Open cups and open caps

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Let X be a set of n points in general position in the plane. By *general position* we mean that no three points lie on a line and no two points have the same x-coordinate. The set $Y \subseteq X$ is a *k-cup* or a *k-cap* if the points lie on the graph of a convex, resp. concave function. The set $Y \subseteq X$ is *open* in X if there is no point $p \in X$ with $x(q_1) < x(p) < x(q_k)$ lying above the polygonal line $p_1 p_2 \dots p_k$.

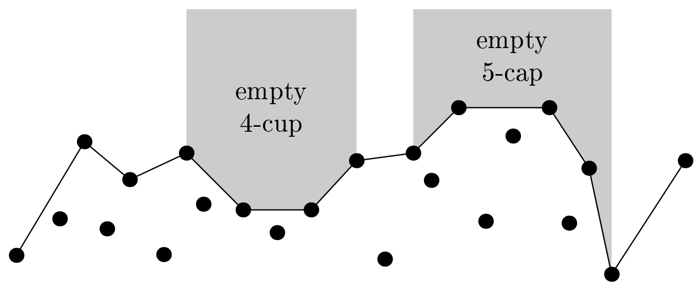

Figure 1: The set of points on the polygonal line is open. There is also the empty 4-cup and the empty 5-cap in the figure.

Erdős-Szekeres theorem says that for every positive integer k there exists positive integer N such that any N-point set contains k points that are vertices of a convex polygon. There are several proofs of the theorem using Ramsey theory and a proof using cups and caps. The latter proof gives much better upper bound on N.

Erdős also asked if for every k there exists N such that any N-point set X contain k vertices of an empty convex polygon. Empty polygon is a polygon with no point of X in its interior. We say that $Y \subseteq X$ is a k-hole if Y lies in the vertices of an empty convex k-gon. Today, this problem is solved and we know that Erdős conjecture holds only for $k \leq 6$.

What is the sufficient condition for existence of k-hole? The set X is l-convex if and only if every triangle determined by points of X contains at most l points of X in its interior. The l-convex sets were introduced by Valtr and he also showed that for every positive integers k and l there exists a positive integer N such that any l-convex N-point set X contains a k-hole.

Denote by $n(k, l)$ the smallest positive integer N such that any l-convex N-point set contains a k-hole. There were many improvements on the bounds of $n(k, l)$ and the last one is surprisingly an corrolary of theorem for open cups and open caps: For every positives integers k and l there exits positive integer N such that any N-point set in the plane contains an open k-cup or an open l-cap.

We show a simple proof of this theorem. Define $g(k, l)$ as the smallest number N such that any N-point set in general position contains an open k-cup or an open l-cap. We also show some new improvements for the bounds of $g(k, l)$. The old, but simply looking bounds are

$$2^{\binom{\lfloor k/2 \rfloor + \lfloor l/2 \rfloor - 2}{\lfloor k/2 \rfloor - 1}} \leq g(k, l) \leq 2^{\binom{k+l-4}{k-2}}.$$