{A}, \mathrm{B}\}$. Once a local bidder say α has quit at p_A, local bidder β finds it dominant to bid for item B up to its true value; if he wins, β buys B at price $t_\gamma - p_A$ (as γ bids up to t_γ) and α buys A at his dropout price p_A. The case where local bidder β quits first is symmetric.

A *dropout price* means the price for a local bidder's valued item at which the bidder quits immediately unless someone has already quit. An (undominated-strategy) *equilibrium* in the package auction corresponds to a pair (s_α, s_β) of dropout strategies such that (i) s_i tells local bidder i what his dropout price is given his type and the current state of the game and (ii) s_α best replies s_β and vice versa, with the global bidder being straightforward.

5.2.3 Simultaneous auctions mimic the package auction

Concerned about the exposure problem of simultaneous auctions, researchers have considered replacing them by a package auction to allocate possibly complementary goods. The discussion has been going on for almost a decade[3] and is still unsettled, because a package auction has the shortcomings of threshold problem and combinatorial complexity. Adding to this discussion, the next proposition says that, at least in our simplistic model, there is no loss of generality to simply use simultaneous ascending auctions.[4]

Proposition 4 *For any equilibrium of the package auction defined in §5.2.2, there is a*

[3]The discussion dates back to at least as early as January 1995, when a version of Bykowsky, Cull, and Ledyard [6] was drafted. A recent paper leading this discussion is Ausubel and Milgrom [3].

[4]A caveat of this surprising message is that our model of package auctions does not allow global bidder γ to jump-bid. Hence the model might preclude a possible equilibrium where γ makes a jump bid for bundle $\{\mathrm{A}, \mathrm{B}\}$ without specifying the jump on each item; that may impose an additional threshold problem on the local bidders, who have to divide the jump between themselves if they do not want to lose right away. This additional problem might not be replicable in simultaneous auctions, because bidder γ in simultaneous auctions has to submit bids for each item and hence his jump is automatically divided between the local bidders. This caveat, however, does not undermine the normative implication of our message, because this possible additional threshold problem would make package auctions even more inefficient.

perfect Bayesian equilibrium in the simultaneous auctions, which allow jump- and cross-biddng, that generates the same allocation.

Proof Pick any equilibrium of the package auction. As noted in §5.2.2, the equilibrium corresponds to a pair (s_α, s_β) of local bidders' dropout strategies that best reply each other, with the global bidder straightforward. We shall prove that the following constitutes a perfect Bayesian equilibrium of the simultaneous auctions:

â. Whether local bidders cross-bid or not, global bidder γ stays for both items until their total price reaches his value or a local bidder jump-bids.

$\hat{\text{b}}$. A local bidder i does not cross-bid and keeps bidding for his valued item until someone else quits or his dropout price prescribed by s_i has been reached; his dropout action is "stop" rather than "withdraw".

ĉ. If a local bidder deviates to cross-bidding, he plays strategy (a) in Lemma 10.

$\hat{\text{d}}$. If a local bidder becomes the first dropout, then the other local bidder and γ play the jump-bidding subgame according to the equilibrium in Lemma 8.

We claim that, when a local bidder say α becomes the first dropout, if global bidder γ does not withdraw in the decisive moment, the other local bidder β's winning event for B is $\{t_\gamma : t_\beta > t_\gamma\}$ and his winning payment is t_γ in expectation, whether he cross-bids or not. The case where he is not cross-bidding is obvious, for then global bidder γ will win A at the end of the decisive moment and bid for B until $p_B = t_\gamma$. In the case where β does cross-bid, his strategy (ĉ) best replies strategy (â), due to the second paragraph in the proof of Proposition 3; thus, with β cross-bidding, it is optimal for γ to stay for both items until $p_A + p_B = t_\gamma$. Hence the claim follows.

Thus, when a local bidder say α becomes the first dropout, the other local bider gets the same payoff if the winner of B is not determined during the decisive moment, whether he is cross-bidding or not. Thus, the conclusion of Lemma 6 holds if strategies (â) and (ĉ) are expected: bidders β and γ each prefer ending the auction for B within the decisive moment. This has four immediate implications: First, the actions prescribed in ($\hat{\text{d}}$) are best replies. Second, global bidder γ never has to buy one item without knowing the price of its complement; hence he always bids until $p_A + p_B = t_\gamma$, and strategy (â) is his best reply. Third, with the global bidder always straightforward, a local bidder is indifferent about cross-bidding and hence not cross-bidding is a best reply. Fourth, any pair of dropout prices determines the same allocation outcome in simultaneous auctions as in the package auction analyzed in §5.2.2.

The fourth implication listed above implies that the simultaneous-auctions game, conditional on strategies (â), (ĉ), and ($\hat{\text{d}}$), is strategically equivalent to the package auction. As the dropout strategies (s_α, s_β) constitute an equilibrium in the package auction, they best reply each in the simultaneous auctions, given (â), (ĉ), and ($\hat{\text{d}}$). Thus, strategies (â)–($\hat{\text{d}}$) constitute an equilibrium, and its allocation is the same as the one generated by the equilibrium (s_α, s_β) of the package auction. ■

Note that any equilibrium of the package auction leads to probable over-concentration but never over-diffusion; this follows from Lemma 9, as (14) obviously holds due to the global bidder's straightforward behavior. Thus, Proposition 4 implies that the simultaneous-auctions game allowing jump- and cross-bidding has an over-concentrating equilibrium, as long as equilibrium exists in the package auction.[5]

6 Extension to resale

We have seen so far that equilibrium allocations of simultaneous auctions are inefficient with a positive probability. Thus, with the same type of arguments in Zhèng [17], we know that resale after the simultaneous auctions, if allowed, must occur with a positive probability. Even if resale is declared illegal, the strict incentive for resale makes the prohibition costly.

[5]When bidders' types and bids are continuous variables, existence of equilibrium in the package auction is an open nontrivial question. The complexity comes from the fact that a local bidder's payoff is discontinuous in his dropout price and the discontinuity contains both a drop and a jump: if local bidder α is quitting at instant x, local bidder β gets zero payoff if he also quits at x (both losing to the global bidder); whereas, β gets positive payoffs if he quits slightly before instant x (then α will bid up to t_α) or quits slightly after x (then β will bid up to t_β). We of course can ensure existence of equilibrium by assuming that types and bids are finite discrete and bearing the cost of messy calculations due to positive probabilities of ties.

Let us extend the model to incorporate resale. The main idea is to allow winners in an auction to select any selling mechanism for possible resale and give winners in his mechanism the same option. That might sound odd at first glance, because it treats the initial auctions exogenously and resale auctions endogenously. However, this formulation is actually most natural. Recall that the goal here is to understand the performance of a given mechanism, simultaneous ascending auctions. Hence it is appropriate to hold this initial auction mechanism as exogenous. Assuming exogenous resale mechanisms, in contrast, would be inappropriate, because renegotiation can take many forms, and we do not know a priori which specific format will be prevalent. It is therefore natural to let resale mechanisms emerge as optimal actions chosen by some players. To reflect the friction in bargaining, we use the standard mechanism-design formulation: allow one player to select a mechanism and commit to it by letting it operated by a neutral trustworthy mediator and, to be even-handed, preserve the privacy of the other players' types unless they are willingly revealed by the players themselves.

6.1 The auction-resale game

There are N periods, with no discounting, and N is an exogenous large number. (The exogenous N is to ensure that the equilibrium concept is well formed, as explained in Zhèng [17]). In period one, the items are auctioned off via simultaneous ascending auctions, which may allow or ban cross-bidding or jump-bidding. In period two, resale among bidders is allowed. If a bidder has won all items in period one, he can pick any mechanism (defined in the next paragraph) and commit to it for possible resale. ("No sale no matter what" is counted as one such mechanism.) If items are sold to different bidders, one of the winners is randomly selected to pick a resale mechanism; if no other winner vetoes it, the mechanism is implemented; else the mechanism is not implemented and every winner commits to a resale price for the item he currently owns. The probability with which a winner is selected to pick a resale mechanism is proportional to the number of items he currently owns. If a resale mechanism results in no-sale or if period N is reached, the game ends; else in the next period a winner is chosen to pick a resale mechanism, as in the current period.

A selling *mechanism* for player i (who is the current seller) is a mapping from the profile of types across players other than i to a payment arrangement and a lottery that assigns the items to the players (including i). The lottery is called *allocation outcome* from i's viewpoint.

6.2 Myerson auctions

If a seller could costlessly prohibit resale after the operation of his mechanism, then his mechanism becomes the endgame and his mechanism is *incentive feasible* if truth-telling is a Bayesian Nash equilibrium in this endgame. The *Myerson auction* for player i maximizes player i's expected profit among all incentive feasible selling mechanisms for i under the assumption that player i can costlessly prohibit resale after the operation of his mechanism. Given a type-profile, the *virtual utility* of an allocation outcome from player i's standpoint is defined to be the ex post gain of trade generated by this outcome minus

$$\sum_{j \notin i} \frac{1 - F_j(t_j)}{f_j(t_j)},$$

where index j ranges through all players but i who are involved in the trade specified by the allocation outcome, and t_j is the realized type of such a player j. For instance, if player γ sells item A to player α and B to β, the virtual utility from γ's viewpoint is equal to

$$t_\alpha + t_\beta - t_\gamma - \frac{1 - F_\alpha(t_\alpha)}{f_\alpha(t_\alpha)} - \frac{1 - F_\beta(t_\beta)}{f_\beta(t_\beta)}. \tag{21}$$

The next lemma, due to Levin [13], characterizes Myerson auctions in our multiple-object environment. It is proved by extending the standard optimal auctions technique and using the assumption that each bidder's type is one-dimensional.

Lemma 14 *Suppose that the hazard rate $f_i/(1 - F_i)$ for every player i ($i = \alpha, \beta, \gamma$) is weakly increasing. Then in any Myerson auction for player i, for almost every type-profile, the allocation outcome maximizes the virtual utility from i's standpoint among all allocation outcomes from i's standpoint.*

6.3 Endogenous separation between primary and resale markets

Since a bidder's action in period one will be used to update information about him, it is obvious that a bidder who expects a positive probability of buying an item at resale has an incentive to conceal his true value by shading his bids in period one. One type of bid shading that facilitates tractability is not to bid in period one at all. This strategy is a best reply if the bidder expects some other bidder to bid for all items no matter how high the prices become. Although such bidding behavior also constitutes an equilibrium in the case where resale is prohibited, it is weakly dominated there. In contrast, when resale is allowed, such bidding behavior is not weakly dominated. That is because a high bidder can consistently believe that his resale revenue can cover his payments for the items, and the bidders who shy away in period one can consistently believe that entering a bid in period one can only result in being charged a higher price at resale. (This is similar to the point made by Garratt and Tröger [7] for a single-good model.)

Next we construct a perfect Bayesian equilibrium of the auction-resale game where the global bidder wins all items in period one and then offers them for resale. Although there may be other equilibria where players other than the global bidder act as the reseller, the equilibrium constructed here is a more plausible focal point: Before bidders learn to exploit resale opportunities, the goods are over-concentrated to the global bidder, as shown in previous sections; hence it seems more plausible that the global bidder assumes the middleman role when the players learn about resale.

Lemma 15 *There is no loss of generality to assume that a Myerson auction for the global player γ has the property that item A never goes to bidder β and B never goes to α.*

Proof Selling A to β, global player γ receives at most zero revenue. Instead, keeping A to himself, γ's payoff is either zero (if γ does not keep item B) or almost surely positive (if γ also keeps B). Hence γ cannot decrease his payoff by keeping item A to himself instead of selling it bidder β. Symmetrically, γ cannot decrease his payoff by keeping B instead of selling it to α. The lemma then follows from the definition of Myerson auction. ■

Proposition 5 *If the hazard rate $f_i/(1-F_i)$ for every local bidder $i \in \{\alpha, \beta\}$ is weakly increasing, then there is a perfect Bayesian equilibrium where the global bidder γ wins both items in period one and offers resale to local bidders via the Myerson auction from his standpoint, and there is no further resale after the operation of γ's mechanism.*

Proof We shall show that the following constitutes a perfect Bayesian equilibrium:

a. In period one, the global bidder continues bidding until he wins both items, and local bidders do not participate in the auctions.

b. If no one deviates in period one, the global bidder in period two offers the items for possible resale via the Myerson auction from his viewpoint, based on the prior beliefs and subject to the property in Lemma 15; if a local bidder wins an item, he chooses not to resell it.

c. If a local bidder deviates to bidding in period one and quits item k at price p_k, the global bidder's resale mechanism in period two is the Myerson auction based on the posterior that the deviant bidder's type is at least as large as p_k (and again subject to the property in Lemma 15).

To verify this equilibrium, we need only to prove four claims: First, a local bidder who wins in the Myerson auction finds it optimal to not offer the item for further resale. Second, expecting no further resale, a local bidder finds it optimal to be truthful in the Myerson auction. Third, the Myerson auction is optimal for the global player conditional on his winning both items in period one. Fourth, given the resale mechanisms specified in (b) and (c), a local bidder finds it optimal not to bid in period one.

If the first claim is true, the second one follows: as the winners in the Myerson auction choose no-resale, the Myerson auction is the last stage of the auction-resale game, so truth-telling in the Myerson auction is optimal for each local bidder, as this auction is incentive feasible when it is the endgame. If the first and second claims are true, the third one follows, since the Myerson auction by definition maximizes the global player's expected profits among all incentive feasible mechanisms under the assumption the global player can costlessly ban resale. The fourth claim is obvious: by definition of the Myerson auction, if the infimum of the support of a bidder's type is higher, he cannot pay less.

Thus, it suffices to prove the first claim. After the Myerson auction, due to its property found in Lemma 15, local bidder α does not get item B and β does not get A. Suppose bidder α gets item A. He cannot profit from selling A to bidder β, who does not value A. Nor can he profit from selling back to player γ, since the fact that γ sells A to α via the Myerson auction implies that the virtual utility $t_\alpha - (1 - F_\alpha(t_\alpha))/f_\alpha(t_\alpha) - t_\gamma \geq 0$ (Lemma 14; note that the formula for virtual utilities is the same in cases (b) and (c)); as $\frac{1-F_i(t_i)}{f_i(t_i)} \geq 0$, we have $t_\alpha \geq t_\gamma$. Can α profit from coordinating with bidder β, in case that β wins B, to resell both items back to player γ? No because this trade leads to no surplus: the fact that the Myerson auction gives the two items to the local bidders implies that the virtual utility (21) is nonnegative and hence $t_\alpha + t_\beta \geq t_\gamma$. Hence bidder α chooses to not resell A. The other local bidder's no-resale incentive is symmetric. This proves the desired claim. ∎

Corollary 2 *At the equilibrium in Proposition 5, the final allocation is over-concentrated with a positive probability and is never over-diffused.*

Proof Since that equilibrium implements the Myerson auction from the global bidder's viewpoint, the equilibrium final allocation is determined by the virtual-utility algorithm characterized in Lemma 14. Since $\frac{1-F_i(t_i)}{f_i(t_i)} > 0$ for almost all types, the virtual utility (21) of selling A to α and B to β is less than the social surplus $t_\alpha + t_\beta - t_\gamma$ of this trade. Thus, if this trade eventually takes place, then its social surplus is positive; but even if its social surplus is positive, the virtual utility of the trade might still be negative and so the trade need not take place. Hence the final allocation is never over-diffused and is probably over-concentrated. ∎

The conclusion of Corollary 2 is true even without the optimal-resale-mechanism assumption, i.e., even if a reseller is not free to select any resale mechanism. As long as his mechanism is individually rational for the reseller from his standpoints both before and after the operation of his mechanism, the global player γ never resells item A to bidder β and never B to α (as in Lemma 15), and he never resells both items while the total revenue is less than his own value. Thus, as in the last paragraph in the proof of Proposition 5, there will be no further resale after γ's resale mechanism. As a monopolist at resale, γ would under-sell the goods if he has some policy instruments such as reserve prices. What the optimal-resale-mechanism assumption buys for us is a prediction of the final outcome of the game that can be directly calculated from the prior distributions.

6.4 Extension to more bidders

Let us extend Propositon 5 to the case where there are n_k copies of the k-bidder, with $k = \alpha, \beta, \gamma$. For each $k \in \{\alpha, \beta, \gamma\}$, assume that all the k-bidders value the same item and their values are independently drawn from the same distribution F_k.

Corollary 3 *Assume monotone hazard rate as in Propositon 5 and assume that, for each* $k \in \{\alpha, \beta, \gamma\}$, *the values of all the* k*-bidders are independently drawn from the same distribution* F_k. *There is a perfect Bayesian equilibrium where all global bidders participate in period-one auctions and all other bidders do not, the global bidder with the highest type wins both items in period one and offers resale to local bidders via the Myerson auction from his standpoint, and there is no further resale after the operation of his mechanism.*

Proof In period one, when another global bidder is staying, a global bidder's maximum bid is equal to his realized type plus the maximum expected profit that he can obtain during the resale stage; he keeps bidding for both items until $p_A + p_B$ reaches his maximum bid. If all but one global bidder have quit, the remaining global bidder will continue bidding forever should there be remaining local bidders. One can prove that the maximum bid is strictly increasing in the gloabl bidder's type by mimicking the envelope theorem argument in the proof of Proposition 2 in Zhèng [17] (from the start of that proof down to its second displayed equation). Hence the winner in period-1 is the one whose type is highest among all global bidders, so he suffers no loss to not include the other global bidders (γ-bidders) as potential buyers in his resale mechanism.

As in Propositon 5, we complete the proof by showing that a local bidder who wins in the global player's Myerson auction finds it optimal to not offer the item for further resale. For each $k \in \{\alpha, \beta, \gamma\}$, let k^* denote the bidder who has the highest type among the k-bidders. Since bidders of the same kind have the same distribution and hence the same strictly increasing virtual utility function, only bidders α^* and β^* have chances to win in the Myerson auction. By Lemma 14, we may assume without loss of generality that α^* does not get item B and β^* does not get A. Suppose bidder α^* gets item A. He cannot profit from selling A to other α-bidders, who value A less, or to β- or γ-bidders with $\gamma \neq \gamma^*$, who do not value A alone. As in the proof of Proposition 5, nor can α^* profit from selling A back to its previous owner γ^*. The only case left is where the two winning local bidders, α^* and β^*, could resell both items to a single γ-bidder. But this trade generates no surplus either, because $t_{\alpha^*} + t_{\beta^*} \geq t_{\gamma^*}$ as in the proof of Propositon 5, and $t_{\gamma^*} \geq t_{\gamma'}$ for any global bidder γ' who lost the period-one auctions to bidder γ^*. The case of β^* is symmetric. ■

Most results in the paper can be extended to this model. For example, the jump-bidding equilibrium of Lemma 8 becomes: when the last α-bidder is quitting A, all the remaining β-bidders jump-bid for B and each remaining γ-bidder either responds with a jump bid or quits. Item B is won by a β- or γ-bidder with the highest value of his kind.

7 Concluding remarks

In the literature, simultaneous auctions were thought to have a tendency of over-diffusing complementary goods to different bidders. To mitigate this problem, researchers have considered replacing simultaneous auctions by a package auction despite the latter's tendency to over-concentrate complementary goods to a global bidder. In contrast, the main finding in this paper is that the dominant feature of simultaneous ascending auctions is over-concentration and not over-diffusion. Specifically, if jump bidding is allowed, the simultaneous auctions can mimic whatever equilibrium outcome a package auction has, which tends to over-concentrate the goods to a single bidder. We also find that over-concentration is a robust feature when resale is allowed, since we have constructed an equilibrium where only the global bidders participate in the initial auctions and then the winner among them offers resale to the local bidders. As a monopolist at resale, the winner under-sells the goods, hence the over-concentration problem remains.

It should be emphasized that jump-biddng and resale, which our prediction is based on, are compelling constructs. As we have seen in this paper, even if the simultaneous ascending auctions prohibit or have no rule about jump-bidding and resale, bidders have a strict incentive to signal among one another via available communication channels and to trade with one another after the initial auctions. That leads to jump-bidding and resale.

References

[1] Gian Luigi Albano, Fabrizio Germano, and Stefano Lovo. A comparison of standard multi-unit auctions with synergies. *Economics Letters*, 71:55–60, 2001.

[2] Lawrence M. Ausubel. An efficient dynamic auction for heterogeneous commodities. Mimeo, Department of Economics, University of Maryland, July 17, 2002.

[3] Lawrence M. Ausubel and Paul Milgrom. Ascending auctions with package bidding. *Frontiers of Theoretical Economics*, 1(1):Article 1, 2002.

[4] Christopher Avery. Strategic jump bidding in English auctions. *Review of Economic Studies*, 65(2):185–210, April 1998.

[5] Sushil Bikhchandani, Sven de Vries, James Schummer, and Rakesh Vohra. Linear programming and Vickrey auctions. Mimeo, MEDS, Kellogg GSB, Northwestern University, July 5, 2001.

[6] Mark Bykowsky, Robert Cull, and John Ledyard. Mutually destructive bidding: The FCC auction design problem. *Journal of Regulatory. Economics*, 17:205–228, 2000.

[7] Rod Garratt and Thomas Tröger. Speculation in second-price auctions with resale. Mimeo, University of California, Santa Barbara, May 14, 2003.

[8] Faruk Gul and Ennio Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 87:95–124, 1999.

[9] Faruk Gul and Ennio Stacchetti. The English auction with differentiated commodities. *Journal of Economic Theory*, 92:66–95, 2000.

[10] Alan Gunderson and Ruqu Wang. Signaling by jump bidding in private values auctions. Mimeo, Institute for Economic Research Working Paper 975, Queen's University, October 1998.

[11] Vijay Krishna and Motty Perry. Efficient mechanism design. Mimeo, Pennsylvania State University and Hebrew University of Jerusalem, May 2000.

[12] Vijay Krishna and Robert W. Rosenthal. Simultaneous auctions with synergies. *Games and Economic Behavior*, 17:1–31, 1996.

[13] Jonathan Levin. An optimal auction for complements. *Games and Economic Behavior*, 18:176–192, 1997.

[14] Paul Milgrom. Putting auction theory to work: The simultaneous ascending auction. *Journal of Political Economy*, 108(2):245–272, April 2000.

[15] Roger B. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, February 1981.

[16] Robert W. Rosenthal and Ruqu Wang. Simultaneous auctions with synergies and common values. *Games and Economic Behavior*, 17:32–55, 1996.

[17] Charles Z. Zhèng. Optimal auction with resale. *Econometrica*, 70(6):2197–2224, November 2002.

Index